Chapter 3

Interval Oscillation Criteria for Impulsive Partial Differential Equations
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3.1 Introduction

In the study conducted by Huang et.al. [56] using Kartsatos technique in the year 2006, the problem of oscillation and non-oscillation of impulsive delay equation of the form

\[ x''(t) + p(t)x(t - \rho) = e(t), \quad t \neq t_k, \]

\[ x(t_k^+) = \alpha_k x(t_k), \quad x'(t_k^+) = \beta_k x'(t_k), \quad k = 1, 2, \cdots. \]

Zhang et.al [151] also used the same approach to consider the oscillation of second order forced FDE with impulses

\[ x''(t) + p(t)f(x(t - \rho)) = e(t), \quad t \neq t_k, \]

\[ x(t_k^+) = \alpha_k x(t_k), \quad x'(t_k^+) = \beta_k x'(t_k), \quad k = 1, 2, \cdots \]

and established some interval oscillation criteria to develop the known results for the equations without delay or impulses [39, 89].

During the last and few decades, interval oscillation of impulsive differential equations arouses the interest of many researchers, see [49, 50, 75, 99, 104, 105, 127]. For further details, one can refer the monographs [13, 72, 141, 148] and reference cited therein. In the existing literature concentrated on interval oscillation criteria for case of without delay and only few papers appeared for case of with delay.
As far as the knowledge of the author it seems that there has been no paper dealing with interval oscillation criteria for impulsive partial differential equations.

Being motivated by this gap, the following impulsive partial differential equations is of the form:

\[
\begin{align*}
\frac{\partial}{\partial t} \left[ h(t) f \left( \frac{\partial}{\partial t} u(x, t) \right) \right] &+ p(x, t) g(u(x, t - \theta)) + \sum_{j=1}^{m} p_j(x, t) g_j(u(x, t - \theta)) \\
&= a(t) \Delta u(x, t) + \sum_{k=1}^{l} a_k(t) \Delta u(x, t - \sigma_s) + E(x, t), \quad t \neq \tau_k, \\
u(x, \tau_k^+) &= a_k(x, \tau_k, u(x, \tau_k)), \quad k = 1, 2, \ldots, \quad (x, t) \in \Omega \times \mathbb{R}_+ \equiv G,
\end{align*}
\]

(3.1.1)

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with a piecewise smooth boundary \( \partial \Omega \), \( \Delta \) is the Laplacian in the Euclidean space \( \mathbb{R}^N \) and \( \mathbb{R}_+ = [0, +\infty) \).

The following condition of Dirichlet boundary condition is enhances in the equation (3.1.1):

\[
u(x, t) = 0, \quad (x, t) \in \partial \Omega \times \mathbb{R}_+.
\]

Next, present the following set of conditions which assume to hold, throughout the chapter.

\((A_1)\) \( h(t) \in C^1(\mathbb{R}_+, (0, +\infty)) \), \( p(x, t), p_j(x, t) \in C(G, \mathbb{R}_+) \), \( p(t) = \min_{x \in \Omega} p(x, t) \), \( p_j(t) = \min_{x \in \Omega} p_j(x, t) \), \( j = 1, 2, \cdots, m \), \( g, g_j \in C(\mathbb{R}, \mathbb{R}) \) are convex in \( \mathbb{R}_+ \) with \( u g(u) > 0 \), \( u g_j(u) > 0 \) and \( \frac{g(u)}{u} \geq \epsilon > 0 \), \( \frac{g_j(u)}{u} \geq \epsilon_j > 0 \) for \( u \neq 0 \), \( j = 1, 2, \cdots, m \), \( 0 \leq t_0 = \tau_0 < \tau_1 < \tau_2 < \cdots \), \( \lim_{t \to +\infty} \tau_k = +\infty, t - \theta < t, t - \sigma_s < t \), \( \lim_{t \to +\infty} \frac{t - \theta}{t - \sigma_s} = +\infty, s = 1, 2, \cdots, l \) and \( E \in C(G, \mathbb{R}) \).

\((A_2)\) \( f \in C(\mathbb{R}, \mathbb{R}) \) are convex in \( \mathbb{R}_+ \) with \( uf(u) > 0 \), \( f(u) \leq \eta u \) for \( u \neq 0 \), \( f^{-1} \in C(\mathbb{R}, \mathbb{R}) \) are continuous functions with \( uf^{-1}(u) > 0 \) for \( u \neq 0 \) and there exist positive constant \( \zeta \) such that \( f^{-1}(uv) \leq \zeta f^{-1}(u)f^{-1}(v) \) for \( uv \neq 0 \) and \( \int_{t_0}^{+\infty} f^{-1}\left(\frac{1}{h(s)}\right) ds = +\infty. \)
(A3) $a(t), a_s(t) \in PC (\mathbb{R}_+, \mathbb{R}_+), s = 1, 2, \ldots, l$,

(i) $PC$ represents the class of functions which are piecewise continuous in $t$;

(ii) these function are discontinuities of first kind only at $t = \tau_k$, $k = 1, 2, \cdots$ and

(iii) left continuous at $t = \tau_k$, $k = 1, 2, \cdots$.

(A4) $u(x, t)$ and its derivative $u_t(x, t)$

(i) are piecewise continuous in $t$;

(ii) these function are discontinuities of first kind only at $t = \tau_k$, $k = 1, 2, \cdots$ and

(iii) left continuous at $t = \tau_k$, $u(x, \tau_k) = u(x, \tau_k^-)$, $u_t(x, \tau_k) = u_t(x, \tau_k^-)$, $k = 1, 2, \cdots$.

(A5) $a_k, b_k \in PC(\bar{\Omega} \times [0, +\infty), \mathbb{R})$, $k = 1, 2, \cdots$, and positive constants exist $\alpha_k, \alpha_k^*, \beta_k, \beta_k^*$ such that $\alpha_k^* \leq \alpha_k \leq \beta_k^* \leq \beta_k$ for $k = 1, 2, \cdots$,

$$
\alpha_k^* \leq \frac{a_k(x, \tau_k, u(x, \tau_k))}{u(x, \tau_k)} \leq \alpha_k, \quad \beta_k^* \leq \frac{b_k(x, \tau_k, u_t(x, \tau_k))}{u_t(x, \tau_k)} \leq \beta_k.
$$

**Definition 3.1.1.** By a solution $u$ of the problem (3.1.1), (3.1.2) is a function $u \in C^2(\bar{\Omega} \times [0, +\infty), \mathbb{R}) \cap C(\bar{\Omega} \times [0, +\infty), \mathbb{R})$ that satisfies (3.1.1), where

$$
t_{-1} := \min \left\{0, \min_{1 \leq s \leq l} \left\{ \inf_{t \geq 0} t - \sigma_s \right\} \right\}, \quad i_{-1} := \min \left\{0, \inf_{t \geq 0} t - \theta \right\}.
$$

**Definition 3.1.2.** By a solution $u$ of the problem (3.1.1), (3.1.2) is said to be oscillatory in the domain $G$ if it has arbitrary large zeros. Otherwise it is non-oscillatory.
For convenience, we introduce the following notations:
\[
v(t) = K \Phi \int_{\Omega} u(x,t) \Phi(x) dx, \quad P(t) = cp(t) + \sum_{j=1}^{m} \epsilon_j p_j(t)
\]
where \( K = \left( \int_{\Omega} \Phi(x) dx \right)^{-1}. \)

This chapter is organized as follows: The main results are given in Section 3.2.
In Section 3.3, one example is considered to illustrate the main results.

