CHAPTER 2

INTEGRAL POINTS ON THE HYPERBOLA

It is well known that binary quadratic Diophantine Equations are rich in variety (Mollin 1996). For example, the binary quadratic equation $3x^2 + xy = 14$ has a limited number of integral solutions (Tekcan 2007) and it is worth mentioning here, that the above equation represents a hyperbola. This result acts as a force-engine to search for infinitely many integral solutions for a hyperbola. This paper concerns with the problem of finding different patterns of nonzero integral solutions on the hyperbolic curve represented by $x^2 - 6xy + y^2 + 6x - 2y + 5 = 0$. Also a few inventive relations satisfied by the solutions of the above hyperbola are presented.

2.1 METHOD OF ANALYSIS

The hyperbola equation under consideration is

$$x^2 - 6xy + y^2 + 6x - 2y + 5 = 0 \quad (2.1)$$

Shifting the origin to the center (0,1), the Equation (2.1) transformed into

$$X^2 - 6XY + Y^2 + 4 = 0 \quad (2.2)$$

Where, $x = X, y = Y + 1 \quad (2.3)$
2.1.1 Pattern-1

Let \( X = M + N, Y = M - N \) \hspace{1cm} (2.4)

the Equation (2.1) reduces to the Pellian equation

\[ M^2 = 2N^2 + 1 \] \hspace{1cm} (2.5)

whose general solution \((\tilde{\alpha}, \tilde{\beta})\) is given by

\[ \tilde{M}_s + \sqrt{2}\tilde{N}_s = \left(3 + 2\sqrt{2}\right)^{s+1}; s = 0, 1, 2, ... \]

Since irrational roots occur in pairs, we have

\[ \tilde{M}_s - \sqrt{2}\tilde{N}_s = \left(3 - 2\sqrt{2}\right)^{s+1}; s = 0, 1, 2, ... \]

From the above two equations, it is obtained that

\[ \tilde{M}_s = \frac{1}{2} \left[ \left(3 + 2\sqrt{2}\right)^{s+1} + \left(3 - 2\sqrt{2}\right)^{s+1} \right] \]

\[ \tilde{N}_s = \frac{1}{2\sqrt{2}} \left[ \left(3 + 2\sqrt{2}\right)^{s+1} - \left(3 - 2\sqrt{2}\right)^{s+1} \right]; s = 0, 1, 2, ... \]

Taking advantage of Equations (2.3) and (2.4), the sequences of

integral solutions of the hyperbola (2.1) are given by
\[ x_s = \tilde{M}_s + \tilde{N}_s; \quad y_s = \tilde{M}_s - \tilde{N}_s + 1 \]

i.e.,
\[ x_s = \frac{1}{4} \left[ 2 \left( \left( 3 + 2\sqrt{2} \right)^{x_{s-1}} + \left( 3 - 2\sqrt{2} \right)^{x_{s-1}} \right) + \sqrt{2} \left( \left( 3 + 2\sqrt{2} \right)^{y_{s-1}} - \left( 3 - 2\sqrt{2} \right)^{y_{s-1}} \right) \right] \quad (2.6) \]

and
\[ y_s = \frac{1}{4} \left[ 2 \left( \left( 3 + 2\sqrt{2} \right)^{y_{s-1}} + \left( 3 - 2\sqrt{2} \right)^{y_{s-1}} \right) - \sqrt{2} \left( \left( 3 + 2\sqrt{2} \right)^{x_{s-1}} - \left( 3 - 2\sqrt{2} \right)^{x_{s-1}} \right) \right] + 4 \]

\[ s = 0, 1, 2, \ldots \]

The above values of \( x_s \) and \( y_s \) satisfy the following recurrence relations respectively.

\[ x_{s-2} - 6x_{s-1} + x_s = 0, \quad x_0 = 1, x_1 = 5 \]

and

\[ y_{s-2} - 6y_{s-1} + y_s = -4, \quad y_0 = 2, y_1 = 2 \]

A few numerical examples are given below:

<table>
<thead>
<tr>
<th>( s )</th>
<th>( x_s )</th>
<th>( y_s )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>29</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>169</td>
<td>30</td>
</tr>
<tr>
<td>4</td>
<td>985</td>
<td>170</td>
</tr>
<tr>
<td>5</td>
<td>5741</td>
<td>986</td>
</tr>
<tr>
<td>6</td>
<td>33461</td>
<td>5742</td>
</tr>
<tr>
<td>7</td>
<td>195025</td>
<td>33462</td>
</tr>
<tr>
<td>8</td>
<td>1136689</td>
<td>195026</td>
</tr>
</tbody>
</table>
2.1.2 Properties

A few properties among the solutions are given below:

1. It is observed that \(x\) values are odd and \(y\) values are even.

2. \(x_s - y_{s+1} + 1 = 0\).

3. \(y_{s+2} - y_{s+1} - x_{s+1} + x_s = 0\).

4. \(x_{3n-2} \equiv 0 \pmod{5}; n = 1, 2, 3, \ldots\)

5. \(y_{n^2 + 3n + 2} \equiv 0 \pmod{6}\)

6. \(x_{s+1}^3 - y_{s+1}^3 = 0 \pmod{4}\)

7. \((x_s + y_s - 1)^2 - 2(x_s - y_s + 1)^2 = 4\)

8. \((x_{2s+1} + y_{2s+1} + 1) = (x_s + y_s - 1)^2\)

9. \((x_{3s+2} + y_{3s+2} - 1) = (x_s + y_s - 1)^3 - 3(x_s + y_s - 1)\)

10. \((x_{4s+3} + y_{4s+3} - 1) = (x_s + y_s - 1)^4 - 4(x_{2s+1} + y_{2s+1} - 1) - 6\)

11. \(x_s^3 - y_{s+1}^3 + 3x_s y_{s+1} + 1 = 0\).

