CHAPTER 6

OBSERVATIONS ON THE OCTIC EQUATION WITH SIX UNKNOWNS

This portion of the research attempts to discuss some of the observations on the octic equation with six unknowns. The works done, particularly, by Gopalan & Sangeetha (2011) and Manjusomnath et al. (2012) deal with special equations of sixth degree with four and five unknowns. In heptic equations with three and five unknowns are analyzed. In the forthcoming part of the study is concerned with the problem of determining non-trivial integral solution of the non-homogeneous equation of eighth degree with six unknowns given by

$$4(x^6 - y^6) - (x^2 - y^2)(x^4 + y^4 + 10x^2y^2) = (z^4 - w^4)p^2 - 4T^4p^2(x^2 - y^2).$$

6.1 METHOD OF ANALYSIS

The Diophantine equation representing the non-homogeneous equation of degree eight is given by

$$4(x^6 - y^6) - (x^2 - y^2)(x^4 + y^4 + 10x^2y^2)(z^4 - w^4)p^2 - 4T^4p^2(x^2 - y^2)$$

(6.1)

Introduction of the transformations

$$x = u + v, \, y = u - v, \, z = u + 2v, \, w = u - 2v, \, p = 2u, \, v > 1$$

(6.2)

in (6.1) leads to
\[ u^2 + v^2 = T^4 \]  \hspace{1cm} (6.3)

The above Equation (6.3) is solved through different approaches and thus, one obtains different sets of solutions to (6.1)

**APPROACH I:**

Let \[ T = a^2 + b^2 \]  \hspace{1cm} (6.4)

Substituting (6.4) in (6.3) and using the method of factorisation, define

\[ (u + iv) = (a + ib)^4 \]  \hspace{1cm} (6.5)

Equating real and imaginary parts in (6.5) we get

\[
\begin{align*}
u &= a^4 - 6a^2b^2 + b^4 + 4a^3b - 4ab^3 \\
v &= 4a^3b - 4ab^3
\end{align*}
\]  \hspace{1cm} (6.6)

In view of (6.2), (6.4) and (6.6), the corresponding values of \( x, y, z, w, p, T \) are represented by

\[
\begin{align*}
x(a, b) &= a^4 - 6a^2b^2 + b^4 + 4a^3b - 4ab^3 \\
y(a, b) &= a^4 - 6a^2b^2 + b^4 - 4a^3b + 4ab^3 \\
z(a, b) &= a^4 - 6a^2b^2 + b^4 + 8a^3b - 8ab^3 \\
w(a, b) &= a^4 - 6a^2b^2 + b^4 - 8a^3b + 8ab^3 \\
p(a, b) &= 2a^4 - 12a^2b^2 + 2b^4 \\
T(a, b) &= a^2 + b^2
\end{align*}
\]  \hspace{1cm} (6.7)

The above values of \( x, y, z, w, p, T \) satisfy the following properties:

1) \[ X(a, a) + y(a, a) + z(a, a) + w(a, a) = 2p(a, a) \]

2) \[ X(a, 1) + y(a, 1) + z(a, 1) + w(a, 1) + 4S + 24T_{2, 2} - 4T_{2, 2} \]  \[ a^4 = 8 \]
3) \[ p(1,b) + T(1,b) \equiv 0 \pmod{3} \]

4) \[ z(1,b) - w(1,b) = 8(T_{2,b} - s_0_b) \]

5) \[ X(a,1) - y(a,1) = 4S_{0a} - 4T_{2,a} \]

**APPROACH II:**

Now, rewrite (6.3) as,

\[ u^2 + v^2 = T^4 + 1 \quad (6.8) \]

Also \(1\) can be written as

\[ 1 = (-i)^n (i)^n \quad (6.9) \]

Substituting (6.4) and (6.9) in (6.8) and using the method of factorisation, define,

\[ (u + iv) = (i)^n (a + ib)^4 \quad (6.10) \]