### 3.2 Oscillation Results

In this section, the intervals \([c_1, d_1]\) and \([c_2, d_2]\) are considered to establish oscillation criteria. So we also assume that

\((A_6)\) \(c_i, d_i \notin \{\tau_k\}, i = 1, 2, k = 1, 2, \ldots, \) with \(c_1 < d_1, c_2 < d_2\) and \(h(t) \geq 0, \)
\(p(t) \geq 0, p_j(t) \geq 0, j = 1, 2, \ldots, m\) for \(t \in [c_1 - \theta, d_1] \cup [c_2 - \theta, d_2]\) and \(E(t)\)
has different signs in \([c_1 - \theta, d_1]\) and \([c_2 - \theta, d_2]\), for instance, let
\[E(t) \leq 0 \quad \text{for} \quad t \in [c_1 - \theta, d_1], \quad \text{and} \quad E(t) \geq 0 \quad \text{for} \quad t \in [c_2 - \theta, d_2].\]

Denote
\[
I(s) := \max \{i : t_0 < t_i < s\}, \quad h_i := \max \{h(t) : t \in [c_i, d_i]\}, \quad i = 1, 2
\]
\[
J_q(c_i, d_i) = \left\{ q \in C^1[c_i, d_i], \; q(t) \neq 0, \; q(c_i) = q(d_i) = 0, \; i = 1, 2 \right\}
\]
\[
J_G(c_i, d_i) = \left\{ G \in C^1[c_i, d_i], \; G(t) \geq 0, \; G(t) \neq 0, \; G(c_i) = G(d_i) = 0, \; G'(t) = 2g(t)\sqrt{G(t)}, \; g(t) \in C[c_i, d_i], \; i = 1, 2 \right\}
\].

**Lemma 3.2.1.** If the impulsive differential inequality
\[
[h(t)f(v'(t))]' + ep(t)v(t - \theta) + \sum_{j=1}^{m} \epsilon_j p_j(t)v(t - \theta) \leq E(t), \quad t \neq \tau_k
\]
\[\alpha_k^* \leq \frac{v'(\tau^+_k)}{v(\tau_k)} \leq \alpha_k; \quad \beta_k^* \leq \frac{v'(\tau^+_k)}{v'(\tau_k)} \leq \beta_k, \quad k = 1, 2, \ldots, \]
\[
\begin{aligned}
& (3.2.1) 
\end{aligned}
\]
has no eventually positive solution, then each solution of the problem defined by (3.1.1), (3.1.2) is oscillatory in $G$.

**Proof.** Suppose to the contrary that there is a non-oscillatory solution $u(x,t)$ of the boundary value problem (3.1.1), (3.1.2). Without loss of generality, we may assume that $u(x,t) > 0$ in $\Omega \times [t_0, +\infty)$ for some $t_0 > 0$, $u(x, t - \theta) > 0$ and $u(x, t - \sigma_s) > 0$, $s = 1, 2, \cdots, l$.

For $t \neq \tau_k, t \geq t_0, k = 1, 2, \cdots$, equation (3.1.1) is multiplied both sides by $K_\Phi \Phi(x) > 0$ and integrating with respect to $x$ over the domain $\Omega$, we obtain

$$
\frac{d}{dt} \left[ h(t) \int_\Omega u(x,t)\Phi(x)dx \right] + K_\Phi \int_\Omega p(x,t)g(u(x, t - \theta))\Phi(x)dx \\
+ \sum_{j=1}^m K_\Phi \int_\Omega p_j(x,t)g_j(u(x, t - \theta))\Phi(x)dx = a(t)K_\Phi \int_\Omega \Delta u(x,t)\Phi(x)dx \\
+ \sum_{s=1}^l a_s(t)K_\Phi \int_\Omega \Delta u(x, t - \sigma_s)\Phi(x)dx + K_\Phi \int_\Omega E(x,t)\Phi(x)dx.
$$

(3.2.2)

By applying Green’s formula and (3.1.2) is the boundary condition,

$$
K_\Phi \int_\Omega \Delta u(x,t)\Phi(x)dx = K_\Phi \int_{\partial \Omega} \left[ \Phi(x) \frac{\partial u}{\partial \eta} - u \frac{\partial \Phi(x)}{\partial \eta} \right] dS + K_\Phi \int_\Omega u(x,t)\Delta \Phi(x)dx \\
= 0 - \lambda_0 v(t) \leq 0,
$$

(3.2.3)

and for $s = 1, 2, \cdots, l$, we have

$$
K_\Phi \int_\Omega \Delta u(x, t - \sigma_s)\Phi(x)dx = K_\Phi \int_{\partial \Omega} \left[ \Phi(x) \frac{\partial u(x, t - \sigma_s)}{\partial \eta} - u(x, t - \sigma_s) \frac{\partial \Phi(x)}{\partial \eta} \right] dS \\
+ K_\Phi \int_\Omega u(x, t - \sigma_s)\Delta \Phi(x)dx \\
= 0 - \lambda_0 v(t - \sigma_s) \leq 0,
$$

(3.2.4)
where $dS$ is surface component on $\partial \Omega$. Furthermore Jensen’s inequality applied for convex functions and by using the hypothesis $(A_1)$, we get that

$$K \Phi \int_\Omega p(x,t)g(u(x,t-\theta))\Phi(x)dx \geq p(t)K \Phi \int_\Omega g(u(x,t-\theta))\Phi(x)dx \geq \epsilon p(t)v(t-\theta),$$

(3.2.5)

and for $j = 1, 2, \ldots, m$

$$\sum_{j=1}^m K \Phi \int_\Omega p_j(x,t)g_j(u(x,t-\theta))\Phi(x)dx \geq \sum_{j=1}^m p_j(t)K \Phi \int_\Omega g_j(u(x,t-\theta))\Phi(x)dx \geq \sum_{j=1}^m \epsilon_j p_j(t)v(t-\theta).$$

(3.2.6)

Take

$$E(t) = K \Phi \int_\Omega E(x,t)\Phi(x)dx.$$  

(3.2.7)

Combining (3.2.2)-(3.2.7), we get that

$$[h(t)f(v'(t))]' + \epsilon p(t)v(t-\theta) + \sum_{j=1}^m \epsilon_j p_j(t)v(t-\theta) \leq E(t).$$

For $t = \tau_k$, $k = 1, 2, \ldots$, equation (3.1.1) is multiplied both sides by $K \Phi \Phi(x) > 0$, integrating with respect to $x$ over the domain $\Omega$, and from $(A_5)$, we obtain

$$\alpha_k^* \leq \frac{u(x,\tau_k^+)}{u(x,\tau_k^-)} \leq \alpha_k, \quad \beta_k^* \leq \frac{\partial u(x,\tau_k^+)}{\partial \tau_k} \leq \beta_k.$$

Since $v(t) = K \Phi \int_\Omega u(x,t)\Phi(x)dx$, we have

$$\alpha_k^* \leq \frac{v(\tau_k^+)}{v(\tau_k^-)} \leq \alpha_k, \quad \beta_k^* \leq \frac{v'(\tau_k^+)}{v'(\tau_k^-)} \leq \beta_k.$$

