CHAPTER 8

ON STAR NUMBERS

Neither time nor custom stale the infinite variety of the ‘Queen of Mathematics’ which is chiefly due to its multitudinal challenges exposed by number theory. Many numbers exhibit fascinating properties, they form sequences, they form patterns and so on. One may consider the “pentagonal numbers” with respect to the general form of the rank of square and recurrence relation satisfied by the solutions. The reference is solicited towards Beiler Albert (1963) and Gopalan & Jayakumar (2005). This section of the study sheds light on analyzing and searching different numbers and determining recurrence relation on star number

A star number is said to be a centered figurate number that stands for a centered-hexagram (6-pointed star). For example, the play by the Chinese checkers. The \( n \)th star number is given by the formula
\[
S_n = 6n(n - 1) + 1.
\]

The first few star numbers are 1, 13, 37, 73, 121 and so on and so forth. The digital root of a star number is always 1 or 4, and progresses in the sequence 1, 4, 1. The last two digits of a star number in base 10 are always 01, 13, 21, 33, 37, 41, 53, 61, 73, 81, or 93. Unique among the star numbers is 35113, since its prime factors (i.e. 13, 37 and 73) are also consecutive star numbers. Geometrically, the \( n \)th star number is made up of a central point and 12 copies of the \((n-1)\)th triangular number — making it numerically equal to the \( n \)th centered dodecagonal number, but differently
arranged. Infinitely many star numbers are also triangular numbers, the first four being $S_1 = 1 = T_1$, $S_7 = 253 = T_{22}$, $S_{91} = 49141 = T_{313}$, and $S_{1261} = 9533161 = T_{4366}$. $S_n$ is given by the formula $S_n = 6n(n - 1) + 1$ for $n \geq 1$.

Here, the determination of the general form of the rank of square star numbers is undertaken. In addition, the recurrence relations satisfied by the solutions are presented.

8.1 GENERAL RANK OF SQUARE STAR NUMBERS

8.1.1 Theorem

The general form of the ranks of square star numbers ($S_m$) are given by

$$m = \frac{1}{4\sqrt{6}} \left( \left( 5 + 2\sqrt{6} \right)^n \left( \sqrt{6} + 2 \right) + \left( 5 - 2\sqrt{6} \right)^n \left( \sqrt{6} - 2 \right) \right) + \frac{1}{2}, n \geq 1$$

Proof: Let $S_m$ be a square star number. We write

$$S_m = t^2, \text{ where } t \text{ is a non-zero integer} \quad (8.1)$$

Using the definition of Star number the above Equation (8.1) is written as

$$m^2 - m + \frac{1}{6} = \frac{t^2}{6}$$

By writing complete square and simplifying,

$$3(2m - 1)^2 - 1 = 2t^2 \quad (8.2)$$

If we take $x = 2m - 1$ then
\[3x^2 - 2t^2 = 1 \quad (8.3)\]

Which is the well-known Pell’s equation whose solutions are given by

\[x_n = \frac{1}{2\sqrt{6}} \left((5 + 2\sqrt{6})^{n+1} (\sqrt{6} + 2) + (5 - 2\sqrt{6})^{n+1} (\sqrt{6} - 2)\right) \quad (8.4)\]

\[t_n = \frac{1}{2\sqrt{6}} \left((5 + 2\sqrt{6})^{n+1} (\sqrt{6} + 3) + (5 - 2\sqrt{6})^{n+1} (\sqrt{6} - 3)\right) \quad (8.5)\]

where \(n = 0, 1, 2, \ldots\)

In view of the Equation (8.2), the rank \(m\) of square Star number and the values of \(t\) are given by

\[m = \frac{1}{4\sqrt{6}} \left((5 + 2\sqrt{6})^n (\sqrt{6} + 2) + (5 - 2\sqrt{6})^n (\sqrt{6} - 2)\right) + \frac{1}{2} \quad (8.6)\]

\[t = \frac{1}{2\sqrt{6}} \left((5 + 2\sqrt{6})^n (\sqrt{6} + 3) + (5 - 2\sqrt{6})^n (\sqrt{6} - 3)\right) \quad (8.7)\]

where \(n = 0, 1, 2, \ldots\)

For simplicity and brevity some values of \(m\), \(t\) and their corresponding star and square numbers are presented in the following table.

<table>
<thead>
<tr>
<th>Values of n</th>
<th>Ranks(m)</th>
<th>Ranks(t)</th>
<th>Star numbers ((S_m))</th>
<th>Square numbers ((t^2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>11</td>
<td>121</td>
<td>121</td>
</tr>
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<td>1079</td>
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<tr>
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<td>4361</td>
<td>10681</td>
<td>114083761</td>
<td>114083761</td>
</tr>
</tbody>
</table>
8.2 RECURRENCE RELATION

8.2.1 Theorem

The ranks $m_n$ and $t_n$ of the Equations (8.6) and (8.7) satisfy the following recurrence relations

\[(i) m_{n+2} - 10m_{n+1} + m_n + 4 = 0\]
\[(ii) t_{n+2} - 10t_{n+1} + t_n = 0 \text{ where } n = 0, 1, 2, \ldots\]

PROOF:

