Chapter 5

Henstock-Kurzweil Laplace Transform

5.1 Introduction

The Laplace transform of a function \( f : [0, \infty) \to \mathbb{R} \), at \( s \in \mathbb{C} \), is defined as the integral
\[
\int_0^\infty e^{-st} f(t) \, dt.
\] (5.1)

Many authors, in some previous decades, showed their interest in studying the Laplace transform in a classical sense; by classical meaning Lebesgue integral and/or Riemann integral on the real line.

In this chapter, we focus on an integral transform that generalizes the classical Laplace transform. Here the generalization is given by the replacement, in the definition of Laplace transform, i.e., in (5.1), of Lebesgue integral with Henstock-Kurzweil (HK) integral. The Henstock-Kurzweil integral is defined in terms of Riemann sums but with slight modification, yet it includes the Riemann, improper Riemann, and Lebesgue integrals as special cases. This integral is equivalent to the Denjoy and Perron integrals.
The first question arises is that whether (or in which conditions) the Laplace transform when treated as HK integral make sense. The sufficient condition for the existence of Laplace transform is that the function \( f \) is locally integrable on \([0, \infty)\), because the product of locally integrable function and bounded measurable function is locally integrable. Since Henstock-Kurzweil integral allows nonabsolute convergence, it makes an ideal setting for the Laplace transform. It was proved [9] that, on a compact interval, the product of nonabsolute integrable function and a bounded variation function is still nonabsolute integrable and this remains valid in case of an unbounded interval [1]. We observed in (5.1) that the kernel function \( e^{-st}, s \in \mathbb{C} \), is not bounded variation function on \([0, \infty)\) hence we cannot claim (as in the classical sense) that the HK Laplace transform exists for any HK integrable function.

The convolution plays an important role in pure and applied mathematics, approximation theory, differential and integral equations and many other areas. Laplace transform has the property that it can interact with convolutions so we obtain various results on convolution. We give necessary and sufficient conditions so that the convolution operator as HK integral is continuous. Finally we solve the problem of inversion by using generalized differentiation.

The results in this chapter are appeared in [2, 14].

5.2 Existential Conditions

In this section, we tackle the problem of existence of Laplace transform as HK integral.

According to Zayed [16], the classical sufficient condition for the existence
of Laplace transform is that the function \( f \) is locally integrable on \([0, \infty)\), i.e., \( f \in L_{\text{loc}}[0, \infty) \). This is because the multiplier for the Lebesgue integrable functions is bounded measurable function and the function \( e^{-st}, \ s \in \mathbb{C} \), is bounded measurable function. We know that the multipliers for HK integrable functions are precisely the functions of bounded variation \([1]\). Note that in (5.1), the function \( e^{-st}, \ s \in \mathbb{C} \), is not of bounded variation on \([0, \infty)\) but it is of bounded variation on any compact interval \( J \subset \mathbb{R} \). Hence we can not consider (5.1) as HK integral directly. So we cannot get any specific condition for the existence of Laplace transform as HK integral. However we do have the following different conditions.

**Theorem 5.2.1.** Let \( f : [0, \infty) \to \mathbb{R} \) be any continuous function such that \( F(x) = \int_0^x f, \ 0 \leq x < \infty, \) is bounded on \([0, \infty)\). Then the Laplace transform \( \mathcal{L}\{f\}(s) \), i.e., \( \mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t) \, dt \), \( \text{Re}.s > 0 \) exists.

**Proof.** Let \( 0 < x < \infty, \ s = \sigma + i\omega, \ \sigma, \omega \in \mathbb{R} \).

We have \( (F(t) e^{-st})' = -s F(t) e^{-st} + f(t) e^{-st} \).

Since \( e^{-st} \) is of bounded variation on any compact interval, say \([a, b]\) and \( F(t) \) is bounded on \([0, \infty)\). Then \( F(t) e^{-st} \) is HK integrable on \([a, b]\).

Therefore \( -s F(t) e^{-st} \) is integrable over \( I = [a, b] \).

Now

\[
\int_a^b \left| (e^{-st})' \right| \, dt = \int_a^b \left| -s e^{-st} \right| \, dt \\
= \int_a^b \sqrt{\sigma^2 + \omega^2} \ e^{-\sigma t} \, dt \\
= \frac{\sqrt{\sigma^2 + \omega^2}}{\sigma} \ [e^{-\sigma a} - e^{-\sigma b}].
\]

Without loss of generality let us take \( a = 0 \). Therefore

\[
\int_0^b \left| (e^{-st})' \right| \, dt = \frac{\sqrt{\sigma^2 + \omega^2}}{\sigma} \ [1 - e^{-\sigma b}] < \infty.
\]
Hence \((e^{-st})'\) is absolutely integrable over \([a, b]\).

Now by Hake’s theorem [1], we can write

\[
\lim_{b \to \infty} \int_0^b \left| (e^{-st})' \right| \, dt = \lim_{b \to \infty} \frac{\sqrt{\sigma^2 + \omega^2}}{\sigma} [1 - e^{-\sigma b}]
\]

\[
= \frac{\sqrt{\sigma^2 + \omega^2}}{\sigma}, \quad \sigma > 0
\]

\[
= \frac{\sqrt{\sigma^2 + \omega^2}}{\sigma}, \quad Re.s > 0.
\]

This shows that \((e^{-st})'\) is absolutely integrable on \([0, \infty)\).

Again \(\lim_{t \to \infty} e^{-st} = 0, Re.s > 0\).

Hence the function \(e^{-st} : [0, \infty) \to \mathbb{R}\) is continuous on \([0, \infty)\) with \(\lim_{t \to \infty} e^{-st} = 0\) and \((e^{-st})'\) is absolutely integrable over \([0, \infty)\) and we have assumed that \(f : [0, \infty) \to \mathbb{R}\) is continuous function such that \(F(x) = \int_0^x f, \quad 0 \leq x < \infty\), is bounded on \([0, \infty)\).

Therefore by Dedekind’s test [10], we can say that the integral

\[
\int_0^\infty e^{-st} f(t) \, dt \quad \text{exists for } Re.s > 0.
\]

\[\square\]

**Remark 5.2.2.** The above result can also be proved by using Du-Bois Reymond’s test [1] as follows:

Take \(\phi(t) = e^{-st}, \quad s = \sigma + i\omega, \quad J = [0, \infty)\).

Then clearly \(\phi(t)\) is differentiable and \(\phi' \in L^1(J)\).

Since \(F\) is bounded, we have \(\lim_{t \to \infty} F(t) e^{-st} = 0, Re.S > 0\).

Hence \(F(x) = \int_0^x f, \quad 0 \leq x < \infty\) is bounded on \(J\), \(e^{-st}\) is differentiable on \(J\), \((e^{-st})' \in L^1(J)\) and \(F(t) e^{-st} \to 0\) as \(t \to \infty, Re.s > 0\).

Therefore by Du-Bois Reymond’s test, \(f(t)e^{-st} \in \mathcal{H}\mathcal{K}(J)\). That is, the Laplace transform of \(f(t), \int_0^\infty e^{-st} f(t) \, dt \) exists, \(Re.s > 0\).
Theorem 5.2.3. If the Laplace transform of $f(t)$, $\int_0^\infty e^{-st}f(t)\,dt$ exists for $\text{Re}.s > 0$, then the function $f \in \mathcal{HK}_{\text{loc}}$.

Proof. Suppose the integral $\int_0^\infty e^{-st}f(t)\,dt$ exists for $\text{Re}.s > 0$.
Note that the function $e^{-st}$, $s \in \mathbb{C}$, as a function of real variable $t$, is not of bounded variation on $[0, \infty)$ but is of bounded variation on any compact interval $I = [a, b]$.
By Hake’s theorem [1], we can write
\[ \int_0^\infty e^{-st}f(t)\,dt = \lim_{b \to \infty} \int_a^b e^{-st}f(t)\,dt. \]
Since the limits on R.H.S. exist, we can say that the HK integral $\int_a^b e^{-st}f(t)\,dt$ exists, $\text{Re}.s > 0$ for some compact interval $[a, b]$.
Hence the function $f(t)$ is HK integrable necessarily on the compact interval $I = [a, b]$, that is, $f \in \mathcal{HK}_{\text{loc}}$. \qed

Theorem 5.2.4. Let $f \in \mathcal{HK}([0, \infty))$. Define $F(x) = \int_x^\infty f(t)\,dt$. Then $\mathcal{L}\{f\}(s)$ exists if and only if $\int_0^\infty e^{-st}F(t)\,dt$ exists at $s \in \mathbb{C}$.

Proof. Let $T > 0$. The function $e^{-st}$, $s \in \mathbb{C}$, is of bounded variation on a compact interval $[0, T]$ and the function $f$ is HK integrable on $[0, \infty)$.
Hence $e^{-st}f(t)$, $s \in \mathbb{C}$, is HK integrable on $[0, T]$ and by integrating by parts, we get
\[ \int_0^T e^{-st}f(t)\,dt = e^{-sT}F(T) - F(0) + s \int_0^T e^{-st}F(t)\,dt. \]
Since the function $f$ is HK integrable on $[0, \infty)$, $F(0)$ is finite, say $A$ and by [9.12(a), [5]], $F$ is continuous function such that $\lim_{T \to \infty} F(T) = 0$.
Therefore by Hake’s theorem [1],
\[ \lim_{T \to \infty} \int_0^T e^{-st}f(t)\,dt = \lim_{T \to \infty} \left\{ e^{-sT}F(T) - F(0) + s \int_0^T e^{-st}F(t)\,dt \right\}. \]
Hence
\[\int_0^\infty e^{-st}f(t)\,dt = s\int_0^\infty e^{-st}F(t)\,dt - A.\]
This shows that \(\mathcal{L}\{f\}(s)\) exists if and only if \(\int_0^\infty e^{-st}F(t)\,dt\) exists at \(s \in \mathbb{C},\ Re.s > 0.\)

\[\square\]

**Theorem 5.2.5. (Uniqueness theorem)** Let \(\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st}f(t)\,dt\) and \(\mathcal{L}\{g\}(s) = \int_0^\infty e^{-st}g(t)\,dt.\) If \(f(t) = g(t)\) a.e. on \([0, \infty),\) then \(\mathcal{L}\{f\}(s) = \mathcal{L}\{g\}(s).\)

### 5.3 Basic Properties

The usual elementary properties such as linearity, dilation, modulation, translation are remains same for the HK Laplace transform, the proofs of which are quite similar to that of classical ones. Also there are some other results that are analogous to the classical one with some modification in the hypothesis which we discuss below.