Therefore $v(t)$ is an eventually positive solution of (3.2.1), which contradicts the hypothesis and completes the proof.  \(\square\)
Theorem 3.2.1. Assume that conditions \((A_1) - (A_5)\) hold, furthermore for any \(T \geq 0\) there exist \(c_i, d_i\) satisfying \((A_6)\) with \(T \leq c_1 < d_1, T \leq c_2 < d_2\) and \(q(t) \in J_q(c_1, d_1)\) such that

\[
\int_{c_i}^{T_{I(c_i) + 1}} P(t)q^2(t)M^i_q(t)dt + \sum_{k=I(c_i) + 1}^{I(d_i) - 1} \int_{\tau_k}^{\tau_{k+1}} P(t)q^2(t)M^i_k(t)dt \\
+ \int_{\tau_{I(d_i)}}^{d_i} P(t)q^2(t)M^i_{I(d_i)}(t) - \int_{c_i}^{d_i} \eta h(t)(q'(t))^2dt \geq \Lambda(q, c_i, d_i) \quad (3.2.8)
\]

where \(\Lambda(q, c_i, d_i) = 0\) for \(I(c_i) = I(d_i)\) and

\[
\Lambda(q, c_i, d_i) = h_i \left\{ q^2(t_{I(c_i) + 1}) - \alpha^i_{I(c_i) + 1}(\tau_{I(c_i) + 1} - c_i) + \sum_{k=I(c_i) + 2}^{I(d_i)} q^2(\tau_k) - \frac{\beta_k - \alpha^i_k}{\alpha^i_k(\tau_k - \tau_{k-1})} \right\}
\]

for \(I(c_i) < I(d_i), i = 1, 2\)

\[
M^i_k(t) = \begin{cases} 
\frac{t - \tau_k}{\alpha_k \theta + \beta_k(t - \tau_k)}, & t \in (\tau_k, \tau_k + \theta) \\
\frac{t - \theta - \tau_k}{t - \tau_k}, & t \in [\tau_k + \theta, \tau_{k+1})
\end{cases}
\]

then each solution of the problem \((3.1.1), (3.1.2)\) is oscillatory in \(G\).

Proof. It is proved in the proceeding of Lemma [3.2.1] to get that

\[
[h(t)f(v'(t))]' \leq E(t) - cp(t)v(t - \theta) - \sum_{j=1}^{m} \epsilon_j p_j(t)v(t - \theta) \quad \text{for} \ t \in [\tau_1, +\infty).
\]

(3.2.9)

Thus \(v'(t) \geq 0\) or \(v'(t) < 0, t \geq \tau_1\) for some \(\tau_1 \geq t_0\). We now claim that

\[
v'(t) \geq 0 \quad \text{for} \ t \geq \tau_1.
\]

(3.2.10)

Suppose not, then \(v'(t) < 0\) and there exists \(\tau_2 \in [\tau_1, +\infty)\) such that \(v'(\tau_2) < 0\). Since \(h(t)f(v'(t))\) is strictly decreasing on \([\tau_1, +\infty)\). It is clear that

\[
h(t)f(v'(t)) < h(\tau_2)f(v'(\tau_2)) := -\mu
\]
where $\mu > 0$ is a constant for $t \in [\tau_2, +\infty)$, we have

$$h(t)f(v'(t)) < -\mu$$

$$v'(t) < f^{-1}\left(\frac{-\mu}{h(t)}\right)$$

$$v'(t) \leq -\zeta_1 f^{-1}\left(\frac{1}{h(t)}\right), \quad \text{where} \quad \zeta_1 = \zeta f^{-1}(\mu) \text{ for } t \in [\tau_2, +\infty).$$

Integrating the above inequality from $\tau_2$ to $t$, we have

$$v(t) \leq v(\tau_2) - \zeta_1 \int_{\tau_2}^{t} f^{-1}\left(\frac{1}{h(s)}\right) ds.$$

Letting $t \to +\infty$, we get

$$\lim_{t \to +\infty} v(t) = -\infty$$

which contradiction proves that (3.2.10) holds. Define the Riccati Transformation

$$y(t) := -\frac{h(t)f(v'(t))}{v(t)}. \quad (3.2.11)$$

It follows from (3.2.9) that $y(t)$ satisfies

$$y'(t) \geq -\frac{E(t)}{v(t)} + \left[\epsilon p(t) + \sum_{j=1}^{m} \epsilon_j p_j(t)\right] \frac{v(t-\theta)}{v(t)} + \frac{y^2(t)}{\eta h(t)}.$$

By the assumption, we can choose $c_1, d_1 \geq t_0$ such that $h(t) \geq 0$, $p(t) \geq 0$ and $p_j(t) \geq 0$ for $t \in [c_1 - \theta, d_1]$, $j = 1, 2, \cdots, m$ and $E(t) \leq 0$ for $t \in [c_1 - \theta, d_1]$ from (3.2.9) we can easily to see that

$$[h(t)f(v'(t))]' - E(t) + \epsilon p(t)v(t-\theta) + \sum_{j=1}^{m} \epsilon_j p_j(t)v(t-\theta) \leq 0$$

for $t \in [c_1, d_1]$, $t \neq \tau_k$, $k = 1, 2, \cdots$, thus

$$y'(t) \geq \frac{y^2(t)}{\eta h(t)} + P(t) \frac{v(t-\theta)}{v(t)}. \quad (3.2.12)$$
For \( t = \tau_k, \ k = 1, 2, \ldots \), one has
\[
y(\tau_k^+) = -\frac{h(\tau_k^+)f(v'(\tau_k^+))}{v(\tau_k^+)} \geq \frac{\beta_k}{\alpha_k}y(\tau_k). \tag{3.2.13}
\]

At first, we consider the case in which \( I(c_1) < I(d_1) \). In this case, all the impulsive moments in \([c_1, d_1]\) are \( \tau_{I(c_1)+1}, \tau_{I(c_1)+2}, \ldots, \tau_{I(d_1)} \). Choose an \( q(t) \in J_q(c_1, d_1) \) and equation (3.2.12) is multiplied both sides by \( q^2(t) \), integrating it from \( c_1 \) to \( d_1 \), we obtain
\[
\int_{c_1}^{\tau_{I(c_1)+1}} q^2(t)y'(t)dt + \int_{\tau_{I(c_1)+1}}^{\tau_{I(c_1)+2}} q^2(t)y'(t)dt + \cdots + \int_{\tau_{I(d_1)}}^{d_1} q^2(t)y'(t)dt \geq \int_{c_1}^{\tau_{I(c_1)+1}} q^2(t)\frac{y^2(t)}{\eta h(t)}dt + \int_{\tau_{I(c_1)+1}}^{\tau_{I(c_1)+2}} q^2(t)\frac{y^2(t)}{\eta h(t)}dt + \cdots + \int_{\tau_{I(d_1)}}^{d_1} q^2(t)\frac{y^2(t)}{\eta h(t)}dt
\]
\[
+ \int_{c_1}^{\tau_{I(c_1)+1}} q^2(t)P(t)\frac{v(t-\theta)}{v(t)}dt + \int_{\tau_{I(c_1)+1}}^{\tau_{I(c_1)+2}} q^2(t)P(t)\frac{v(t-\theta)}{v(t)}dt + \cdots + \int_{\tau_{I(d_1)-1+\theta}}^{\tau_{I(d_1)}} q^2(t)P(t)\frac{v(t-\theta)}{v(t)}dt
\]
\[
+ \int_{\tau_{I(d_1)}}^{d_1} q^2(t)P(t)\frac{v(t-\theta)}{v(t)}dt.
\]

The integration by parts is being used on the left-hand side, and noting that the condition \( q(c_1) = q(d_1) = 0 \), we get
\[
\sum_{k=I(c_1)+1}^{I(d_1)-1} q^2(\tau_k)\left[y(\tau_k) - y(\tau_k^+)\right] \geq \int_{c_1}^{\tau_{I(c_1)+1}} \frac{\eta}{h(t)} \left[h(t)q'(t) + \frac{q(t)y(t)}{\eta} \right]^2 dt
\]
\[
+ \sum_{k=I(c_1)+1}^{I(d_1)-1} \int_{\tau_k}^{\tau_k+\theta} \frac{\eta}{h(t)} \left[h(t)q'(t) + \frac{q(t)y(t)}{\eta} \right]^2 dt + \int_{\tau_{I(d_1)}}^{d_1} \frac{\eta}{h(t)} \left[h(t)q'(t) + \frac{q(t)y(t)}{\eta} \right]^2 dt
\]
\[
+ \int_{c_1}^{\tau_{I(c_1)+1}} q^2(t)P(t)\frac{v(t-\theta)}{v(t)}dt + \sum_{k=I(c_1)+1}^{I(d_1)-1} \int_{\tau_k}^{\tau_k+\theta} q^2(t)P(t)\frac{v(t-\theta)}{v(t)}dt
\]
\[
+ \int_{\tau_{I(d_1)}}^{d_1} q^2(t)P(t)\frac{v(t-\theta)}{v(t)}dt - \int_{c_1}^{d_1} \eta h(t)(q'(t))^2 dt. \tag{3.2.14}
\]
To estimate \( \frac{v(t - \theta)}{v(t)} \), there has several cases to be consider:

**Case 1:** For \( t \in (\tau_k, \tau_{k+1}] \subset [c_1, d_1] \). If \( t \in (\tau_k, \tau_{k+1}] \subset [c_1, d_1] \), since \( \tau_{k+1} - \tau_k > \theta \), we consider two sub cases:

**Case 1.1:** If \( t \in [\tau_k + \theta, \tau_{k+1}] \), then \( t - \theta \in [\tau_k, \tau_{k+1} - \theta] \) and there are no impulsive moments in \((t - \theta, t)\), then for any \( t \in [\tau_k + \theta, \tau_{k+1}] \), one has

\[
v(t) - v(\tau_k^+) = v'(\xi_1)(t - \tau_k), \quad \xi_1 \in (\tau_k, t).
\]

Since \( h(t)f(v'(t)) \) is decreasing

\[
v(t) \geq v'(\xi_1)(t - \tau_k) > \frac{h(t)f(v'(t))}{h(\xi_1)}(t - \tau_k).
\]

From the fact that \( h(t) \) is increasing, we get

\[
\frac{h(t)f(v'(t))}{v(t)} < \frac{h(\xi_1)}{t - \tau_k} < \frac{h(t)}{t - \tau_k}.
\]

We obtain

\[
\frac{v'(t)}{v(t)} < \frac{1}{t - \tau_k}.
\]

Integrating it from \( t - \theta \) to \( t \), we have

\[
\frac{v(t - \theta)}{v(t)} > \frac{t - \theta - \tau_k}{t - \tau_k}.
\]

**Case 1.2:** If \( t \in (\tau_k, \tau_k + \theta) \) then \( t - \theta \in (\tau_k - \theta, \tau_k) \) and there is an impulsive moment \( \tau_k \) in \((t - \theta, t)\). Similar to Case 1.1, we obtain

\[
v(t) - v(\tau_k - \theta) = v'(\xi_2)(t - \tau_k + \theta), \quad \xi_2 \in (\tau_k - \theta, \tau_k]
\]

or

\[
\frac{v'(t)}{v(t)} < \frac{1}{t - \tau_k + \theta}.
\]
Integrating it from \( t - \theta \) to \( t \), we get
\[
\frac{v(t - \theta)}{v(\tau_k)} > \frac{t - \tau_k}{\theta} \geq 0, \quad t \in (\tau_k, \tau_k + \theta). \tag{3.2.15}
\]

For any \( t \in (\tau_k, \tau_k + \theta) \), we have
\[
v(t) - v(\tau_k^+) < v'(\tau_k^+)(t - \tau_k).
\]

Using the impulsive conditions in equation (3.2.1), we get
\[
v(t) - \alpha_k v(\tau_k) < \beta_k v'(\tau_k)(t - \tau_k)
\]
\[
\frac{v(t)}{v(\tau_k)} < \frac{v'(\tau_k)}{v(\tau_k)} \beta_k(t - \tau_k) + \alpha_k.
\]

Using \( \frac{v'(\tau_k)}{v(\tau_k)} < \frac{1}{\theta} \), we obtain
\[
\frac{v(t)}{v(\tau_k)} < \alpha_k + \frac{1}{\theta} \beta_k(t - \tau_k).
\]

That is,
\[
\frac{v(\tau_k)}{v(t)} > \frac{\theta}{\alpha_k \theta + \beta_k(t - \tau_k)}. \tag{3.2.16}
\]

From (3.2.15) and (3.2.16), we get
\[
\frac{v(t - \theta)}{v(t)} > \frac{t - \tau_k}{\alpha_k \theta + \beta_k(t - \tau_k)} \geq 0.
\]

**Case 2:** If \( t \in [c_1, \tau_{I(c_1)+1}] \), we consider three sub cases:

**Case 2.1:** If \( \tau_{I(c_1)} > c_1 - \theta \) and \( t \in [\tau_{I(c_1)} + \theta, \tau_{I(c_1)+1}] \) then \( t - \theta \in [\tau_{I(c_1)}, \tau_{I(c_1)+1} - \theta] \) and there are no impulsive moments in \( (t - \theta, t) \). Similarly analysis the Case 1.1 and using Mean-value Theorem on \( (\tau_{I(c_1)}, \tau_{I(c_1)+1}] \), we get
\[
\frac{v(t - \theta)}{v(t)} > \frac{t - \theta - \tau_{I(c_1)}}{t - \tau_{I(c_1)}} \geq 0.
\]
Case 2.2: If $\tau_{I(c_1)} > c_1 - \theta$ and $t \in [c_1, \tau_{I(c_1)} + \theta)$, then $t - \theta \in [c_1 - \theta, \tau_{I(c_1)})$ and there is an impulsive moments $\tau_{I(c_1)}$ in $(t - \theta, t)$. Similar analysis of the Case 1.2, it is shown as

$$\frac{v(t - \theta)}{v(t)} > \frac{t - \tau_{I(c_1)}}{\alpha_{I(c_1)} \theta + \beta_{I(c_1)} (t - \tau_{I(c_1)})} \geq 0.$$ 

Case 2.3: If $\tau_{I(c_1)} < c_1 - \theta$, then for any $t \in [c_1, \tau_{I(c_1)} + 1]$, $t - \theta \in [c_1 - \theta, \tau_{I(c_1)} + 1 - \theta]$ and there are no impulsive moments in $(t - \theta, t)$. Similar analysis of the Case 1.1, it is proved as

$$\frac{v(t - \theta)}{v(t)} > \frac{t - \theta - \tau_{I(c_1)}}{t - \tau_{I(c_1)}} \geq 0.$$ 

Case 3: For $t \in (\tau_{I(d_1)}, d_1]$, there are three sub cases:

Case 3.1: If $\tau_{I(d_1)} + \theta < d_1$ and $t \in [\tau_{I(d_1)} + \theta, d_1]$ then $t - \theta \in [\tau_{I(d_1)}, d_1 - \theta]$ and there are no impulsive moments in $(t - \theta, t)$. Similar analysis of the Case 2.1, it is shown as

$$\frac{v(t - \theta)}{v(t)} > \frac{t - \theta - \tau_{I(d_1)}}{t - \tau_{I(d_1)}} \geq 0.$$ 

Case 3.2: If $\tau_{I(d_1)} + \theta < d_1$ and $t \in [\tau_{I(d_1)}, \tau_{I(d_1)} + \theta)$, then $t - \theta \in [\tau_{I(d_1)} - \theta, \tau_{I(d_1)})$ and there is an impulsive moments $\tau_{I(d_1)}$ in $(t - \theta, t)$. Similar analysis of the Case 2.2, it is proved as

$$\frac{v(t - \theta)}{v(t)} > \frac{t - \tau_{I(d_1)}}{\alpha_{I(d_1)} \theta + \beta_{I(d_1)} (t - \tau_{I(d_1)})} \geq 0.$$ 