12. Each of the following expressions is a Nasty number.

   (i) \(3[16 \ x_s(y_s - 1) - 8]\)

   (ii) \(6[x_{2s-1} + y_{2s-1} - 3]\)

   (iii) \(3[16(y_s - 1)(y_{s+1} - 1) - 8]\)

2.1.3 Pattern II:

Treating (1) as a quadratic in \(y\) and solving for \(y\), we get
\[ y = (3x + 1) \pm 2 \sqrt{2x^2 - 1} \quad (2.7) \]

To eliminate the square root on the RHS of (2.7), assume

\[ \beta^2 = 2x^2 - 1 \quad (2.8) \]

The Smallest positive integer solution \((\beta_0, x_0)\) of (2.7) is

\[ \beta_0 = 1, \quad x_0 = 1 \]

To obtain the other solutions of (2.7) consider the pellian equation

\[ \beta^2 = 2x^2 + 1 \quad (2.9) \]

whose general solutions \((\tilde{\beta}, \tilde{x})\) are given by

\[
\begin{align*}
\tilde{\beta}_s + \sqrt{2}\tilde{x}_s &= \left(3 + 2\sqrt{2}\right)^{s-1}; S = 0, 1, 2, \ldots \\
\tilde{\beta}_s - \sqrt{2}\tilde{x}_s &= \left(3 - 2\sqrt{2}\right)^{s-1}; S = 0, 1, 2, \ldots
\end{align*}
\]

From the above two equations it is obtained that

\[
\begin{align*}
\tilde{\beta}_s &= \frac{1}{2} \left[ \left(3 + 2\sqrt{2}\right)^{s-1} + \left(3 - 2\sqrt{2}\right)^{s-1} \right] \\
\tilde{x}_s &= \frac{1}{2\sqrt{2}} \left[ \left(3 + 2\sqrt{2}\right)^{s-1} - \left(3 - 2\sqrt{2}\right)^{s-1} \right]; S = 0, 1, 2, \ldots
\end{align*}
\]

Applying Brammaguptha lemma between the solutions

\((\beta_0, x_0)\) and \((\tilde{\beta}_s, \tilde{x}_s)\) the general solution \((\bar{\beta}_s, \bar{x}_s)\) of (2.8) is found to be

\[
\begin{align*}
\bar{x}_{s+1} &= \tilde{\beta}_s + \tilde{x}_s \\
\bar{\beta}_{s+1} &= \tilde{\beta}_s + 2\tilde{x}_s
\end{align*}
\]
and thus, from (2.7), it is drawn that

\[ y_{s+1} = 5 \beta_s + 7 x_s + 1 \] (taking positive root in (2.7))

\[ s = 0, 1, 2, \ldots \]

Therefore, the non-zero integral solutions of (2.1) are given by

\[
x_{s+1} = \frac{1}{2} \left[ \left( \left( 3 + 2 \sqrt{2} \right)^{s+1} + \left( 3 - 2 \sqrt{2} \right)^{s+1} \right) + \frac{1}{2 \sqrt{2}} \left( 3 + 2 \sqrt{2} \right)^{s+1} \\
+ \left( 3 - 2 \sqrt{2} \right)^{s+1} \right]
\]

And

\[
y_{s+1} = \frac{5}{2} \left[ \left( \left( 3 + 2 \sqrt{2} \right)^{s+1} + \left( 3 - 2 \sqrt{2} \right)^{s+1} \right) + \frac{7}{2 \sqrt{2}} \left( 3 + 2 \sqrt{2} \right)^{s+1} \\
+ \left( 3 - 2 \sqrt{2} \right)^{s+1} \right]
\]

\[ s = 0, 1, 2, \ldots \]

It is worth mentioning that the negative root on the right hand side of (2.7) does not lead to any new result. Observations

(i) \( (7x_{s+1} - y_{s+1} + 4)^2 - 2(y_{s+1} - 5x_{s+1} - 1)^2 = 4 \)

(ii) \( (7x_{s+2} - y_{s+2} + 6) = (7x_{s+1} - y_{s+1} + 4)^2 \)

(iii) \( y_{2s} + y_{2s+1} \) is a perfect square
2.1.4 Pattern : III

Treating Equation (2.1) as a quadratic in $x$ and following the procedure presented in the above pattern I, the solutions of (2.1) are found to be

$$x_{s+1} = \frac{1}{2} \left[ \left( \left( 3 + 2\sqrt{2} \right)^{s+1} + \left( 3 - 2\sqrt{2} \right)^{s+1} \right) + \frac{1}{2\sqrt{2}} \left( 3 + 2\sqrt{2} \right)^{s+1} \\
+ \left( 3 - 2\sqrt{2} \right)^{s+1} \right]$$

And

$$y_{s+1} = \frac{5}{2} \left[ \left( \left( 3 + 2\sqrt{2} \right)^{s+1} + \left( 3 - 2\sqrt{2} \right)^{s+1} \right) + \frac{7}{2\sqrt{2}} \left( 3 + 2\sqrt{2} \right)^{s+1} \\
+ \left( 3 - 2\sqrt{2} \right)^{s+1} \right]$$

$s=0,1,2,..............$

The following relations are observed

(i) $\left( x_{s+1} - 5y_{s+1} + 5 \right)^2 - 2\left( x_{s+1} - 5y_{s+1} - 5 \right)^2 = 4$

(ii) $\left( x_{2s+2} - 7y_{2s+2} + 9 \right) = \left( x_{s+1} - 7y_{s+1} + 7 \right)$