**Theorem 5.3.1. (HK Laplace transform of derivative)** Let \(f : [0, \infty) \to \mathbb{R}\) be a function which is in \(ACG_\delta(\mathbb{R})\) such that \(\lim_{t \to \infty} \frac{f(t)}{e^{st}} = 0.\) Then for \(Re.s > 0\) both \(\mathcal{L}\{f\}(s)\) and \(\mathcal{L}\{f'\}(s)\) fail to exist or \(\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0).\)

**Proof.** Let \(T > 0.\) Consider the integral \(J = \int_0^T e^{-st}f'(t)\,dt.\) This integral is valid since the function \(e^{-st},\ s \in \mathbb{C},\) is bounded variation on any compact interval \([0, T]\) and the function \(f'\) is HK integrable on \(\mathbb{R}.\) By integrating by parts, we get
\[J = e^{-sT}f(T) - f(0) + s\int_0^T e^{-st}f(t)\,dt.\]
By Hake’s theorem [1],
\[\lim_{T \to \infty} \int_0^T e^{-st}f'(t)\,dt = \lim_{T \to \infty} e^{-sT}f(T) - f(0) + \lim_{T \to \infty} s\int_0^T e^{-st}f(t)\,dt\]
\[ = s \int_0^\infty e^{-st} f(t) \, dt - f(0). \]

Therefore
\[ \int_0^\infty e^{-st} f'(t) \, dt = s \int_0^\infty e^{-st} f(t) \, dt - f(0). \]

That is, \( \mathcal{L}\{f'(s)\} = s \mathcal{L}\{f\}(s) - f(0) \).

\[ \square \]

**Remark 5.3.2.** The above result can be generalized by imposing the conditions on \( f \) as:

For \( k = 0, 1, 2, \ldots, (n - 1) \), \( f^{(k)} \) are in \( \mathcal{ACG}_\delta(\mathbb{R}) \) and \( \lim_{t \to \infty} \frac{f^{(k)}}{e^{st}} = 0 \), \( \text{Re}.s > 0 \).

Then \( \mathcal{L}\{f^{(n)}\}(s) \) and \( \mathcal{L}\{f\}(s) \) fail to exist or
\[ \mathcal{L}\{f^{(n)}\}(s) = s^n \mathcal{L}\{f\}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1)}(0). \]

**Theorem 5.3.3.** (Differentiation of HK Laplace transform) Let \( f : [0, \infty) \to \mathbb{R} \) be a function such that \( \mathcal{L}\{f\}(s) \) exists on a compact interval, say \( J = [\alpha, \beta] \), \( \alpha, \beta \in \mathbb{R} \). Define \( g(t) = t f(t) \) and assume that \( g \in \mathcal{HK}(\mathbb{R}^+) \).

Then \( \mathcal{L}\{g\}(s) \) exists and \( \mathcal{L}\{g\}(s) = -(\mathcal{L}\{f\}(s))' \) a.e. Here \( s \in \mathbb{R} \).

**Proof.** Suppose
\[ \mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t) \, dt \quad \text{exists on } [\alpha, \beta], \ \alpha, \beta \in \mathbb{R}. \]

Define \( g(t) = t f(t) \).

We know the necessary and sufficient condition for differentiating under the HK integral is that
\[ \int_{t=0}^{\infty} \int_{s=a}^{b} e^{-st} t f(t) \, ds \, dt = \int_{s=a}^{b} \int_{t=0}^{\infty} e^{-st} t f(t) \, dt \, ds \quad (5.2) \]

for all \([a, b] \subset [\alpha, \beta]\) [12].

Now
\[ \int_{t=0}^{\infty} \int_{s=a}^{b} e^{-st} t f(t) \, ds \, dt = \int_{t=0}^{\infty} (e^{-at} - e^{-bt}) f(t) \, dt \]

123
\[
\mathcal{L}\{f\}(a) - \mathcal{L}\{f\}(b)
\]

which exists.

Therefore the double HK integral \(\int_{t=0}^{\infty} \int_{s=a}^{b} e^{-st} tf(t) \, ds \, dt\) exists.

Hence by [lemma 25(a), [13]], the equality in (5.2) holds and we can differentiate under the HK integral. By (5.2), we have

\[
\int_{s=a}^{b} \int_{t=0}^{\infty} e^{-st} tf(t) \, dt \, ds = \mathcal{L}\{f\}(a) - \mathcal{L}\{f\}(b)
= - \int_{a}^{b} (\mathcal{L}\{f\}(s))' \, ds.
\]

Therefore

\[
\int_{s=a}^{b} \int_{t=0}^{\infty} e^{-st} tf(t) \, dt \, ds = - \int_{a}^{b} (\mathcal{L}\{f\}(s))' \, ds
\]

for all \([a, b] \subset [\alpha, \beta]\). Which implies that

\[
\int_{0}^{\infty} e^{-st} tf(t) \, dt = -(\mathcal{L}\{f\}(s))' \quad \text{a.e. on } [\alpha, \beta].
\]

That is, \(\mathcal{L}\{tf(t)\}(s) = -(\mathcal{L}\{f\}(s))' \), a.e. on \([\alpha, \beta]\).

Hence the derivative of the HK Laplace transform of \(f(t)\) exists a.e. on \([\alpha, \beta]\).

\[\square\]

**Remark 5.3.4.** Let \(G(s) = \mathcal{L}\{tf(t)\}(s)\) and \(H(s) = -\mathcal{L}\{f(t)\}(s)\).

Then \((H(s))' = G(s)\) a.e. on \([\alpha, \beta]\). And, we know that [5], A function \(f : [a, b] \to \mathbb{R}\) is HK integrable on \([a, b]\) if and only if there exists an \(\mathcal{ACG}_\delta\) function \(F\) on \([a, b]\) such that \(F' = f\) a.e. on \([a, b]\).

Again, by fundamental theorem of calculus [10], \((H(s))'\) is HK integrable, i.e., every derivative is HK integrable. Hence \(G(s)\) is also HK integrable. Consequently, the function \(H(s)\) is \(\mathcal{ACG}_\delta\) on \([\alpha, \beta]\).

Continuing in this way \(n\)-times, we can say that if \(t^n f(t)\) is HK integrable, then any order of derivative of \(\mathcal{L}\{f\}(s)\) exists a.e. So \(\mathcal{L}\{f\}(s)\) is analytic a.e.
Theorem 5.3.5. (Multiplication by $\frac{1}{t}$) Let $f : [0, \infty) \to \mathbb{R}$ be a function such that $\mathcal{L}\{f\}(s)$ exists and assume that $\frac{f(t)}{t} \in \mathcal{H}\mathcal{K}([0, \infty))$. Then $\mathcal{L}\left\{\frac{f(t)}{t}\right\}(s)$ exists and

$$
\mathcal{L}\left\{\frac{f(t)}{t}\right\}(s) = \int_s^\infty \mathcal{L}\{f\}(u) \, du.
$$

Here $s \in \mathbb{R}$.

Proof. Assume that $\frac{f(t)}{t} \in \mathcal{H}\mathcal{K}(\mathbb{R}^+)$. Since the function $e^{-st}$ is of bounded variation on any compact interval $I = [a, b]$, we have $e^{-st} \frac{f(t)}{t} \in \mathcal{H}\mathcal{K}([a, b])$. That is, $\int_a^b e^{-st} \frac{f(t)}{t} \, dt$ exists.

Without loss of generality, let us take $a = 0$ so that $\int_0^b e^{-st} \frac{f(t)}{t} \, dt$ exists.

By Hake’s theorem [1], $\lim_{b \to \infty} \int_0^b e^{-st} \frac{f(t)}{t} \, dt$ exists.

Hence $\mathcal{L}\left\{\frac{f(t)}{t}\right\}(s)$ exists.

Now consider the integrals

$$
I_1 = \int_0^\infty \int_s^\infty e^{-ut} f(t) \, du \, dt, \quad I_2 = \int_s^\infty \int_0^\infty e^{-ut} f(t) \, dt \, du.
$$

Then

$$
I_1 = \int_0^\infty \left[ e^{-st} - 0 \right] \frac{f(t)}{t} \, dt = \int_0^\infty e^{-st} \frac{f(t)}{t} \, dt \text{ which exists.}
$$

Therefore $I_1$ exists on $\mathbb{R} \times \mathbb{R}$.

Also, note that the function $e^{-st}$ is of bounded variation on any compact interval $J \subset \mathbb{R}$. Therefore $\int_J V_J(e^{-st}) \, dt < M_J$, for some constant $M_J > 0$.