Case 3.3: If $\tau_{I(d_1)} + \theta \geq d_1$, then for any $t \in (\tau_{I(d_1)}, d_1]$, we get $t - \theta \in (\tau_{I(d_1)} - \theta, d_1 - \theta]$ and there is an impulsive moments $\tau_{I(d_1)}$ in $(t - \theta, t)$. Similar analysis of the Case 3.2, it is shown as

$$\frac{v(t - \theta)}{v(t)} > \frac{t - \tau_{I(d_1)}}{\alpha_{I(d_1)} \theta + \beta_{I(d_1)} (t - \tau_{I(d_1)})} \geq 0.$$
Combining all these cases, we have

\[
v(t - \theta) > \begin{cases} 
M^1_{I_1(c_1)}(t) & \text{for } t \in [c_1, \tau_{I_1(c_1)}] \\
M^1_k(t) & \text{for } t \in (\tau_k, \tau_{k+1}], \quad k = I(c_1) + 1, \ldots, I(d_1) - 1 \\
M^1_{I_1(d_1)}(t) & \text{for } t \in (\tau_{I_1(d_1)-1}, d_1]. 
\end{cases}
\]

Hence by (3.2.14), we have

\[
\sum_{k=I_1(c_1)+1}^{I(d_1)} q^2(\tau_k) \left[ y(\tau_k) - y(\tau_k^+) \right] \geq \int_{c_1}^{\tau_{I_1(c_1)+1}} q^2(t) P(t) M^1_{I_1(c_1)}(t) dt \\
+ \sum_{k=I_1(c_1)+1}^{I(d_1)-1} \int_{\tau_k}^{\tau_{k+1}} q^2(t) P(t) M^1_k(t) dt \\
+ \int_{\tau_{I_1(d_1)}}^{d_1} q^2(t) P(t) M^1_{I_1(d_1)}(t) dt - \int_{c_1}^{d_1} \eta h(t)(q')(t)^2 dt. 
\]

(3.2.17)

Since \([h(t)f'(v(t))] < 0\) for all \(t \in (c_1, \tau_{I_1(c_1)+1}], h(t)f'(v'(t))\) is decreasing in \((c_1, \tau_{I_1(c_1)+1}].\)

Thus

\[
v(t) > v(t) - v(c_1) = v'(\xi_4)(t - c_1) \geq \frac{h(t)f(v'(t))}{h(\xi_4)}(t - c_1), \quad \xi_4 \in (c_1, t)
\]

and hence \(\frac{h(t)f(v'(t))}{v(t)} < \frac{h(\xi_4)}{t - c_1}.\) Letting \(t \to \tau_{I_1(c_1)+1}^-,\) it follows that

\[
y(\tau_{I_1(c_1)+1}) \geq -\frac{h_1}{\tau_{I_1(c_1)+1} - c_1}. 
\]

(3.2.18)

Similarly we can prove that on \((\tau_{k-1}, \tau_k], k = I(c_1) + 2, \ldots, I(d_1),\)

\[
y(\tau_k) \geq -\frac{h_1}{\tau_k - \tau_{k-1}}. 
\]

(3.2.19)

Hence (3.2.18) and (3.2.19), we have

\[
\sum_{k=I_1(c_1)+1}^{I(d_1)} q^2(\tau_k) y(\tau_k) \left[ \frac{\beta_k - \alpha^*_k}{\alpha^*_k} \right] \geq -h_1 \left[ q^2(\tau_{I_1(c_1)+1}) \frac{\beta_{I_1(c_1)+1} - \alpha^*_{I_1(c_1)+1}}{\alpha^*_{I_1(c_1)+1}} \frac{1}{\tau_{I_1(c_1)+1} - c_1} \\
+ \sum_{k=I_1(c_1)+2}^{I(d_1)} q^2(\tau_k) \frac{\beta_k - \alpha^*_k}{\alpha^*_k} \frac{1}{\tau_k - \tau_{k-1}} \right] \\
\geq -\Lambda(q, c_1, d_1).
\]
Thus we have
\[ \sum_{k=I(c_1)+1}^{I(d_1)} q^2(\tau_k)g(\tau_k) \left[ \frac{\alpha_k^* - \beta_k}{\alpha_k^*} \right] \leq \Lambda(q, c_1, d_1). \]
Therefore (3.2.17), we get
\[
\int_{c_1}^{I(c_1)+1} q^2(t) P(t) M_{I(c_1)}^1(t) dt + \sum_{k=I(c_1)+1}^{I(d_1)-1} \int_{\tau_k}^{\tau_{k+1}} q^2(t) P(t) M_k^1(t) dt \\
+ \int_{\tau_{I(d_1)}}^{d_1} q^2(t) P(t) M_{I(d_1)}^1(t) dt - \int_{c_1}^{d_1} \eta h(t)(q'(t))^2 dt < \Lambda(q, c_1, d_1)
\]
which contradicts (3.2.8).

If \( I(c_1) = I(d_1) \) then \( \Lambda(q, c_1, d_1) = 0 \) and there are no impulsive moments in \([c_1, d_1]\). To prove it similarly, the proof of (3.2.17) is result as, we obtain
\[
\int_{c_1}^{d_1} [q^2(t) P(t) M_{I(c_1)}(t) - \eta h(t)(q'(t))^2] dt < 0.
\]
This again contradicts our assumption. Finally if \( v(t) \) is eventually negative, we can consider \([c_2, d_2]\) and reach similar contradiction. The proof of theorem is complete.

\[
\square
\]

**Theorem 3.2.2.** Suppose that \((A_1) - (A_3)\) hold, furthermore for any \( T > 0 \) there exist \( c_i, d_i \) satisfying \((A_6)\) with \( T \leq c_1 < d_1, T \leq c_2 < d_2 \) and \( G(t) \in JG(c_i, d_i) \) such that
\[
\int_{c_i}^{\tau_{I(c_i)}+1} G(t) P(t) M_{I(c_i)}^1(t) dt + \sum_{k=I(c_i)+1}^{I(d_i)-1} \int_{\tau_k}^{\tau_{k+1}} G(t) P(t) M_k^1(t) dt \\
+ \int_{\tau_{I(d_i)}}^{d_i} G(t) P(t) M_{I(d_i)}^1(t) dt - \int_{c_i}^{d_i} \eta h(t) g^2(t) dt \geq \Theta(G, c_i, d_i) \tag{3.2.20}
\]
where \( \Theta(G, c_i, d_i) = 0 \) for \( I(c_i) = I(d_i) \) and
\[
\Theta(G, c_i, d_i) = h_i \left\{ \frac{\beta_{I(c_i)+1} - \alpha_{I(c_i)+1}^*}{\alpha_{I(c_i)+1}^*(\tau_{I(c_i)}+1) - c_i} \sum_{k=I(c_i)+2}^{I(d_i)} G(\tau_k) \frac{\beta_k - \alpha_k^*}{\alpha_k^*(\tau_k - \tau_{k-1})} \right\}
\]
for \( I(c_i) < I(d_i) \), \( i = 1, 2 \), then each solution of the problem (3.1.1), (3.1.2) is oscillatory in \( G \).
Proof. To prove is similarly to the proof of Theorem 3.2.1 suppose $v(t - \theta)$ for $t \geq t_0$. If $I(c_i) < I(d_i)$, multiplying $G(t)$ throughout (3.2.12) and integrating over $[c_i, d_i]$, we get

