CHAPTER 4

LAT shite Points

ON THE TERNARY CUBIC EQUATION

The full-bodied gravitation of research towards Diophantine Equations is unequivocal. Mollin (1996) recounts the Diophantine equations, homogeneous and non-homogeneous, have aroused the interest of numerous mathematicians since antiquity (Mollin1996). The problem of finding all integer solutions of a Diophantine Equation with three or more variables and degree at least three, in general, presents a good deal of difficulties. There is a vast general theory of homogeneous quadratic equations with three variables. A lot is known about equations in two variables in higher degrees. For equations with more than three variables and degree at least three, very little is known. It is worth to note that undesirability appears in equations, even perhaps at degree four with fairly small coefficients. Gopalan et al. (2008),Gopalan &Janaki (2008), Gopalan & Vijayashankar(2010) have exhibited a few Diophantine Equations of degree three in relation to general form of integral solutions. It is therefore towards this end, it is considered that the non homogeneous Diophantine equation of degree three with three unknowns represented by $x^3 + y^3 + 7xy(x + y) = 16z^2$ and analyses for its non-zero integral solutions. A few interesting properties among the solutions are presented.
4.1 METHOD OF ANALYSIS

The ternary cubic equation under consideration is

\[ x^3 + y^3 + 7xy(x + y) = 16z^2 \]  \hspace{1cm} (4.1)

PATTERN

Two solution patterns are depicted here in solving (4.1)

Introducing the linear transformations

\[ x = u + v; \quad y = u - v; \quad z = u \quad (u \neq v \neq 0) \]  \hspace{1cm} (4.2)

Equation (4.1) is simplified to the well known pythagorian equation

\[ (2u - 1)^2 = 2v^2 + 1 \]

Let \[ y = 2u - 1 \]

\[ y^2 = 2v^2 + 1 \]  \hspace{1cm} (4.3)

The smallest positive integer solution of (4.3) is \[ v_0 = 2, \quad y_0 = 3, \]

\[ \tilde{y}_s + \sqrt{2}\tilde{v}_s = (3 + 2\sqrt{2})^s + 1; \quad s = 0, 1, 2, \ldots \]

since irrational roots occur in pairs, the outcome is

\[ \tilde{y}_s - \sqrt{2}\tilde{v}_s = (3 - 2\sqrt{2})^s + 1; \quad s = 0, 1, 2, \ldots \]
From the above two equations, it is obtained that

\[ \tilde{y}_s = \frac{1}{2} \left[ (3 + 2\sqrt{2})^s + 1 - (3 - 2\sqrt{2})^s + 1 \right] \]

\[ \tilde{v}_s = \frac{1}{2\sqrt{2}} \left[ (3 + 2\sqrt{2})^s + 1 - (3 - 2\sqrt{2})^s + 1 \right] \]

\[ \tilde{y}_s = 2\tilde{u}_s - 1 \]

\[ \tilde{u}_s = \frac{1}{4} \left[ (3 + 2\sqrt{2})^s + 1 + (3 - 2\sqrt{2})^s + 1 \right] + \frac{1}{2} \]

where \( s = 0, 1, \ldots \)

Taking advantage of Equation (4.2), the sequence of integral solutions of the Equation (4.1) are given by

\[ x_s = \frac{1}{4} \left[ (3 + 2\sqrt{2})^s + 1 + (3 - 2\sqrt{2})^s + 1 \right] + \frac{1}{2} \]

\[ + \frac{1}{2\sqrt{2}} \left[ (3 + 2\sqrt{2})^s + 1 - (3 - 2\sqrt{2})^s + 1 \right] \]

\[ y_s = \frac{1}{4} \left[ (3 + 2\sqrt{2})^s + 1 + (3 - 2\sqrt{2})^s + 1 \right] + \frac{1}{2} \]

\[ - \frac{1}{2\sqrt{2}} \left[ (3 + 2\sqrt{2})^s + 1 - (3 - 2\sqrt{2})^s + 1 \right] \]

\[ z_s = \frac{1}{4} \left[ (3 + 2\sqrt{2})^s + 1 + (3 - 2\sqrt{2})^s + 1 \right] + \frac{1}{2} \]

where \( s = 0, 1, 2, 3, \ldots \)

The above values of \( x_s, y_s \) and \( z_s \) satisfy the following recurrence relations respectively
\[ x_{s+2} - 6x_{s+1} + x_s + 2 = 0, x_0 = 4, x_1 = 21 \]

\[ y_{s+2} - 6y_{s+1} + y_s + 2 = 0, y_0 = 0, y_1 = -3 \]

and

\[ z_{s+2} - 6z_{s+1} + z_s + 2 = 0, z_0 = 2, z_1 = 9 \]

A few numerical examples are given below:

<table>
<thead>
<tr>
<th>s</th>
<th>( x_s )</th>
<th>( y_s )</th>
<th>( z_s )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>0</td>
<td>2</td>
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<tr>
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<tr>
<td>5</td>
<td>23661</td>
<td>-4059</td>
<td>9801</td>
</tr>
</tbody>
</table>

**PROPERTIES:**

A few properties among the solutions are given below:

1) \(x_s + y_{s+1} - 1 = 0\).
2) \(z_{2s+1}\) is a perfect square.
3) \(x_{2s} \equiv 0(\text{mod } 4)\)
4) \(y_{2s} \equiv 0(\text{mod } 4)\)
5) \(2(x_{2s+1} + y_{2s+1} - 1) = 4(x_s + y_s - 1)^2 - 2\).
6) \(x_{s+1}y_s - x_sy_{s+1} \equiv 0(\text{mod } 4)\)
7) \(4z_{2s} = 2(x_s + y_s)\)

**PATTERN:** 2
It is worth to observe that there exists another pattern of solutions of (4.1) obtained as follows:

Introducing the transformations

\[ x = u + v, \quad y = u - v, \quad z = ku \]  \hspace{1cm} (4.5)

where \( k \) takes values 1, 2, 3, …

Simplifying Equation (4.1) to the well-known Pythagorean equation

\[(2u - k^2)^2 = 2v^2 + k^4; \quad k = 1, 2, 3, \ldots \]  \hspace{1cm} (4.6)

Let \( y = 2u - k^2 \)

To obtain the other solutions of (4.6), consider the Pellian equation

\[ y^2 = 2v^2 + 1 \]

The smallest positive integer solution of the above equation is \( v_0 = 2, \; y_0 = 3 \).

Now,

\[ \tilde{y}_s + \sqrt{2} \tilde{v}_s = (3 + 2\sqrt{2})^{s-1}; \quad s = 0, 1, 2, \ldots \]

since irrational roots occur in pairs, the outcome is

\[ \tilde{y}_s - \sqrt{2} \tilde{v}_s = (3 - 2\sqrt{2})^{s-1}; \quad s = 0, 1, 2, \ldots \]

From the above two equations, it is obtained that
where \( k = 1,2,3, \ldots \) and \( s = 0,1,2,3, \ldots \)

Taking advantage of Equation (4.5), the sequences of integral solutions of the Equation (4.1) are given by

\[
\tilde{y}_s = \frac{k^2}{2} \left[ (3 - 2\sqrt{2})^{s+1} + (3 + 2\sqrt{2})^{s+1} \right]
\]

\[
\nu_s = \frac{k^2}{2\sqrt{2}} \left[ (3 + 2\sqrt{2})^{s+1} - (3 - 2\sqrt{2})^{s+1} \right]
\]

\[
\tilde{y}_s = 2\bar{u}_s - k^2
\]

\[
\bar{u}_s = \frac{k^2}{4} \left[ (3 + 2\sqrt{2})^{s+1} + (3 - 2\sqrt{2})^{s+1} \right] + \frac{k^2}{2}
\]

where \( k = 1,2,3, \ldots \) and \( s = 0,1,2,3, \ldots \)

The above values of \( x_s, y_s \) and \( z_s \) satisfy the following recurrence relations respectively

\[
x_{s+2} - 6x_{s+1} + x_s + 2k^2 = 0, x_0 = 4k^2, x_1 = 21k^2
\]

\[
y_{s+2} - 6y_{s+1} + y_s + 2k^2 = 0, y_0 = 0, y_1 = -3k^2
\]

and

\[
z_{s+2} - 6z_{s+1} + z_s + 2k^2 = 0, z_0 = 2k^3, z_1 = 9k^3
\]

where \( k = 1,2,3, \ldots \) and \( s = 0,1,2,3, \ldots \)
A few numerical examples are given below:

<table>
<thead>
<tr>
<th>s</th>
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Properties

A few properties among the solutions are given below:

1) \(x_s + y_{s,1} - k^2 = 0\).

2) \(\frac{z_{2s-1}}{k}\) is a perfect square.

3) \(x_{2s}(k) \equiv 0 \pmod{4}\)

4) \(y_{2s}(k) \equiv 0 \pmod{4}\)

5) \(2(x_s + y_s - 1)^2 - (x_s - y_s)^2 = 2k^4\)

6) \(x_{2s-1}(k)y_s(k) - x_s(k)y_{s,1}(k) \equiv 0 \pmod{4}\)

7) \(4z_s = 2k(x_s + y_s)\).

8) \(\frac{32x_s}{k^2} + \frac{32y_s}{k^2} - \frac{16x_s y_s}{k^4}\) is a perfect square.

9) \(z_{s,1}(k) - z_s(k)\) is a \(R_s\) Number.

10) \(x_{2s-1}(k) + y_{2s-1}(k) - 2k^2\) is a perfect square.