Hence by [lemma 25(a), [13]], $I_2$ exists on $\mathbb{R} \times \mathbb{R}$ and $I_1 = I_2$ on $\mathbb{R} \times \mathbb{R}$. So

$$
\int_0^\infty e^{-st} \frac{f(t)}{t} \, dt = \int_s^\infty \int_0^\infty e^{-ut} f(t) \, dt \, du
$$

$$
= \int_s^\infty \mathcal{L}\{f\}(u) \, du.
$$

That is, $\mathcal{L}\left\{\frac{f(t)}{t}\right\}(s) = \int_s^\infty \mathcal{L}\{f\}(u) \, du$. \qed
Theorem 5.3.6. (HK Laplace transform of integral) Let $f: [0, \infty) \to \mathbb{R}$ be HK integrable function such that $\mathfrak{L}\{f\}(s)$ exists. Then $\mathfrak{L}\left\{\int_0^t f(u) \, du\right\}(s)$ exists and $\mathfrak{L}\left\{\int_0^t f(u) \, du\right\}(s) = \frac{1}{s} \mathfrak{L}\{f\}(s)$.

Proof. Define $F(t) = \int_0^t f(u) \, du$, $t \in \mathbb{R}^+$. Then $F'(t) = f(t)$ a.e. on $\mathbb{R}^+$ [9.12(b), [5]].

Since $f \in \mathcal{H}K(\mathbb{R}^+)$, the integral

$$\int_0^\infty f(u) \, du$$

exists on $\mathbb{R}^+$.

That is,

$$\lim_{t \to \infty} \int_0^t f(u) \, du$$

exists on $\mathbb{R}^+$.

Hence $\lim_{t \to \infty} F(t)$ is bounded on $\mathbb{R}^+$.

Now for $T > 0$, consider

$$\int_0^T e^{-st} f(t) \, dt = \int_0^T e^{-st} F'(t) \, dt.$$ 

This integral exists since $e^{-st}$ is of bounded variation on $[0, T]$ and the function $f(t)$ is HK integrable on $\mathbb{R}^+$.

By integrating by parts, we get

$$\int_0^T e^{-st} f(t) \, dt = e^{-sT} F(T) + s \int_0^T e^{-st} F(t) \, dt.$$ 

Since $\lim_{T \to \infty} F(T)$ is bounded on $\mathbb{R}$, set $\lim_{T \to \infty} F(T) = A$.

By Hake’s theorem [1], we can write

$$\int_0^\infty e^{-st} f(t) \, dt = \lim_{T \to \infty} \int_0^T e^{-st} f(t) \, dt$$

$$= \lim_{T \to \infty} e^{-sT} F(T) + s \lim_{T \to \infty} \int_0^T e^{-st} F(t) \, dt$$

$$= s \int_0^\infty e^{-st} F(t) \, dt \quad Re.s > 0.$$ 

126
Therefore
\[ \int_0^\infty e^{-st} F(t) \, dt = \frac{1}{s} \int_0^\infty e^{-st} f(t) \, dt, \quad \text{Re}.s > 0. \]
That is,
\[ \int_0^\infty e^{-st} \int_0^t f(u) \, du \, dt = \frac{1}{s} \mathcal{L}\{f\}(s), \quad \text{Re}.s > 0. \]
Hence \( \mathcal{L}\left\{ \int_0^t f(u) \, du \right\} = \frac{1}{s} \mathcal{L}\{f\}(s), \quad \text{Re}.s > 0. \)

### 5.4 Convolution

In this section we shall discuss some classical results of Laplace convolution in the HK integral setting, called HK-convolution. First we give two results about the existence of HK-convolution.

**Theorem 5.4.1.** Let \( f \in \mathcal{HK}(\mathbb{R}^+) \) and \( g \in \mathcal{BV}(\mathbb{R}^+) \). Then we have \( f * g \) exists on \( \mathbb{R}^+ \).

*Proof.* By definition [4],
\[
f * g(t) = \int_0^t f(t - \tau) g(\tau) \, d\tau
\]
which exists as HK-integral by multiplier theorem [1].

By integrating by parts, we get
\[
|f * g(t)| \leq A \|f\| \left\{ \inf_{\mathbb{R}^+} |g| + V_{[0,t]} g(\tau) \right\}.
\]
Hence \( f * g \) exists on \( \mathbb{R}^+ \). \( \square \)

**Theorem 5.4.2.** Let \( f \in \mathcal{HK}_{loc} \) and \( g \in \mathcal{BV} \) such that the support of \( g \) is in the compact interval \( I = [a, b] \). Then
\[
|f * g(t)| \leq A \|f\|_{[t-b,t-a]} \left\{ \inf_{[a,b]} |g| + V_{[a,b]} g(\tau) \right\}.
\]
Proof. Suppose supp$(g) \subset [a, b]$. Then we have

$$f \ast g(t) = \int_a^b f(t - \tau)g(\tau) \, d\tau.$$ 

Integrate by parts, we get

$$|f \ast g(t)| \leq \inf_{\tau \in [a, b]} |g(\tau)| \cdot \left| \int_a^b f(t - \tau) \, d\tau \right| + \int_a^b f(t - \tau) \, d\tau \cdot V_{[a, b]} g(\tau) \leq \left\{ \inf_{\tau \in [a, b]} |g(\tau)| + V_{[a, b]} g(\tau) \right\} \sup_{[t-b,t-a]} \left| \int_{t-b}^{t-a} f(u) \, du \right|.$$ 

Therefore

$$|f \ast g(t)| \leq A\|f\|_{[t-b,t-a]} \left\{ \inf_{[a,b]} |g| + V_{[a,b]} g(\tau) \right\}.$$ 

The following result shows that the HK-convolution is bounded.

**Theorem 5.4.3.** Let $f \in \mathcal{H} \mathcal{K}(\mathbb{R}^+)$ and $g \in L^1(\mathbb{R}^+) \cap BV(\mathbb{R}^+)$. Then we have $A\|f \ast g\| \leq A\|f\| \|g\|_1$.

**Proof.** For $0 \leq a < b \leq \infty$, we have

$$\int_a^b f \ast g(t) \, dt = \int_a^b \int_0^t f(t - \tau)g(\tau) \, d\tau \, dt.$$ 

Let

$$I_1 = \int_a^b \int_0^t f(t - \tau)g(\tau) \, d\tau \, dt \quad \text{and} \quad I_2 = \int_0^t \int_a^b f(t - \tau)g(\tau) \, d\tau \, dt.$$ 

Consider

$$I_2 = \int_0^t \int_a^b f(t - \tau)g(\tau) \, d\tau \, dt.$$ 

Since $f \in \mathcal{H} \mathcal{K}(\mathbb{R}^+)$, we have $\int_a^b f(t - \tau) \, dt$ exists and finite.

Let

$$\int_{t-a}^{t-b} f(\tau) \, d\tau = M.$$
Therefore
\[ |I_2| \leq M \int_0^t |g(\tau)| \, d\tau. \]

Since \( g \in L^1(\mathbb{R}^+) \), we have for some finite \( K \),
\[ \int_0^t |g(\tau)| \, d\tau = K. \]

So \( |I_2| \leq MK \) and hence \( I_2 \) exists on \( \mathbb{R}^+ \times \mathbb{R}^+ \).

Now as \( g \in L^1(\mathbb{R}^+) \), we have
\[ \|g\|_1 \leq K_1 \]
and also
\[ \int_{\mathbb{R}^+} V_{[0,t]} g \leq M_1. \]

Therefore by lemma 25, [13], \( I_1 = I_2 \) on \( \mathbb{R}^+ \times \mathbb{R}^+ \).

Hence
\[
\left| \int_a^b f \ast g(t) \, dt \right| \leq \sup_{[t-a,t-b]} \left| \int_{t-a}^{t-b} f(u) \, du \right| \cdot \int_0^t |g(\tau)| \, d\tau \leq A \|f\|_{[t-a,t-b]} \cdot \|g\|_1.
\]

Therefore
\[
\sup_{[a,b]} \left| \int_a^b f \ast g(t) \, dt \right| \leq A \|f\|_{[t-a,t-b]} \cdot \|g\|_1.
\]

That is, \( A \|f \ast g\| \leq A \|f\| \|g\|_1. \) \hfill \( \Box \)

Now we give some basic elementary properties of HK-convolution.

**Theorem 5.4.4.** Suppose \( f \in H\mathcal{K}(\mathbb{R}^+) \) and \( g \in BV(\mathbb{R}^+) \). Then the following holds:

I) \( f \ast g(t) = g \ast f(t) \) for all \( t \in \mathbb{R}^+ \).

II) \( (\lambda f) \ast g = f \ast (\lambda g) = \lambda (f \ast g) \) for some \( \lambda \in \mathbb{C} \).

III) \( (f \ast g)_x(t) = f_x \ast g(t) = f \ast g_x(t) \).

IV) \( h \ast (f + g)(t) = h \ast f(t) + h \ast g(t) \) for some \( h \in H\mathcal{K}(\mathbb{R}^+) \).