$$
\sum_{k=I(c_i)+1}^{I(d_i)} G(\tau_k) \left[ y(\tau_k) - y(\tau_k^+) \right] \geq \int_{c_i}^{\tau(c_i)+1} \frac{1}{\eta} \left[ \sqrt{\frac{G(t)}{h(t)}} y(t) + \eta g(t) \sqrt{h(t)} \right]^2 dt \\
+ \sum_{k=I(c_i)+1}^{I(d_i)-1} \int_{\tau_k}^{\tau_k^+} \frac{1}{\eta} \left[ \sqrt{\frac{G(t)}{h(t)}} y(t) + \eta g(t) \sqrt{h(t)} \right]^2 dt \\
+ \int_{\tau I(d_i)}^{d_1} \frac{1}{\eta} \left[ \sqrt{\frac{G(t)}{h(t)}} y(t) + \eta g(t) \sqrt{h(t)} \right]^2 dt + \int_{c_i}^{\tau(c_i)+1} G(t) P(t) \frac{v(t - \theta)}{v(t)} dt \\
+ \sum_{k=I(c_i)+1}^{I(d_i)-1} \int_{\tau_k}^{\tau_{k+1}} G(t) P(t) \frac{v(t - \theta)}{v(t)} dt + \int_{\tau_{k+1}}^{\tau_k + \theta} G(t) P(t) \frac{v(t - \theta)}{v(t)} dt \\
+ \int_{\tau I(d_i)}^{d_1} G(t) P(t) \frac{v(t - \theta)}{v(t)} dt - \int_{c_i}^{d_1} \eta h(t) g^2(t) dt \\
\geq \int_{c_i}^{\tau(c_i)+1} G(t) P(t) M_{r(c_i)}^1 dt + \sum_{k=I(c_i)+1}^{I(d_i)-1} \int_{\tau_k}^{\tau_{k+1}} G(t) P(t) M_{r(d_i)}^1 dt \\
+ \int_{\tau I(d_i)}^{d_1} G(t) P(t) M_{r(d_i)}^1 dt - \int_{c_i}^{d_1} \eta h(t) g^2(t) dt. \tag{3.2.21}
$$

On the other hand, from the proof of Theorem 3.2.1 we have

$$
y(\tau I(c_i)+1) \geq -\frac{h_1}{\tau I(c_i)+1 - c_i}, \quad y(\tau_k) \geq -\frac{h_1}{\tau_k - \tau_k-1}
$$

for $k = I(c_i) + 2, \cdots, I(d_i)$. We get

$$
\sum_{k=I(c_i)+1}^{I(d_i)} G(\tau_k) y(\tau_k) \left[ \frac{\beta_k - \alpha_k^*}{\alpha_k^*} \right] \geq -h_1 \left[ G(\tau I(c_i)+1) \left( \frac{\beta I(c_i)+1 - \alpha I(c_i)+1}{\alpha I(c_i)+1} \right) \frac{1}{\tau I(c_i)+1 - c_i} \right. \\
+ \sum_{k=I(c_i)+2}^{I(d_i)} G(\tau_k) \left( \frac{\beta_k - \alpha_k^*}{\alpha_k^*} \right) \frac{1}{\tau_k - \tau_k-1} \right] \\
\geq -\Theta(G, c_i, d_i).
$$

Thus we have

$$
\sum_{k=I(c_i)+1}^{I(d_i)} G(\tau_k) y(\tau_k) \left[ \frac{\alpha_k^* - \beta_k}{\alpha_k^*} \right] \leq \Theta(G, c_i, d_i).
$$
Therefore (3.2.21), we get

\[ \int_{\tau_{I(c_1)}}^{\tau_{I(d_1)}} G(t) P(t) M_{I(c_1)}^1(t) dt + \sum_{k=I(c_1)+1}^{I(d_1)-1} \int_{\tau_k}^{\tau_{k+1}} G(t) P(t) M_{k}^1(t) dt \]

\[ + \int_{\tau_{I(d_1)}}^{d_1} G(t) P(t) M_{I(d_1)}^1(t) dt - \int_{c_1}^{d_1} \eta r(t) g^2(t) dt \leq \Theta(G, c_1, d_1) \]

which contradicts (3.2.20). If \( I(c_1) = I(d_1) \), the proof is similar to that the Theorem 3.2.1, and so it is omitted here. The proof of theorem is complete. \( \square \)

Next, we will establish Kemenev type oscillation criteria for (3.1.1) following the ideas of [64] and [108]. Let \( \mathbb{D} = \{(t,s) : t_0 \leq s \leq t \} \), then a function \( H \in C(\mathbb{D}, \mathbb{R}) \) is said to belong to the class \( \mathcal{H} \) if

\( (A_7) \) \( H(t,t) = 0, H(t,s) > 0 \) for \( t > s \) and

\( (A_8) \) \( H \) has partial derivative \( \frac{\partial H}{\partial t} \) and \( \frac{\partial H}{\partial s} \) on \( \mathbb{D} \) such that

\[ \frac{\partial H}{\partial t} = 2h_1(t,s)\sqrt{H(t,s)}, \quad \frac{\partial H}{\partial s} = -2h_2(t,s)\sqrt{H(t,s)} \]

where \( h_1, h_2 \in L_{loc}(\mathbb{D}, \mathbb{R}). \)

The following two lemmas are needed to prove our theorem.

**Lemma 3.2.2.** Suppose that \( (A_1) - (A_5) \) hold and \( v(t) \) is a solution of (3.1.1), (3.1.2). If there exist \( \theta_i \in (c_i, d_i), \theta_i \notin \{\tau_k\}, i = 1, 2 \) such that \( v(t) > 0 \) on \( [\theta_1, d_1] \) and \( v(t) < 0 \) on \( [\theta_2, d_2] \) then for any \( H \in \mathcal{H} \)

\[ \int_{\theta_i}^{\tau_{I(\theta_i)}+1} H(d_i, s) P(s) M_{I(\theta_i)}^1(t) dt + \sum_{k=I(\theta_i)+1}^{I(d_i)-1} \int_{\tau_k}^{\tau_{k+1}} H(d_i, s) P(s) M_{k}^1(t) dt \]

\[ + \int_{\tau_{I(d_i)}}^{d_i} H(d_i, s) P(s) M_{I(d_i)}^1(t) dt \leq \sum_{k=I(\theta_i)+1}^{I(d_i)} H(d_i, \tau_k) \frac{\alpha_k - \beta_k}{\alpha_k^*} y(\tau_k) - H(d_i, \theta_i) y(\theta_i) \]

\[ + \int_{\theta_i}^{d_i} \eta h(s) h_2^2(d_i, s) ds. \quad (3.2.22) \]
**Proof.** Using the Theorem 3.2.1 as a proof to proceed, and multiply (3.2.12) by \( H(t,s) \). Integrating with respect to \( s \) from \( \theta_i \) to \( t \) for \( t \in [\theta_i, d_i) \), \( i = 1, 2 \), it follows from \((A_7)\) and \((A_8)\) that

\[
\int_{\theta_i}^{t} H(t,s) P(s) \frac{v(t - \theta)}{v(t)} \, ds \leq \int_{\theta_i}^{t} H(t,s) y'(s) \, ds - \int_{\theta_i}^{t} H(t,s) \frac{y^2(s)}{\eta h(s)} \, ds
\]

\[
= \left( \int_{\theta_i}^{\tau_{I(\theta_i)+1}} + \int_{\tau_{I(\theta_i)+1}}^{\tau_{I(\theta_i)+2}} + \cdots + \int_{\tau_{I(\theta_i)}}^{t} \right) H(t,s) \, ds
\]

\[
- \left( \int_{\theta_i}^{\tau_{I(\theta_i)+1}} + \int_{\tau_{I(\theta_i)+1}}^{\tau_{I(\theta_i)+2}} + \cdots + \int_{\tau_{I(\theta_i)}}^{t} \right) H(t,s) \frac{y^2(s)}{\eta h(s)} \, ds
\]

\[
\leq \sum_{k=I(\theta_i)+1}^{I(t)} H(t,\tau_k)y(\tau_k) \left( \frac{\alpha_k - \beta_k}{\alpha_k^*} \right) - H(t,\theta_i)y(\theta_i)
\]

\[
+ \int_{\theta_i}^{t} \eta h(s) h^2(t, s) \, ds.
\]

Now, let \( t \to d_i \), which yields that (3.2.22) holds. Thus the lemma is proved. \( \square \)