Equating real and imaginary parts in (10) we get

\[ u = \cos \left( \frac{n\pi}{2} \right) (a^4 - 6a^2b^2 + b^4) + \sin \left( \frac{n\pi}{2} \right) (4ab^3 - 4a^3b) \]
\[ v = \cos \left( \frac{n\pi}{2} \right) (4ab^3 - 4a^3b) + \sin \left( \frac{n\pi}{2} \right) (a^4 - 6a^2b^2 + b^4) \quad (6.11) \]

In view of (6.2), (6.4) and (6.11), the corresponding values of \( x, y, z, w, p, T \) are represented
\[ x = \cos\left(\frac{\pi}{2}\right)(a^3 - 6a^2b^2 + b^4 + 4ab^3 - 4a^2b) + \sin\left(\frac{\pi}{2}\right)(a^3 - 6a^2b^2 + b^4 + 4ab^3 - 4a^2b) \]
\[ y = \cos\left(\frac{\pi}{2}\right)(a^3 - 6a^2b^2 + b^4 - 4ab^3 + 4a^2b) + \sin\left(\frac{\pi}{2}\right)(4ab^2 - 4a^2b - a^3 + 6a^2b^2 - b^4) \]
\[ z = \cos\left(\frac{\pi}{2}\right)(a^3 - 6a^2b^2 + b^4 + 8ab^2 - 8a^2b) + \sin\left(\frac{\pi}{2}\right)(4ab^2 - 4a^2b + 2a^4 - 12a^2b^2 + 2b^4) \]
\[ w = \cos\left(\frac{\pi}{2}\right)(a^3 - 6a^2b^2 + b^4 - 8ab^2 + 8a^2b) + \sin\left(\frac{\pi}{2}\right)(4ab^2 - 4a^2b - 2a^4 + 12a^2b^2 - 2b^4) \]
\[ p = \cos\left(\frac{\pi}{2}\right)(2a^3 - 12a^2b^2 + 2b^4) + \sin\left(\frac{\pi}{2}\right)(8ab^3 - 8a^2b) \]
\[ T = a^2 + b^2 \]

**APPROACH III**

Here, 1 can also be written as

\[ 1 = \frac{[(m^2 - n^2) + i2mn][(m^2 - n^2) - i2mn]}{(m^2 + n^2)^2}(6.12) \]

Following the same procedure as above we get the integral solution of (6.1) as

\[ x = (m^2 + n^2)^2 \left\{ (m^2 - n^2)(a^4 - 6a^2b^2 + b^4 + 4a^3b - 4ab^3) + \right\} \]
\[ y = (m^2 + n^2)^2 \left\{ (m^2 - n^2)(a^4 - 6a^2b^2 + b^4 - 4a^3b + 4ab^3) + \right\} \]
\[ z = (m^2 + n^2)^2 \left\{ (m^2 - n^2)(a^4 + 6a^2b^2 + b^4 + 8a^3b - 8ab^3) + \right\} \]
\[ w = (m^2 + n^2)^2 \left\{ (m^2 - n^2)(2a^4 - 12a^2b^2 + 2b^4 - 4a^3b + 4ab^3) + \right\} \]
\[ p = (m^2 + n^2)^2 \left\{ (m^2 - n^2)(2a^4 - 12a^2b^2 + 2b^4) + \right\} \]
\[ T = (m^2 + n^2)^2(a^2 + b^2) \]