**Proof.** The proof is similar to that of the classical case. \hfill \( \Box \)

129
The associative property of HK-convolution is given in the next theorem.

**Theorem 5.4.5.** If $f \in \mathcal{HK}(\mathbb{R}^+)$, $g \in \mathcal{BV}(\mathbb{R}^+)$ and $h \in L^1(\mathbb{R}^+)$, then

$$(f * g) * h(t) = f * (g * h)(t), \quad \forall \, t \in \mathbb{R}^+.$$  

**Proof.** Consider

$$(f * g) * h(t) = \int_0^t \int_0^\tau f(\xi) \ g(\tau - \xi) \ h(t - \tau) \, d\xi \, d\tau.$$  

By changing the order of integration, we get

$$(f * g) * h(t) = \int_{\xi=0}^t \int_{\tau=0}^{\tau-\xi} f(\xi) \ g(u) \ h(t - u - \xi) \, du \, d\xi.$$  

$$= \int_{\xi=0}^t f(\xi) \ g * h(t - \xi) \, d\xi.$$  

$$= f * (g * h)(t).$$  

Now consider

$$I_1 = \int_{\tau=\xi}^t \int_{\xi=0}^t f(\xi) \ g(\tau - \xi) \ h(t - \tau) \, d\xi \, d\tau$$

and

$$I_2 = \int_{\xi=0}^t \int_{\tau=\xi}^t g(\tau - \xi) \ h(t - \tau) \, d\tau \, d\xi.$$  

Let

$$J = \int_{\tau=\xi}^t g(\tau - \xi) \ h(t - \tau) \, d\tau.$$  

Integrate by parts, we get

$$|J| \leq \left| g(t - \xi) \right| \int_{\tau=\xi}^t h(t - \tau) \, d\tau \right| + \left| \int_{\tau=\xi}^t h(t - \tau) \, d\tau \right| \cdot V_{[\xi,t]} \left| g(\tau - \xi) \right|$$  

$$\leq \inf_{[\xi,t]} |g| \cdot \left| \int_{\tau=\xi}^t h(t - \tau) \, d\tau \right| + \left| \int_{\tau=\xi}^t h(t - \tau) \, d\tau \right| \cdot V_{[\xi,t]} \left| g(\tau - \xi) \right|$$  

$$\leq \inf_{[\xi,t]} |g| \cdot \left| h(t - \tau) | \, d\tau \right| + \left| \int_{\tau=\xi}^t h(t - \tau) \, d\tau \right| \cdot V_{[\xi,t]} \left| g(\tau - \xi) \right|$$  

$$\leq \inf_{[\xi,t]} |g| \cdot \|h\|_1 + \|h\|_1 \cdot V_{[\xi,t]} \left| g(\tau - \xi) \right|$$  

130
\[ \leq \|h\|_1 \left\{ \inf_{[\xi,t]} |g| + V_{[\xi,t]} g(\tau - \xi) \right\}. \]

Therefore
\[ I_2 \leq \sup_{\mathbb{R}^+} \left| \int_0^t f(\xi) \, d\xi \right| \cdot \left\{ \|h\|_1 \left\{ \inf_{[\xi,t]} |g| + V_{[\xi,t]} g(\tau - \xi) \right\} \right\} \]
\[ \leq A\|f\| \cdot \|h\|_1 \left\{ \inf_{[\xi,t]} |g| + V_{[\xi,t]} g(\tau - \xi) \right\}. \]

Hence \( I_2 \) exists on \( \mathbb{R}^+ \times \mathbb{R}^+ \). And, by \([\text{lemma 25, [13]})\] we are through. \( \square \)

Now we show that the HK-Laplace transform of convolution of two functions is the product of their HK-Laplace transforms.

**Theorem 5.4.6.** Let \( f_1 \in \mathcal{H}\mathcal{K}(\mathbb{R}^+) \), \( f_2 \in \mathcal{L}^1(\mathbb{R}^+) \cap \mathcal{B}\mathcal{V}(\mathbb{R}^+) \). Then \( f \ast g \) exists and if \( \mathcal{L}\{f_1(t)\} \) and \( \mathcal{L}\{f_2(t)\} \) exist at \( s \in \mathbb{C} \), then \( \mathcal{L}\{f_1 \ast f_2(t)\} \) exists at \( s \in \mathbb{C} \) and \( \mathcal{L}\{f_1 \ast f_2(t)\}(s) = \mathcal{L}\{f_1(t)\}(s) \cdot \mathcal{L}\{f_2(t)\}(s) \).

**Proof.** Consider
\[ \mathcal{L}\{f_1 \ast f_2(t)\}(s) = \int_0^\infty e^{-st} f_1 \ast f_2(t) \, dt \]
\[ = \int_0^\infty \int_0^t e^{-st} f_1(\tau) e^{-s(t-\tau)} f_2(t - \tau) \, d\tau \, dt. \]

By interchanging the iterated integrals, we have
\[ \mathcal{L}\{f_1 \ast f_2(t)\}(s) = \int_0^\infty e^{-st} f_1(\tau) \int_{\tau}^\infty e^{-s(t-\tau)} f_2(t - \tau) \, dt \, d\tau \]
\[ = \int_0^\infty \int_{\tau}^\infty e^{-st} f_1(\tau) \, d\tau \, e^{-su} f_2(u) \, du. \]

But by hypothesis both the integrals
\[ \int_0^\infty e^{-st} f_1(\tau) \, d\tau \quad \text{and} \quad \int_0^\infty e^{-st} f_2(\tau) \, d\tau \]
exist at \( s \in \mathbb{C} \).

Therefore \( \mathcal{L}\{f_1 \ast f_2(t)\} \) exists at \( s \in \mathbb{C} \) and
\( \mathcal{L} \{ f_1 * f_2(t) \} (s) = \mathcal{L} \{ f_1(t) \} (s) \mathcal{L} \{ f_2(t) \} (s) \).

Now it remains to show the justification for interchanging the iterated integrals.

Since the function \( f_2 \in L^1(\mathbb{R}^+) \), \( \exists K_I \) such that \( \| f_2 \|_1 < K_I \), where \( I = [a, b] \subset \mathbb{R}^+ \) is any compact interval.

Clearly, \( V_{[a,b]} e^{-s(t-\tau)} f_2(t-\tau) \leq e^{-\sigma u} V_{[t-b,t-a]} f_2(\tau) + |f_2(\tau)| 2 e^{-\sigma r} \) for some reals \( u \) and \( r \) and so

\[
\int_0^\infty V_{[t-b,t-a]} e^{-s\tau} f_2(\tau) \, d\tau \leq e^{-\sigma u} \int_0^\infty V_{[t-b,t-a]} f_2(\tau) \, d\tau + 2 e^{-\sigma r} \int_0^\infty |f_2(\tau)| \, d\tau.
\]

Therefore

\[
\int_0^\infty V_{[t-b,t-a]} e^{-s\tau} f_2(\tau) \, d\tau \leq e^{-\sigma u} \int_0^\infty V_{[t-b,t-a]} f_2(\tau) \, d\tau + 2 K_I e^{-\sigma r}.
\]

Since \( f_2 \in BV(\mathbb{R}^+) \), there exists a constant \( M_I \) such that

\[
\int_0^\infty V_{[t-b,t-a]} f_2(\tau) \, d\tau < M_I.
\]

Then

\[
\int_0^\infty V_{[t-b,t-a]} e^{-s\tau} f_2(\tau) \, d\tau \leq e^{-\sigma u} M_I + 2 K_I e^{-\sigma r}.
\]

Thus for each compact interval \( I = [a, b] \subset \mathbb{R}^+ \), the integral

\[
\int_0^\infty V_{[t-b,t-a]} e^{-s\tau} f_2(\tau) \, d\tau
\]

is finite and \( \| f_2 \|_1 \leq K_I \).

Hence the integral

\[
\int_0^\infty \int_0^\infty e^{-st} f_1(t) f_2(t-\tau) \, dt \, d\tau
\]

exists on \( \mathbb{R}^+ \times \mathbb{R}^+ \). And, by [lemma 25, [13]], we can interchange the iterated integrals. 

\( \square \)
Theorem 5.4.7. In addition to the hypothesis of the above theorem if $f_1 \ast f_2(t)$ is in $\mathcal{H}K(\mathbb{R}^+)$ and $f_2$ is continuous, then $f_1 \ast f_2(t)$ is continuous with respect to the Alexiewicz norm.

Proof. Choose any $\delta > 0$ such that $0 < \delta \leq 1$ (The case for negative $\delta$ is analogous).

Define $D(t, \delta) = f_1 \ast f_2(t + \delta) - f_1 \ast f_2(t) = I_1 + I_2$.

Where

\[ I_1 = \int_0^t f_1(\tau) [f_2(t + \delta - \tau) - f_2(t - \tau)] d\tau, \quad I_2 = \int_t^{t + \delta} f_1(\tau) f_2(t + \delta - \tau) d\tau. \]

Since $f_2 \in BV(\mathbb{R}^+)$, $f_2$ is bounded on $\mathbb{R}^+$.

So there exists $M_2$ such that $|f_2(x)| \leq M_2$, for all $x \in \mathbb{R}^+$. Then

\[ |I_2| \leq M_2 \left| \int_t^{t + \delta} f_1(\tau) d\tau \right|. \]

But $f_1 \in \mathcal{H}K(\mathbb{R}^+)$ implies that

\[ \lim_{\delta \to 0} \left| \int_t^{t + \delta} f_1(\tau) d\tau \right| \to 0. \]

Therefore $|I_2| \to 0$ as $\delta \to 0$.