**Lemma 3.2.3.** Suppose that \((A_1) - (A_3)\) hold and \( v(t) \) is a solution of \((3.1.1)\), \((3.1.2)\). If there exist \( \theta_i \in (c_i, d_i) \), \( \theta_i \notin \{\tau_k\}, i = 1, 2 \) such that \( v(t) > 0 \) on \([c_1, \theta_1)\) and \( v(t) < 0 \) on \([c_2, \theta_2)\) then for any \( H \in \mathcal{H} \)

\[
\int_{c_i}^{\tau_{I(c_i)+1}} H(s,c_i) P(s) M_i^i(t) \, dt + \sum_{k=I(c_i)+1}^{I(\theta_i)-1} \int_{\tau_k}^{\tau_{I(c_i)+1}} H(s,c_i) P(s) M_k^i(t) \, dt
\]

\[
+ \int_{\tau_{I(\theta_i)}}^{\theta_i} H(s,c_i) P(s) M_{I(\theta_i)}^i(t) \, dt \leq \sum_{k=I(c_i)+1}^{I(\theta_i)} H(\tau_k,c_i) \frac{\alpha_k^* - \beta_k}{\alpha_k^*} y(\tau_k) + H(\theta_i,c_i) y(\theta_i)
\]

\[
+ \int_{c_i}^{\theta_i} \eta h(s) h^2(t, s) \, ds.
\] (3.2.23)

**Proof.** Similarly to the proof of Lemma 3.2.2 multiplying (3.2.12) by \( H(s,t) \) for \( i = 1, 2 \),
we have
\[
\int_{t}^{\theta_i} H(s, t) P(s) \frac{v(t - \theta)}{v(t)} \leq \int_{t}^{\theta_i} H(s, t) y'(s) ds - \int_{t}^{\theta_i} H(s, t) \frac{w^2(s)}{\eta h(s)} ds
\]
\[
= \left( \int_{t}^{\tau_{I(t)} + 1} + \int_{\tau_{I(t)} + 1}^{\tau_{I(t)} + 2} + \cdots + \int_{\tau_{I(\theta_i)}}^{\theta_i} \right) H(s, t) d(y(s))
\]
\[
- \left( \int_{t}^{\tau_{I(t)} + 1} + \int_{\tau_{I(t)} + 1}^{\tau_{I(t)} + 2} + \cdots + \int_{\tau_{I(\theta_i)}}^{\theta_i} \right) H(s, t) \frac{y^2(s)}{\eta h(s)} ds
\]
\[
\leq \sum_{k=I(I(t))}^{I(\theta_i)} H(\tau_k, t) y(\tau_k) \left( \frac{\alpha_k^* - \beta_k}{\alpha_k^*} \right) + H(\theta_i, t) y(\theta_i)
\]
\[
+ \int_{t}^{\theta_i} \eta h(s) h_2^2(s, t) ds.
\]
Thus, it yields (3.2.23) by letting \( t \to c_i^+ \). The lemma is complete.

**Theorem 3.2.3.** Suppose that \((A_1) - (A_5)\) hold. Assume that there are \( \theta_i \in (c_i, d_i) \), \( i = 1, 2 \), and \( H \in \mathcal{H} \) such that

\[
\frac{1}{H(d_i, \theta_i)} \left[ \int_{\theta_i}^{\tau_{I(\theta_i)} + 1} H(d_i, s) P(s) M_i^I(\theta_i)(t) dt + \sum_{k=I(\theta_i) + 1}^{I(d_i) - 1} \int_{\tau_k}^{\tau_{k+1}} H(d_i, s) P(s) M_i^k(t) dt \right]
\]
\[
+ \int_{\tau_{I(d_i)}}^{d_i} H(d_i, s) P(s) M_i^I(d_i)(t) dt - \int_{\theta_i}^{d_i} \eta h(s) h_2^2(d_i, s) ds
\]
\[
\frac{1}{H(\theta_i, c_i)} \left[ \int_{c_i}^{\tau_{I(c_i)} + 1} H(s, c_i) P(s) M_i^I(c_i)(t) dt + \sum_{k=I(c_i) + 1}^{I(\theta_i) - 1} \int_{\tau_k}^{\tau_{k+1}} H(s, c_i) P(s) M_i^k(t) dt \right]
\]
\[
+ \int_{\tau_{I(c_i)}}^{\theta_i} H(s, c_i) P(s) M_i^I(c_i)(t) dt - \int_{c_i}^{\theta_i} \eta h(s) h_2^2(s, c_i) ds \geq \Xi(H, c_i, d_i) \tag{3.2.24}
\]

where \( \Xi(H, c_i, d_i) = 0 \) for \( I(c_i) = I(d_i) \) and

\[
\Xi(H, c_i, d_i)
\]
\[
= \frac{h_i}{H(d_i, \theta_i)} \left[ H(d_i, \tau_{I(\theta_i)} + 1) \frac{\beta_{I(\theta_i)+1} - \alpha_{I(\theta_i)+1}}{\alpha_{I(\theta_i)+1}(\tau_{I(\theta_i)} + 1 - \theta_i)} + \sum_{k=I(\theta_i) + 2}^{I(d_i)} H(d_i, \tau_k) \frac{\beta_k - \alpha_k^*}{\alpha_k^*(\tau_k - \tau_{k-1})} \right]
\]
\[
+ \frac{h_i}{H(\theta_i, c_i)} \left[ H(\tau_{I(c_i)} + 1, c_i) \frac{\beta_{I(c_i)+1} - \alpha_{I(c_i)+1}^*}{\alpha_{I(c_i)+1}^*(\tau_{I(c_i)} + 1 - c_i)} + \sum_{k=I(c_i) + 2}^{I(\theta_i)} H(\tau_k, c_i) \frac{\beta_k - \alpha_k^*}{\alpha_k^*(\tau_k - \tau_{k-1})} \right]
\]
for $I(c_i) < I(d_i)$, $i = 1, 2$ then each solution of the problem (3.1.1), (3.1.2) is oscillatory in $G$.

**Proof.** From Lemmas 3.2.2 and 3.2.3 we see that (3.2.22) and (3.2.23) with $i = 1$ hold. Dividing (3.2.22) and (3.2.23) by $H(d_1, \theta_1)$ and $H(\theta_1, c_1)$, respectively, then adding them, we obtain

$$
\frac{1}{H(d_1, \theta_1)} \left[ \int_{\theta_1}^{T(\theta_1) + 1} H(d_1, s) P(s) M_k^1(t) dt + \sum_{k=I(\theta_1) + 1}^{I(d_1) - 1} \int_{\tau_k}^{\tau_{k+1}} H(d_1, s) P(s) M_k^1(t) dt \right]
$$

$$
+ \int_{\tau(\theta_1)}^{d_1} H(d_1, s) P(s) M_k^1(t) dt - \int_{\theta_1}^{d_1} \eta h(s) h_2^2(d_1, s) ds
$$

$$
+ \frac{1}{H(\theta_1, c_1)} \left[ \int_{c_1}^{T(c_1) + 1} H(s, c_1) P(s) M_k^1(t) dt + \sum_{k=I(c_1) + 1}^{I(\theta_1) - 1} \int_{\tau_k}^{\tau_{k+1}} H(s, c_1) P(s) M_k^1(t) dt \right]
$$

$$
+ \int_{\tau(\theta_1)}^{\theta_1} H(s, c_1) P(s) M_k^1(t) dt - \int_{c_1}^{\theta_1} \eta h(s) h_1^2(s, c_1) ds
$$

$$
\leq \frac{1}{H(d_1, \theta_1)} \left[ \sum_{k=I(\theta_1) + 1}^{I(d_1)} H(d_1, \tau_k) \frac{\alpha_k}{\beta_k} \alpha_k - \beta_k y(\tau_k) - \int_{\tau_1}^{d_1} h_2^2(d_1, s) \eta h(s) ds \right]
$$

$$
+ \frac{1}{H(\theta_1, c_1)} \left[ \sum_{k=I(c_1) + 1}^{I(\theta_1)} H(\tau_k, c_1) \frac{\alpha_k}{\beta_k} \alpha_k - \beta_k y(\tau_k) - \int_{c_1}^{\theta_1} h_1^2(s, c_1) \eta h(s) ds \right].
$$