**APPROACH IV**

Writing 1 as

\[ 1 = \frac{[(m^2 - n^2) + i2mn][(m^2 - n^2) - i2mn]}{(m^2 + n^2)^2} \]
Following the same procedure as above we get the integral solution of (6.1) as

\[
\begin{align*}
    x &= (m^2 + n^2)^2 \left( \frac{2mn(a^4 - 6a^2b^2 + b^4 + 4a^3b - 4ab^3)}{(m^2 - n^2)(a^4 - 6a^2b^2 + b^4 - 4a^3b + 4ab^3)} \right) \\
    y &= (m^2 + n^2)^2 \left( \frac{2mn(a^4 - 6a^2b^2 + b^4 - 4a^3b + 4ab^3)}{(m^2 - n^2)(-a^4 + 6a^2b^2 - b^4 - 4a^3b + 4ab^3)} \right) \\
    z &= (m^2 + n^2)^2 \left( \frac{2mn(a^4 - 6a^2b^2 + b^4 + 8a^3b - 8ab^3)}{(m^2 - n^2)(2a^4 - 12a^2b^2 + 2b^4 - 4a^3b + 4ab^3)} \right) \\
    w &= (m^2 + n^2)^2 \left( \frac{2mn(a^4 - 6a^2b^2 + b^4 - 8a^3b + 8ab^3)}{(m^2 - n^2)(-2a^4 + 12a^2b^2 - 2b^4 - 4a^3b + 4ab^3)} \right) \\
    p &= (m^2 + n^2)^2 \left( \frac{2mn(2a^4 - 12a^2b^2 + 2b^4)}{(m^2 - n^2)(-8a^3b + 8ab^3)} \right) \\
    T &= (m^2 + n^2)^2(a^2 + b^2)
\end{align*}
\]

**APPROACH V**

The solution of (6.3) can also be obtained as

\[
U = (m^2 - n^2)(m^2 + n^2)v = 2mn(m^2 + n^2)
\]

In view of (6.16) and (6.2), the integral solutions of (6.1) is obtained as

\[
\begin{align*}
    x &= (m^2 + n^2)(m^2 - n^2 + 2mn) \\
    y &= (m^2 + n^2)(m^2 - n^2 - 2mn) \\
    z &= (m^2 + n^2)(m^2 - n^2 + 4mn) \\
    w &= (m^2 + n^2)(m^2 - n^2 - 4mn) \\
    p &= 2(m^2 + n^2)(m^2 - n^2) \\
    T &= (m^2 + n^2)
\end{align*}
\]

The above values of \(x, y, z, w, p\) and \(T\) satisfies the following properties:

1) \(x(a, a) + y(a, a) + z(a, a) + w(a, a) + 3T(a, a)\) is nasty number
2) \( x(a,1) - y(a,1) + z(a,1) - w(a,1) - 2P_Ra^4 - 24T_{4,a} - 12P_Ra \\
= 03)p(a,1) + T(a,1) = 130A_0 + 2T_{2,a} - 1 \)

3) \( x(a, a) + y(a, a) + z(a, a) + w(a, a) + p(a, a) = 0 \)

4) \( x(1, 2^n) + y(1, 2^n) + z(1, 2^n) + w(1, 2^n) \\
= 4(ky_n + 2 - 2^{n+1})(j_n + 1 - (-1)^n)(1 - 3j_n - (-1)^n) \)

**APPROACH VI**

Writing 1 as

\[
1 = \frac{(3+4i)(3-4i)}{\sqrt{2}}
\] (6.14)

In view of (6.14) and (6.2), the integral solutions of (6.1) is obtained as

\[
\begin{align*}
x(A,B) &= 125(7A^4 - 42A^2B^2 + 7B^4 - 4A^3B + 4AB^3) \\
y(A,B) &= 125(-A^4 + 6A^2B^2 - 2B^4 - 28A^3B + 28AB^3) \\
z(A,B) &= 125(11A^4 - 66A^2B^2 + 11B^4 + 8A^3B - 8AB^3) \\
w(A,B) &= 125(-5A^4 + 30A^2B^2 - 5B^4 - 40A^3B + 40AB^3) \\
p(A,B) &= 125(6A^4 - 36A^2B^2 + 6B^4 - 32A^3B + 32AB^3) \\
T(A,B) &= 25(A^2 + B^2)
\end{align*}
\] 6.15