Now consider

\[ I_1 = \int_0^t f_1(\tau) [f_2(t + \delta - \tau) - f_2(t - \tau)] d\tau. \]

By integrating by parts, we get

\[ |I_1| \leq \left| [f_2(t + \delta - \tau) - f_2(t - \tau)] \int_0^t f_1(\tau) d\tau \right| \]

\[ + \left| \int_0^t \int_0^u f_1(\tau) d\tau d(f_2(u + \delta - \tau) - f_2(u - \tau)) \right| \]

\[ \leq |f_2(t + \delta - \tau) - f_2(t - \tau)| \cdot \sup_{t \in \mathbb{R}^+} \left| \int_0^t f_1(\tau) d\tau \right| \]

\[ + \sup_{u \in \mathbb{R}^+} \left| \int_0^u f_1(\tau) d\tau \right| \cdot \left| \int_0^t d(f_2(u + \delta - \tau) - f_2(u - \tau)) \right|. \]

133
\[
= |f_2(t + \delta - \tau) - f_2(t - \tau)| \cdot \sup_{t \in \mathbb{R}^+} \left| \int_0^t f_1(\tau) \, d\tau \right|
+ \sup_{u \in \mathbb{R}^+} \left| \int_0^u f_1(\tau) \, d\tau \right| \cdot |f_2(u + \delta - \tau) - f_2(u - \tau)|.
\]

Therefore

\[
|I_1| \leq |f_2(t + \delta - \tau) - f_2(t - \tau)| \cdot \sup_{t \in \mathbb{R}^+} \left| \int_0^t f_1(\tau) \, d\tau \right|
+ \sup_{u \in \mathbb{R}^+} \left| \int_0^u f_1(\tau) \, d\tau \right| \cdot |f_2(u + \delta - \tau) - f_2(u - \tau)|.
\]

Since \(f_2\) is continuous on \(\mathbb{R}^+\), for given \(\epsilon > 0\) there exists \(\eta > 0\) such that

\[
|I_1| \leq \epsilon, \forall \delta < \eta.
\]

Therefore \(|D(t, \delta)| \leq |I_1| + |I_2| < \epsilon, \forall \delta < \eta.\)

That is, for given \(\epsilon > 0\) there exists \(\eta > 0\) such that \(|D(t, \delta)| < \epsilon, \forall \delta < \eta.\)

Hence \(f_1 * f_2\) is continuous at \(t \in \mathbb{R}^+.\)

Now, let \(\alpha, \beta \in \mathbb{R}^+.\)

Consider

\[
\int_{\alpha}^{\beta} [f_1 * f_2(t + \delta) - f_1 * f_2(t)] \, dt = \int_{\alpha}^{\alpha+\delta} f_1 * f_2(t) \, dt - \int_{\beta}^{\beta+\delta} f_1 * f_2(t) \, dt.
\]

Write \(F_{1,2}(x) = \int_{\alpha}^{x} f_1 * f_2(t) \, dt.\) Then

\[
\sup_{\alpha, \beta \in \mathbb{R}^+} \left| \int_{\alpha}^{\beta} [f_1 * f_2(t + \delta) - f_1 * f_2(t)] \, dt \right| \leq \sup_{\alpha \in \mathbb{R}^+} \left| F_{1,2}(\alpha + \delta) - F_{1,2}(\alpha) \right|
+ \sup_{\beta \in \mathbb{R}^+} \left| F_{1,2}(\beta + \delta) - F_{1,2}(\beta) \right|.
\]

Since \(F_{1,2}\) is an indefinite HK integral of \(f_1 * f_2\) on \(\mathbb{R}^+\), it is continuous on \(\mathbb{R}^+\) [5]. Therefore for given \(\epsilon > 0\) there exists \(\eta > 0\) such that

\[
|F_{1,2}(\xi + \delta) - F_{1,2}(\xi)| < \frac{\epsilon}{2}, \forall \delta < \eta \text{ and for all } \xi \in \mathbb{R}^+.
\]

Hence we have

\[
\sup_{\alpha, \beta \in \mathbb{R}^+} \left| \int_{\alpha}^{\beta} [f_1 * f_2(t + \delta) - f_1 * f_2(t)] \, dt \right| < \epsilon, \forall \delta < \eta.
\]

134
Thus for given $\epsilon > 0$ there exists $\eta > 0$ such that $\|A_1 f_1 * f_2(t+\delta) - f_1 * f_2(t)\| < \epsilon$, $\forall \delta < \eta$.

This shows that $f_1 * f_2(t)$ is continuous at $t \in \mathbb{R}^+$ with respect to the Alexiewicz norm.

The following result concerns with necessary and sufficient conditions on $f_n$ and $g_n : [0, t] \to \mathbb{R}$ so that the HK-convolution as an operator is continuous. It involves either uniform boundedness or uniform convergence of the indefinite HK integral of $f_n$.

**Theorem 5.4.8.** Let $\{f_n\}$ be a sequence of HK-integrable functions, $f_n : [0, t] \to \mathbb{R}$, $\forall n$ such that $\int_0^t f_n = \int_0^t f$ as $n \to \infty$ for some HK-integrable function $f : [0, t] \to \mathbb{R}$. Define $F_n(x) = \int_0^x f_n$, $F(x) = \int_0^x f$.

Let $\{g_n\}$ be a sequence of uniform bounded variation functions, $g_n : [0, t] \to \mathbb{R}$, $\forall n$, such that $g_n \to g$ pointwise on $[0, t]$, where $g : [0, t] \to \mathbb{R}$. Then $f_n * g_n(t) \to f * g(t)$ as $n \to \infty$ if and only if

i) $F_n \to F$ uniformly on $[0, t]$ as $n \to \infty$.

ii) $F_n \to F$ pointwise on $[0, t]$, $\{F_n\}$ is uniformly bounded and $V(g_n - g) \to 0$.

**Proof.** i) Suppose $F_n \to F$ uniformly on $[0, t]$ as $n \to \infty$.

By assumption and multiplier theorem [1], $f_n * g_n(t)$ exists at $t \in \mathbb{R}^+$ and

$$f_n * g_n(t) = \int_0^t f_n(u) g_n(t-u) du$$

and $f * g(t) = \int_0^t f(u) g(t-u) du$.

Consider $f_n * g_n(t) - f * g(t) = I_1 + I_2$,

where

$$I_1 = \int_0^t [f_n(u) - f(u)] g_n(t-u) du, \quad I_2 = \int_0^t [g_n(t-u) - g(t-u)] f(u) du.$$

135
By integrating by parts, we get

\[
I_1 = g_n(t - u) \int_0^t [f_n(u) - f(u)] \, du - \int_0^t \int_0^y [f_n(u) - f(u)] \, dg_n
\]

\[
= g_n(t - u)[F_n(t) - F(t)] - \int_0^t [F_n(u) - F(u)] \, dg_n.
\]

Then

\[
|I_1| \leq Max_{0 \leq u \leq t} |F_n(u) - F(u)| \int_0^t dg_n + M |F_n(t) - F(t)|
\]

\[
\quad \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty \quad \text{(Since} \quad F_n \rightarrow F \text{ uniformly)}.
\]

Hence \(I_1 \rightarrow 0\) as \(n \rightarrow \infty\).

And by integrating by parts, we get

\[
I_2 = \{g_n(t - u) - g(t - u)\} \int_0^t f - \int_0^t F \, dg_n + \int_0^t F \, dg.
\]

By assumption the first term tends to 0 and since \(F\) is continuous and each \(g_n\) is of bounded variation with \(g_n \rightarrow g\) as \(n \rightarrow \infty\), so we have \(\int_0^t F \, dg_n \rightarrow \int_0^t F \, dg\). Hence \(I_2 \rightarrow 0\) as \(n \rightarrow \infty\). Therefore \(f_n * g_n(t) \rightarrow f * g(t)\) as \(n \rightarrow \infty\).

Now for the converse part, suppose that either \(F_n \not\rightarrow F\) on \([0, t]\) or \(F_n - F \rightarrow 0\) not uniformly on \([0, t]\).

Then there is a sequence \(\{x_i\}_{i \in \Lambda}\) in \([0, t]\) on which \(F_n \not\rightarrow F\) uniformly. So, by Bolzano-Weierstrass theorem [7], we can find a subsequence \(\{y_i\}_{i \in \Gamma}\) of \(\{x_i\}_{i \in \Lambda}\), \(\Gamma \subset \Lambda\), such that \(F_n(y_i) \not\rightarrow F(y_i)\) for all \(i\) and \(y_i \rightarrow y\) as \(i \rightarrow \infty\), i.e., \(F_n(y_i) - F(y_i) \not\rightarrow 0\).

Let us assume without loss of generality that \(0 < y_n \leq y \leq t\). Let \(H\) be Heaviside step function.

Define

\[
g_n(t - u) = \begin{cases} 
H(u - y_n), & n \in \Gamma \\
H(u - y), & \text{otherwise}
\end{cases}
\]
and \( g(t-u) = H(u-y) \).

Then we have, for \( n \in \Gamma \), \( f_n * g_n(t) = F_n(t) - F_n(y_n) \) and \( f * g(t) = F(t) - F(y) \).

Since \( F_n(y_n) \to F(y_n) \) as \( n \to \infty \), \( y_n \to y \) as \( n \to \infty \) and \( F \) is continuous, we have \( F_n(y_n) \to F(y) \).

But \( F_n(t) \to F(t) \) as \( n \to \infty \). Therefore \( f_n * g_n(t) \to f * g(t) \) as \( n \to \infty \) which is contradiction. Hence we must have \( F_n \to F \) uniformly on \([0, t]\).

ii) Suppose \( F_n \to F \) pointwise on \([0, t]\), \( F_n \) is uniformly bounded and \( V(g_n - g) \to 0 \).