The right-hand side of the above inequality the first term is replaced by $c_1$ by $\theta_1$ and the second term $d_1$ by $\theta_1$ in (3.2.18) and (3.2.19). Then

$$
\frac{1}{H(d_1, \theta_1)} \left[ \int_{\theta_1}^{T(\theta_1) + 1} H(d_1, s) P(s) M_k^1(t) dt + \sum_{k=I(\theta_1) + 1}^{I(d_1) - 1} \int_{\tau_k}^{\tau_{k+1}} H(d_1, s) P(s) M_k^1(t) dt \right]
$$

$$
+ \int_{\tau(\theta_1)}^{d_1} H(d_1, s) P(s) M_k^1(t) dt - \int_{\theta_1}^{d_1} \eta h(s) h_2^2(d_1, s) ds
$$

$$
+ \frac{1}{H(\theta_1, c_1)} \left[ \int_{c_1}^{T(c_1) + 1} H(s, c_1) P(s) M_k^1(t) dt + \sum_{k=I(c_1) + 1}^{I(\theta_1) - 1} \int_{\tau_k}^{\tau_{k+1}} H(s, c_1) P(s) M_k^1(t) dt \right]
$$

$$
+ \int_{\tau(\theta_1)}^{\theta_1} H(s, c_1) P(s) M_k^1(t) dt - \int_{c_1}^{\theta_1} \eta h(s) h_1^2(s, c_1) ds \right].
$$
there is an impulsive movement

\[ \Omega = (0, T < \infty) \]

\[ \theta = \pi = 1 \]

\[ F(x, t) = 9 \sin x \cos(t - \frac{\pi}{8}) + m \sin x \cos(t - \frac{\pi}{8}) \]

which contradicts (3.2.24). The proof is complete.

\[ \square \]

3.3 Example

In this section, we present an example to illustrate our results established in Section 3.2.

**Example 3.3.1.** Consider the following impulsive partial differential equations of the form

\[
\begin{align*}
\frac{\partial}{\partial t} \left[ 6 \left( \frac{\partial}{\partial t} u(x, t) \right) \right] + mu \left( x, t - \frac{\pi}{8} \right) + 4u \left( x, t - \frac{\pi}{8} \right) &= 6\Delta u(x, t) + 5\Delta u \left( x, t - \frac{\pi}{8} \right) + E(x, t), \quad t \neq 2k\pi \pm \frac{\pi}{4}, \\
u(x, \tau_k^+) = \frac{1}{3} u(x, \tau_k), \quad u_t(x, \tau_k^+) = \frac{2}{3} u_t(x, \tau_k), \quad k = 1, 2, \ldots,
\end{align*}
\]

for \((x, t) \in (0, \pi) \times \mathbb{R}_+\), with the boundary condition

\[
u(0, t) = u(\pi, t) = 0, \quad t \neq 2k\pi \pm \frac{\pi}{4}, \quad k = 1, 2, \ldots.
\]

Here \(\Omega = (0, \pi), \ N = 1, \ \alpha_k = \alpha_k^* = \frac{1}{3}, \ \beta_k = \beta_k^* = \frac{2}{3}, \ h(t) = 3, \ p(t) = m, \)

\(p_1(t) = 4, \ f(u) = 2u, \ g(u) = g_1(u) = u, \ a(t) = 6, \ a_1(t) = 5, \ \sigma_1 = \frac{\pi}{8}, \ \eta = 2, \)

\(F(x, t) = 9 \sin x \cos(t - \frac{\pi}{8}) + m \sin x \cos(t - \frac{\pi}{8}) \) and \(m\) is a positive constant. Also \(\theta = \frac{\pi}{8}, \ \tau_{k+1} - \tau_k = \pi/2 > \pi/8\). For any \(T > 0\), we choose \(k\) large enough such that \(T < c_1 = 4k\pi - \frac{\pi}{2} < d_1 = 4k\pi\) and \(c_2 = 4k\pi + \frac{\pi}{8} < d_2 = 4k\pi + \frac{\pi}{2}, \ k = 1, 2, \ldots\). Then there is an impulsive movement \(\tau_k = 4k\pi - \frac{\pi}{2} \in [c_1, d_1]\) and an impulsive moment \(\tau_{k+1} = 4k\pi + \frac{\pi}{4} \in [c_2, d_2]\). For \(\epsilon = \epsilon_1 = 1\), we have \(P(t) = m + 4, \) and we take
\[ q(t) = \sin 16t \in J_{q(c_i, d_i)}, \ i = 1, 2, \ \tau_{I(c_1)} = 4k\pi - \frac{7\pi}{4}, \ \tau_{I(d_1)} = 4k\pi - \frac{\pi}{4}, \text{ then by using simple calculation, the left side of Equation (3.2.8) is the following:} \]

\[
\int_{c_1}^{\tau_{I(c_1)} + 1} P(t)q^2(t)M^1_{I(c_1)}(t)dt + \sum_{k=I(c_1)+1}^{I(d_1)-1} \int_{\tau_k}^{\tau_{k+1}} P(t)q^2(t)M^1_k(t)dt \\
+ \int_{t_{I(d_1)}}^{d_1} P(t)q^2(t)M^1_{I(d_1)}(t) - \int_{c_1}^{d_1} \eta h(t)(q'(t))^2 dt \\
\geq (m + 4) \left[ \int_{4k\pi - \frac{\pi}{4}}^{4k\pi - \frac{\pi}{2}} \sin^2(16t) \left( \frac{t - \frac{\pi}{8} - 4k\pi + \frac{7\pi}{4}}{t - 4k\pi + \frac{7\pi}{4}} \right) dt \\
+ \int_{4k\pi - \frac{\pi}{8}}^{4k\pi - \frac{\pi}{4}} \sin^2(16t) \left( \frac{t - 4k\pi + \frac{\pi}{4}}{\frac{\pi}{8} \left( \frac{1}{3} + \frac{2}{3} \right) \left( t - 4k\pi + \frac{\pi}{4} \right)} \right) dt \\
+ \int_{4k\pi - \frac{\pi}{8}}^{4k\pi - \frac{\pi}{4}} \sin^2(16t) \left( \frac{t - \frac{\pi}{8} - 4k\pi + \frac{\pi}{4}}{t - 4k\pi + \frac{\pi}{4}} \right) dt \right] - 3(16)^2 \int_{4k\pi - \frac{\pi}{2}}^{4k\pi} (1 + \cos 32t) dt \\
\simeq (m + 4)(0.27685) - 603.18578.
\]

for \( m \) large enough. On the other hand, note that \( I(c_1) = k + 1, I(d_1) = k, \ r_1 = 3 \), we have \( \Lambda(q, c_1, d_1) = 0. \) Therefore the equation (3.2.8) is satisfied in \([c_1, d_1]\).

Similarly, we can prove that for \( t \in [c_2, d_2] \). Hence by Theorem 3.2.1, each solution of (3.3.1), (3.3.2) is oscillatory. One such solution is \( u(x, t) = \sin x \cos t. \)

We conclude this chapter with the following remark.

**Remark 3.3.1.** In this chapter, we have obtained some sufficient conditions to the impulsive partial differential equations. The improvement factors impulses, delay and forcing term that affect the interval qualitative properties of solution in the sequence of subintervals in \( \mathbb{R}_+ \), were taken into account together. Our newly obtained results in this chapter have improved and extended some of the results already prevailing in the existing literature.