Let $m = \frac{1}{4\sqrt{6}} \left( (5 + 2\sqrt{6})^n (\sqrt{6} + 2) + (5 - 2\sqrt{6})^n (\sqrt{6} - 2) \right) + \frac{1}{2}(8.8)$

This Equation (8.8) can be written as

\[4\sqrt{6} m_n - 2\sqrt{6} = \left( (5 + 2\sqrt{6})^n (\sqrt{6} + 2) + (5 - 2\sqrt{6})^n (\sqrt{6} - 2) \right) \quad (8.9)\]

If we take $A = 5 + 2\sqrt{6}$, then $A - 4\sqrt{6} = 5 - 2\sqrt{6}$, $B = \sqrt{6} + 2$,

then $B - 4 = \sqrt{6} - 2$ and the Equation (8.9) becomes

\[4\sqrt{6} m_n - 2\sqrt{6} = \left( A^n B + (A - 4\sqrt{6})^n (B - 4) \right) (8.10)\]

Replacing $n$ by $n+1$, $n+2$ successively in (10), we get

\[2\sqrt{6}(2m_{n+1} - 1) = \left( A^{n+1} B + (A - 4\sqrt{6})^{n+1} (B - 4) \right) (8.11)\]
\[2\sqrt{6}(2m_{n+2} - 1) = \left( A^{n+2} B + (A - 4\sqrt{6})^{n+2} (B - 4) \right) (8.12)\]
Multiplying the Equation (8.10) by \( A \) and then subtracting from the Equation (8.11) we get

\[
(2m_n - 1)A - 2m_{n+1} + 1 = 2\left(A - 4\sqrt{6}\right)^2(B - 4) (8.13)
\]

Multiplying the Equation (8.11) by \( A \) and then subtracting from the equation (8.12), to obtain

\[
(2m_{n+1} - 1)A - 2m_{n+2} + 1 = 2\left(A - 4\sqrt{6}\right)^{n+1}(B - 4) (8.14)
\]

Multiplying the Equation (8.13) by \( A - 4\sqrt{6} \) and subtracting from the Equation (8.14) it is obtained that

\[
(2m_{n+1} - 1)A - 2m_{n+2} + 1 - (2m_n - 1)A(A - 4\sqrt{6}) - 2m_{n+1}(A - 4\sqrt{6}) + (A - 4\sqrt{6}) = 0
\]

On simplification,

\[-2m_{n+2} + 20m_{n+1} - 2m_n - 8 = 0\]

On dividing by\(-2\),

\[m_{n+2} - 10m_{n+1} + m_n + 4 = 0 (8.15)\]

This is the recurrence relation satisfied by the ranks \( m \). It is observed that the values of \( m : (1,5,45), (5,45,441) \) and \( (45,441,4361) \) are satisfied by the Equation(8.15)

**Proof of (ii)**

Let \( t_n = \frac{1}{2\sqrt{6}}\left((5 + 2\sqrt{6})^n(\sqrt{6} + 3) + (5 - 2\sqrt{6})^n(\sqrt{6} - 3)\right)(8.16)\)
This Equation (8.16) can be written as

\[ 2\sqrt{6} t_n = \left( (5 + 2\sqrt{6})^n (\sqrt{6} + 3) + (5 - 2\sqrt{6})^n (\sqrt{6} - 3) \right) \quad (8.17) \]

If we take \( A = 5 + 2\sqrt{6}, \) then \( A - 4\sqrt{6} = 5 - 2\sqrt{6}, \) \( C = \sqrt{6} + 3, \)

then \( C - 6 = \sqrt{6} - 3 \) and the Equation (8.17) becomes

\[ 2\sqrt{6} t_n = \left( (A)^n (C) + (A - 4\sqrt{6})^n (C - 6) \right) \quad (8.18) \]

Replacing \( n \) by \( n + 1, n + 2 \) successively in (8.18), the outcome is

\[ 2\sqrt{6} t_{n+1} = \left( (A)^{n+1} (C) + (A - 4\sqrt{6})^{n+1} (C - 6) \right) \quad (8.19) \]

\[ 2\sqrt{6} t_{n+2} = \left( (A)^{n+2} (C) + (A - 4\sqrt{6})^{n+2} (C - 6) \right) \quad (8.20) \]

Multiplying the equation (8.18) by \( A, \) and then subtracting from the Equation (8.19) to get

\[ At_n - t_{n+1} = 2(A - 4\sqrt{6})^n (C - 6) \quad (8.21) \]

Multiplying the Equation (8.18) by \( A, \) and then subtracting from the Equation (8.20) to get

\[ At_{n+1} - t_{n+2} = 2(A - 4\sqrt{6})^n (C - 6) \quad (8.22) \]

Multiplying the Equation (8.21) by \( A - 4\sqrt{6}, \) and then subtracting from the Equation (8.22) to get

\[ (A - 4\sqrt{6})(At_n - t_{n+1}) - (At_{n+1} - t_{n+2}) = 0 \quad (8.23) \]
On simplifying, the following is obtained

\[ t_{n+2} - 10t_{n+1} + t_n = 0 \quad (8.24) \]

This is the recurrence relation satisfied by the ranks \( t \). It is observed that the values of \( t(1,11,109); (11,109,1079); (109,1079,10681) \) are satisfied by the equation (8.24)