Consider \( f_n * g_n(t) - f * g(t) = I_1 + I_2 \),

where

\[
I_1 = \int_0^t f_n(u) [g_n(t-u) - g(t-u)] \, du, \quad I_2 = \int_0^t [f_n(u) - f(u)] g(t-u) \, du.
\]

By integrating by parts, we get

\[
I_1 = [g_n(t-u) - g(t-u)] \int_0^t f_n - \int_0^t \int_0^y f_n \, d(g_n - g)
\]

\[
= [g_n(t-u) - g(t-u)] F_n(t) - \int_0^t F_n(y) \, d(g_n - g).
\]

Therefore

\[
|I_1| \leq |g_n(t-u) - g(t-u)| \, |F_n(t)| + |F_n(y)| \, V(g_n - g).
\]

Since \( \{F_n\} \) is uniformly bounded, we have \( |F_n| \leq M \), \( \forall n \) and for some constant \( M \).

Therefore

\[
|I_1| \leq |g_n(t-u) - g(t-u)| \, M + M \, V(g_n - g).
\]

By assumption \( g_n \to g \) as \( n \to \infty \) and also \( V(g_n - g) \to 0 \).

Hence \( I_1 \to 0 \) as \( n \to \infty \).

Again by integrating by parts, we write

\[
I_2 = g(t-u) \{F_n(t) - F(t)\} - \int_0^t (F_n - F) \, dg. \quad (5.3)
\]
By assumption $F_n(t) \to F(t)$, so the first term tends to 0. And, since $|F_n| \leq M$, $F_n \to F$ and $g$ is of bounded variation, by the dominated convergence theorem for Riemann-Stieltjes integral [6], we have

$$\int_0^t F_n \, dg \to \int_0^t F \, dg \text{ as } n \to \infty.$$ 

Therefore by (5.3), $I_2 \to 0$ as $n \to \infty$.

Hence $f_n * g_n(t) \to f * g(t)$ as $n \to \infty$.

Now for the converse part suppose there is $c \in (a, b)$ such that $F_n(c) - F(c) \not\to 0$ as $n \to \infty$.

Let $g_n(t-x) = g(t-x) = H(x-c)$.

Then

$$\int_0^t f_n(u) g_n(t-u) \, du = F_n(t) - F_n(c), \quad \int_0^t f(u) g(t-u) \, du = F(t) - F(c).$$

Since $F_n(t) \to F(t)$ and $F_n(c) \not\to F(c)$, we have $F_n(t) - F_n(c) \to F(t) - F(c)$ as $n \to \infty$. That is,

$$f_n * g_n(t) \not\to f * g(t) \text{ as } n \to \infty$$

which is contradiction. Hence we must have $F_n \to F$ pointwise on $[0, t]$.

Now if $\{F_n\}$ is not uniformly bounded, then there is a sequence $\{x_i\}_{i \in \Lambda}$ in $[0, t]$ on which $|F_n| \to \infty$.

Therefore by Bolzano-Weierstrass theorem [7], we can find a subsequence $\{y_i\}_{i \in \Gamma}$ of $\{x_i\}_{i \in \Lambda}$, $\Gamma \subset \Lambda$, such that $F_n(y_i) \geq 1$ for all $n$ and $F_n(y_i) \to \infty$ and $y_i \to y$.

Without loss of generality let us take $0 \leq y_i \leq y \leq t$.

Define $g_n(t-u) = \frac{H(u-y_i)}{\sqrt{F_n(y_i)}}$.

Then $V(g_n) \leq 1$, $g = 0$ and $V(g_n - g) = 0$ and

$$\int_0^t f_n(u) g_n(t-u) \, du = \int_0^t f_n(u) \frac{H(u-y_i)}{\sqrt{F_n(y_i)}} \, du$$
\[
\int_t^{y_t} \frac{f_n(u)}{\sqrt{F_n(y_t)}} \, du = \frac{F_n(t)}{\sqrt{F_n(y_t)}} - \sqrt{F_n(y_t)}.
\]

Therefore

\[
\int_0^t f_n(u)g_n(t-u) \, du \to -\infty \quad \text{as } n \to \infty
\]

and

\[
\int_0^t f(u)g(t-u) \, du = 0.
\]

Hence \(f_n g_n \to fg\) as \(n \to \infty\) which is contradiction and so we must have \(\{F_n\}\) is uniformly bounded sequence.

\section{5.5 Inversion}

Since in case of Henstock-Kurzweil integral, the integration process and differentiation process are inverse of each other. Here we shall establish the inverse of Henstock-Kurzweil Laplace transform by using Post’s generalized differentiation method [8]. For this we shall need following lemmas:

\textbf{Lemma 5.5.1.} Let \(\eta\) be such that \(0 < \eta < b - a\), and \(h(x) \in C^2(a \leq x \leq a + \eta)\), \(h'(a) = 0\), \(h''(a) < 0\), \(h(x)\) is nonincreasing on \((a, b]\). Then

\[
\int_a^b e^{kh(x)} \, dx \to e^{kh(a)} \left(\frac{-\pi}{2k h''(a)}\right)^{\frac{1}{2}}\quad k \to \infty.
\]

\textbf{Proof.} Since \(h(x) \in C^2(a \leq x \leq a + \eta)\), \(h''(x)\) is continuous.

Let \(\epsilon > 0\) be such that \(0 < \epsilon < -h''(a)\).

Choose \(\delta > 0\), \(\delta < \eta\) such that \(|h''(x) - h''(a)| < \epsilon\), \(a \leq x \leq a + \delta\).

That is,

\[
h''(a) - \epsilon < h''(x) < h''(a) + \epsilon, \quad a \leq x \leq a + \delta.
\]

\[139\]
Consider the integral
\[ I_k = \int_a^b e^{k[h(x) - h(a)]} \, dx. \]
We write \( I_k = I_k' + I_k'' \),
where
\[ I_k' = \int_a^{a + \delta} e^{k[h(x) - h(a)]} \, dx, \quad I_k'' = \int_{a + \delta}^b e^{k[h(x) - h(a)]} \, dx. \]
Since \( h \) is nonincreasing on \((a, b]\), we have \( h(x) \leq h(a + \delta) \) for all \( x > a + \delta \).
Therefore
\[ I_k'' \leq \int_{a + \delta}^b e^{k[h(a + \delta) - h(a)]} \, dx \]
and
\[ |I_k''| \leq e^{k[h(a + \delta) - h(a)]}(b - a - \delta). \]
Hence \( I_k'' \to 0 \) as \( k \to \infty \).
Now consider
\[ I_k' = \int_a^{a + \delta} e^{k[h(x) - h(a)]} \, dx. \]
By Taylor’s series with remainder, we have
\[ h(x) - h(a) \approx h''(\xi) \frac{(x - a)^2}{2}, \quad a \leq \xi \leq a + \delta. \]
Therefore
\[ I_k' = \int_a^{a + \delta} e^{k[h''(\xi) \frac{(x - a)^2}{2}]} \, dx. \]
So by (5.4), we can write
\[
\int_a^{a + \delta} e^{k[h''(\xi) \frac{(x - a)^2}{2}]} \, dx \leq \int_a^{a + \delta} e^{k h''(\xi) \frac{(x - a)^2}{2}} \, dx \leq \int_a^{a + \delta} e^{k h''(\xi) \frac{(x - a)^2}{2}} \, dx.
\]
That is,
\[
\int_a^{a + \delta} e^{k h''(\xi) \frac{(x - a)^2}{2}} \, dx \leq I_k' \leq \int_a^{a + \delta} e^{k h''(\xi) \frac{(x - a)^2}{2}} \, dx. \quad (5.5)
\]
Consider
\[ J = \int_a^{a + \delta} e^{k[h''(\xi) - \epsilon] \frac{(x - a)^2}{2}} \, dx. \]
140
After substituting \( x - a = u \), we get

\[ J = \int_{0}^{\delta} e^{\frac{k}{2} (h''(a) - \epsilon)} u^2 \, du \]

\[ = \frac{1}{\sqrt{-\frac{k}{2} (h''(a) - \epsilon)}} \int_{0}^{\delta} e^{-\frac{k}{2} u^2} e^{\frac{k}{2} (h''(a) - \epsilon)} u^2 \, du \]

\[ \longrightarrow \frac{1}{\sqrt{-\frac{k}{2} (h''(a) - \epsilon)}} \int_{0}^{\infty} e^{-u^2} du \quad \text{as} \quad k \to \infty \]

\[ = \frac{1}{\sqrt{-\frac{k}{2} (h''(a) - \epsilon)}} \frac{\sqrt{\pi}}{2} \quad \text{as} \quad k \to \infty. \]

Therefore

\[ \int_{a}^{a+\delta} e^{k (h''(a) - \epsilon) \frac{(x-a)^2}{2}} \, dx = \left( \frac{\pi}{-2k (h''(a) - \epsilon)} \right)^{\frac{1}{2}} \quad \text{as} \quad k \to \infty. \]

Similarly,

\[ \int_{a}^{a+\delta} e^{k (h''(a) + \epsilon) \frac{(x-a)^2}{2}} \, dx = \left( \frac{\pi}{-2k (h''(a) + \epsilon)} \right)^{\frac{1}{2}} \quad \text{as} \quad k \to \infty. \]

Therefore (5.5) becomes

\[ \left( \frac{\pi}{-2k (h''(a) - \epsilon)} \right)^{\frac{1}{2}} \leq I'_k \leq \left( \frac{\pi}{-2k (h''(a) + \epsilon)} \right)^{\frac{1}{2}}. \]

Since \( \epsilon > 0 \) is arbitrary, we have

\[ \left( \frac{\pi}{-2k h''(a)} \right)^{\frac{1}{2}} \leq I'_k \leq \left( \frac{\pi}{-2k h''(a)} \right)^{\frac{1}{2}}. \]

Hence

\[ I'_k \longrightarrow \left( \frac{\pi}{-2k h''(a)} \right)^{\frac{1}{2}} \quad \text{as} \quad k \to \infty. \]

So

\[ \int_{a}^{b} e^{k (h(x) - h(a))} \, dx \longrightarrow \left( \frac{\pi}{-2k h''(a)} \right)^{\frac{1}{2}} \quad \text{as} \quad k \to \infty. \]

That is,

\[ \int_{a}^{b} e^{k h(x)} \, dx \longrightarrow e^{k h(a)} \left( \frac{\pi}{-2k h''(a)} \right)^{\frac{1}{2}} \quad \text{as} \quad k \to \infty. \]
Lemma 5.5.2. Let $\eta$ be such that $0 < \eta < b - a$, and $h(x) \in C^2(b - \eta \leq x \leq b)$, $h'(b) = 0$, $h''(b) < 0$, $h(x)$ is nonincreasing on $[a,b)$. Then
\[
\int_a^b e^{kh(x)} \, dx \to e^{kh(b)} \left( -\frac{\pi}{2} \frac{h''(b)}{h''(b)} \right) \frac{1}{2} \text{ as } k \to \infty.
\]
Proof. The proof is similar to that of previous lemma.

Lemma 5.5.3. Let $\eta$ be such that $0 < \eta < b - a$, and $h(x) \in C^2(a \leq x \leq a + \eta)$, $h'(a) = 0$, $h''(a) < 0$, $h(x)$ is nonincreasing on $(a,b]$. Suppose $\phi(x) \in \mathcal{HK}([a,b])$. Then
\[
\int_a^b \phi(x) e^{kh(x)} \, dx \to \phi(a) e^{kh(a)} \left( -\frac{\pi}{2} \frac{h''(a)}{h''(a)} \right) \frac{1}{2} \text{ as } k \to \infty.
\]
Proof. Since $h(x) \in C^2(a \leq x \leq a + \eta)$, we have $h''(x)$ is continuous.

Let $\epsilon > 0$ be such that $0 < \epsilon < -h''(a)$.

Choose $\delta > 0$, $\delta < \eta$ such that $|h''(x) - h''(a)| < \epsilon$, $a \leq x \leq a + \delta$.

That is,
\[
h''(a) - \epsilon < h''(x) < h''(a) + \epsilon, \quad a \leq x \leq a + \delta.
\]

Consider the integral
\[
I_k = \int_a^b [\phi(x) - \phi(a)] e^{kh(x) - h(a)} \, dx.
\]
We show that $I_k \to 0$ as $k \to \infty$.

Let
\[
g_k(x) = \begin{cases} 
  e^{kh(x) - h(a)}, & x \in [a, b] \\
  0, & x = a.
\end{cases}
\]

We claim that $g_k \to 0$ as $k \to \infty$.

Since $h$ is nonincreasing on $(a,b]$, we have $h(x) - h(a) < 0$ for all $x \in (a,b]$.

Therefore $g_k(x) = e^{kh(x) - h(a)} \to 0$ as $k \to \infty$.

Set $g(x) = 0$ for all $x \in [a,b]$. Then
\[
g_k(x) = e^{kh(x) - h(a)} \to 0 = g(x) \text{ for all } x \in [a,b], \text{ as } k \to \infty.
\]
Since $\phi(x) \in \mathcal{H}\mathcal{K}([a, b])$, the integral
\[ \int_a^b [\phi(x) - \phi(a)] \, dx \]
exists.

Observe that the sequence $\{g_k(x)\}$ is of uniform bounded variation on $[a, b]$ with $g_k(x) \to 0$ as $k \to \infty$.

Then by [Cor. 3.2, [11]], we have
\[
\int_a^b [\phi(x) - \phi(a)] e^{k[h(x) - h(a)]} \, dx \to \int_a^b [\phi(x) - \phi(a)] \cdot 0 \, dx \text{ as } k \to \infty \\
\to 0 \text{ as } k \to \infty.
\]

Therefore
\[
\int_a^b \phi(x) e^{k[h(x) - h(a)]} \, dx = \phi(a) \int_a^b e^{k[h(x) - h(a)]} \, dx.
\]

That is,
\[
\int_a^b \phi(x) e^{kh(x)} \, dx = \phi(a) \int_a^b e^{kh(x)} \, dx.
\]

But by lemma (5.5.1), we have
\[
\int_a^b e^{kh(x)} \, dx \to e^{kh(a)} \left( \frac{-\pi}{2kh''(a)} \right)^{\frac{1}{2}} \text{ as } k \to \infty.
\]

Therefore
\[
\int_a^b \phi(x) e^{kh(x)} \, dx \to \phi(a) e^{kh(a)} \left( \frac{-\pi}{2kh''(a)} \right)^{\frac{1}{2}} \text{ as } k \to \infty.
\]

\[\square\]

**Lemma 5.5.4.** Let $\eta$ be such that $0 < \eta < b - a$, and $h(x) \in C^2(b - \eta \leq x \leq b)$, $h'(b) = 0$, $h''(b) < 0$, $h(x)$ is nonincreasing on $[a, b)$ and suppose $\phi(x) \in \mathcal{H}\mathcal{K}([a, b])$. Then
\[
\int_a^b \phi(x) e^{kh(x)} \, dx \to \phi(b) e^{kh(b)} \left( \frac{-\pi}{2kh''(b)} \right)^{\frac{1}{2}} \text{ as } k \to \infty.
\]
Proof. The proof is similar to that of the previous lemma.

Now we are ready to obtain the inversion theorem for Henstock-Kurzweil Laplace transform.

**Theorem 5.5.5. (Inversion Theorem)** If \( f(t) \in \mathcal{HK}(\mathbb{R}^+) \) and if the HK-Laplace transform of \( f(t) \), \( \mathcal{L}\{f\}(s) \) exists, then

\[
\lim_{k \to \infty} \frac{1}{k!} \left( \frac{k}{t} \right)^{k+1} \int_0^\infty e^{-\frac{k}{t} u} u^k f(u) \, du = f(t).
\]

**Proof.** The inversion theorem follows immediately, if we can prove the following:

I)  
\[
\lim_{k \to \infty} \frac{1}{k!} \left( \frac{k}{t} \right)^{k+1} \int_t^\infty e^{-\frac{k}{t} u} u^k f(u) \, du = \frac{f(t)}{2} \tag{5.6}
\]

II)  
\[
\lim_{k \to \infty} \frac{1}{k!} \left( \frac{k}{t} \right)^{k+1} \int_0^t e^{-\frac{k}{t} u} u^k f(u) \, du = \frac{f(t)}{2}. \tag{5.7}
\]

I) We have the relation

\[
\int_a^b f(x) e^{k \phi(x)} \, dx \longrightarrow f(a) e^{k \phi(a)} \left( \frac{-\pi}{2k \phi''(a)} \right)^{\frac{1}{2}} \quad \text{as} \quad k \to \infty.
\]

Take \( a = t \) and \( \phi(x) = \ln(x) - \frac{r}{t} \) for all \( x \in [t, \infty) \). Then we have

\[
\int_t^b f(x) e^{k \left[ \ln(x) - \frac{r}{t} \right]} \, dx \longrightarrow f(t) e^{k \left[ \ln(t) - 1 \right]} \left( \frac{-\pi}{2k \left( \frac{1}{\pi^2} \right)} \right)^{\frac{1}{2}} \quad \text{as} \quad k \to \infty.
\]

Therefore

\[
\int_t^b f(x) x^k e^{\frac{-k}{t} x} \, dx \longrightarrow f(t) t^k e^{-k \left( \frac{\pi t^2}{2k} \right)^{\frac{1}{2}}} \quad \text{as} \quad k \to \infty.
\]

That is,

\[
\int_t^b f(x) x^k e^{\frac{-k}{t} x} \, dx \longrightarrow f(t) t^{k+1} e^{-k \left( \frac{\pi}{2k} \right)^{\frac{1}{2}}} \quad \text{as} \quad k \to \infty.
\]
Therefore as \( k \to \infty \), we have

\[
\frac{1}{k!} \left( \frac{k}{t} \right)^{k+1} \int_t^b f(x) x^k e^{-\frac{kx}{t}} \, dx = \frac{1}{k!} \left( \frac{k}{t} \right)^{k+1} f(t) \left( \frac{\pi}{2k} \right)^{\frac{1}{2}}
\]

\[
= \left( \frac{k}{e} \right)^k \cdot \frac{k}{k!} f(t) \left( \frac{\pi}{2k} \right)^{\frac{1}{2}}.
\]

The Stirling’s formula is given by \( \left( \frac{n}{e} \right)^n \sqrt{2\pi n} \to n! \), using this we can write

\[
\frac{1}{k!} \left( \frac{k}{t} \right)^{k+1} \int_t^b f(x) x^k e^{-\frac{kx}{t}} \, dx = \frac{k}{\sqrt{2\pi k}} f(t) \left( \frac{\pi}{2k} \right)^{\frac{1}{2}} \text{ as } k \to \infty
\]

\[
= \frac{f(t)}{2} \text{ as } k \to \infty.
\]

This is true for every \( b \in \mathbb{R}, \ b > t \). Therefore by Hake’s theorem [1], we have

\[
\lim_{k \to \infty} \frac{1}{k!} \left( \frac{k}{t} \right)^{k+1} \int_t^\infty f(x) x^k e^{-\frac{kx}{t}} \, dx = \frac{f(t)}{2}.
\]

II) Again we have the relation

\[
\int_a^b f(x) e^{k \varphi(x)} \, dx \to f(b) e^{k \varphi(b)} \left( -\frac{\pi}{2k \varphi'(b)} \right)^{\frac{1}{2}} \text{ as } k \to \infty.
\]

Take \( b = t \) and \( \varphi(x) = \ln(x) - \frac{x}{t} \), for all \( x \in (0, t] \). Then we get

\[
\lim_{k \to \infty} \frac{1}{k!} \left( \frac{k}{t} \right)^{k+1} \int_a^t f(x) x^k e^{-\frac{kx}{t}} \, dx = \frac{f(t)}{2}.
\]

By Hake’s theorem [1], we have

\[
\lim_{k \to \infty} \frac{1}{k!} \left( \frac{k}{t} \right)^{k+1} \int_0^t f(x) x^k e^{-\frac{kx}{t}} \, dx = \frac{f(t)}{2}.
\]

Therefore from (5.6) and (5.7), we can write

\[
\lim_{k \to \infty} \frac{1}{k!} \left( \frac{k}{t} \right)^{k+1} \int_0^t f(x) x^k e^{-\frac{kx}{t}} \, dx = f(t).
\]

\[\square\]
5.6  Examples

Here we shall give some examples of HK-Laplace transformable functions whose classical Laplace transform do not exist. However we could not able to find the exact form of the HK-Laplace transform of these functions.

**Example 5.6.1.** Consider the function \( g(t) = \frac{e^{-st}}{\ln(t+2)}, \ s \in \mathbb{C}. \)

Let \( f(t) = e^{-st} \sin t \) and \( \phi(t) = \frac{1}{\ln(t+2)}. \)

Observe that the function \( \phi(t) \) is differentiable on \( \mathbb{R}^+ \) and \( \phi' \in L^1(\mathbb{R}^+) \) and if \( F(t) = \int_0^t e^{-su} \sin u \, du, \ Re.s > 0, \) then \( |F(t)| \leq M \) for all \( t \in (0, \infty), \ Re.s > 0 \) for some constant \( M > 0. \)

Now

\[
F(t) \phi(t) = \left\{ \frac{e^{-st} (s \sin t - \cos t)}{s^2 + 1} + \frac{1}{s^2 + 1} \right\} \cdot \frac{1}{\ln(t+2)}.
\]

Then

\[
\lim_{t \to \infty} F(t) \phi(t) = \lim_{t \to \infty} \left\{ \frac{e^{-st} (s \sin t - \cos t)}{s^2 + 1} + \frac{1}{s^2 + 1} \right\} \cdot \frac{1}{\ln(t+2)} \to 0.
\]

Therefore \( \lim_{t \to \infty} F(t) \phi(t) \) exists.

Hence by Du-Bois-Reymond’s test [1], we have \( \frac{e^{-st} \sin t}{\ln(t+2)} \in \mathcal{HK}(\mathbb{R}^+), \ Re.s > 0. \)

That is,

\[
\int_0^\infty \frac{e^{-st} \sin t}{\ln(t+2)} \, dt \text{ exists, } Re.s > 0.
\]

Hence the HK-Laplace transform of the function \( g(t) = \frac{\sin t}{\ln(t+2)} \) exists, \( Re.s > 0. \)

**Example 5.6.2.** Consider the function \( f(t) = \frac{\cos t}{\sqrt{t}}. \)

We have

\[
F(t) = \int_0^t \cos u \, du = 2 \int_0^{\sqrt{t}} \cos u^2 \, du.
\]
Let $\phi(t) = e^{-st}$, $s \in \mathbb{C}$. Then $\phi'(t) = -se^{-st} \in L^1(\mathbb{R}^+)$, $Re.s > 0$.

Now

$$F(t) \phi(t) = 2 e^{-st} \int_0^{\sqrt{t}} \cos u^2 \, du.$$  

Then

$$\lim_{t \to \infty} F(t) \phi(t) = 2 \lim_{t \to \infty} e^{-st} \int_0^{\sqrt{t}} \cos u^2 \, du \quad \Rightarrow \quad 0,$$

Therefore by Du-Bois-Reymond’s test [1], we have $f(t) \phi(t) \in \mathcal{H}\mathcal{K}(\mathbb{R}^+)$, $Re.s > 0$.

Hence the HK-Laplace transform of the function $f(t) = \frac{\cos t}{\sqrt{t}}$ exists for $Re.s > 0$.

**Example 5.6.3.** Let $\phi(t) = \frac{e^{-st}}{t}$, $t \in \mathbb{R}^+$, $\sigma > 0$.

This function is continuous on $\mathbb{R}^+$ and

$$\phi(t + 1) = \frac{1}{(t + 1) e^{\sigma(t+1)}} < \frac{1}{t e^{\sigma t}} = \phi(t) \quad \text{for all } t \in \mathbb{R}^+$$

and also $\lim_{t \to \infty} \phi(t) = \lim_{t \to \infty} \frac{1}{t e^{\sigma t}} = 0$, $\sigma > 0$.

Therefore $\phi(t)$ is decreasing to zero as $t \to \infty$.

Observe that the integral

$$\int_0^\infty \frac{e^{-\sigma t}}{t} \, dt, \sigma > 0$$

diverges.

Now let $n_0 \in \mathbb{N}$ be such that $0 < (1 + 4n_0) \frac{\pi}{4\omega}$.

For $t \in \mathbb{R}$, we have $|\sin \omega t| \geq \frac{1}{\sqrt{2}}$ if and only if $\omega t \in \bigcup_{n=0}^\infty [(1 + 4n)\frac{\pi}{4}, (3 + 4n)\frac{\pi}{4}]$.  

147
If \( n \in \mathbb{N} \), then \((3 + 4n) \frac{\pi}{4} < (1 + n) \pi\).

\[
\int_0^{(1+n)\pi} \frac{e^{-\sigma t}}{t} \left| \sin t \right| \, dt = \sum_{j=n_0}^{n} \int_0^{(3+4j)\pi} \frac{e^{-\sigma t}}{t} \left| \sin \omega t \right| \, dt \\
\geq \sum_{j=n_0}^{n} \int_0^{(3+4j)\pi} \frac{e^{-\sigma t}}{t} \frac{1}{\sqrt{2}} \, dt \\
\geq \frac{1}{\sqrt{2}} \sum_{j=n_0}^{n} \int_0^{(3+4j)\pi} \frac{4}{(3+4j) \pi} e^{\sigma (3+4j) \frac{\pi}{4}} \, dt \\
= \frac{1}{\sqrt{2}} \sum_{j=n_0}^{n} \frac{\pi}{(3+4j) \pi} \left( 1 + j \right) e^{\sigma (1+j) \pi}.
\]

On the other hand, we have

\[
\int_0^{(1+n)\pi} \frac{e^{-\sigma t}}{t} \, dt = \int_0^{n_0\pi} \frac{e^{-\sigma t}}{t} \, dt + \int_{n_0\pi}^{(1+n)\pi} \frac{e^{-\sigma t}}{t} \, dt \\
= \int_0^{n_0\pi} \frac{e^{-\sigma t}}{t} \, dt + \sum_{j=n_0}^{n} \int_{j\pi}^{(1+j)\pi} \frac{e^{-\sigma t}}{t} \, dt \\
\leq \int_0^{n_0\pi} \frac{e^{-\sigma t}}{t} \, dt + \sum_{j=n_0}^{n} \int_{j\pi}^{(1+j)\pi} \frac{e^{-\sigma j\pi}}{j\pi} \, dt \\
= \int_0^{n_0\pi} \frac{e^{-\sigma t}}{t} \, dt + \sum_{j=n_0}^{n} \frac{e^{-\sigma j\pi}}{j\pi} \cdot \pi.
\]

But since \( \frac{e^{-\sigma t}}{t} \in \mathcal{H} \mathcal{K}(\mathbb{R}^+) \), we have \( \int_0^{\infty} \frac{e^{-\sigma t}}{t} \, dt = \infty \).

Therefore \( \sum_{j=n_0}^{\infty} \frac{e^{-\sigma j\pi}}{j\pi} = \infty \).

So \( \frac{e^{-\sigma t}}{t} \sin \omega t \notin \mathcal{L}^1(\mathbb{R}^+) \), \( \sigma > 0 \).

Similarly, \( \frac{e^{-\sigma t}}{t} \cos \omega t \notin \mathcal{L}^1(\mathbb{R}^+) \), \( \sigma > 0 \).

Now let \( f(t) = \sin \omega t, \phi(t) = \frac{e^{-\sigma t}}{t}, t \in \mathbb{R} \).

Observe that \( \phi(t) \) is monotone function and \( \phi(t) \to 0 \) as \( t \to \infty \).

And \( \left| \int_0^t \sin \omega u \, du \right| \leq \frac{2}{|\omega|} \), \( \omega \neq 0 \) for all \( t \in \mathbb{R}^+ \). That is, \( F \) is bounded.

Therefore by Chartier-Dirichlet’s test [1], we have \( \frac{e^{-\sigma t}}{t} \sin \omega t \in \mathcal{H} \mathcal{K}(\mathbb{R}^+) \).
Similarly, \( \frac{e^{-st}}{t} \cos \omega t \in \mathcal{HK}(\mathbb{R}^+) \).

Hence \( \frac{e^{-st}}{t} \{ \cos \omega t - i \sin \omega t \} \in \mathcal{HK}(\mathbb{R}^+) \).

That is, \( \int_{0}^{\infty} \frac{e^{-st}}{t} e^{-i\omega t} \, dt \) exists, \( \sigma > 0 \).

Thus \( \int_{0}^{\infty} \frac{e^{-st}}{t} dt \) exists, \( Re.s > 0 \) as a HK integral, where \( s = \sigma + i\omega \).
References


