In this chapter, the notion of the some new classes of functions like:

- Totally na-Feebly regular continuous function and some of its related functions
- Feebly regular door symmetrical space
- Discussion of the new mapping biclop.na-continuous function with similar type of functions

**Definition 5.1:**

The topological space $X$ is a F.reg.door space if and only if every subset of $X$ is either F.reg.open or F.reg.closed.

**Definition 5.2:**

A F.reg.door space $(X, \tau)$ is said to be feebly regular door symmetrical space (briefly F.reg.door symmetrical space) if for $x$ and $y$ in $X$,

$$x \in \text{F.reg.cl}\{y\} \Rightarrow y \in \text{F.reg.cl}\{x\}.$$
Theorem 5.3:

Let \((X, \tau)\) be a F.reg.door symmetrical space and \(Y\) is a F.reg.door space of \(X\). Then \(Y\) is F.reg.door space with respective topology \(\tau\) in \(Y\).

Proof:

Let \(S \subseteq Y\). Then \(S \subseteq X\).

So \(S\) is either F.reg.open in \(X\) or F.reg.closed in \(X\).

Hence \(S \cap Y\) is either F.reg.open in \(Y\) or F.reg.closed in \(Y\). But \(S \cap Y = S\).

Then \(S\) is either F.reg.open in \(Y\) or F.reg.closed in \(Y\).

Thus \((Y, \tau_Y)\) is a F.reg.door symmetrical space.

Theorem 5.4:

The property of being a F.reg.door symmetrical space is a topological property.

Proof:

Let \(X\) be a F.reg.door symmetrical space and let a function \(f\) from the topological space \(X\) to \(Y\) be an homeomorphism.

Let \(S \subseteq Y\), consider \(f^{-1}(S) \subseteq X\), since \(X\) is a F.reg.door symmetrical space.

Then \(f^{-1}(S)\) is either F.reg.open or F.reg.closed in \(X\).

Now \(f(f^{-1}(S)) = S\).

Then \(S\) is either F.reg.open or F.reg.closed in \(Y\). Thus \(Y\) is F.reg.door symmetrical space.

Theorem 5.5:

Let \((X, \tau)\) be a F.reg.door symmetrical space, let \(Y \subseteq X\) be a F.reg.clopen subset of \(X\) then \((Y, \tau_Y)\) is also a F.reg.door symmetrical space.
**Proof:**

Let \( M \subseteq Y \) be a subset of \( Y \). Now \( M \subseteq X \). But \( X \) is a F.reg.door symmetrical space.

Then \( M \) is either F.reg.open or F.reg.closed in \( X \).

Since \( Y \) is either F.reg.open and F.reg.closed, \( M \) is either F.reg.open or F.reg.closed in \( Y \).

Then \( Y \) is F.reg.door symmetrical space.

**Definition 5.6:**

A function \( f \) from the F.reg.door symmetrical topological space \( X \) to \( Y \) is said to be totally na-F.reg.continuous if the inverse image of every F.reg.open set in \( Y \) is \( \delta \)-clopen in \( X \).

**Theorem 5.7:**

A function \( f : X \rightarrow Y \) is a totally na-F.reg.continuous function if and only if the inverse image of every F.reg.closed subset of \( Y \) is \( \delta \)-clopen in \( X \).

**Proof:**

Let \( F \) be any F.reg.closed in \( Y \).

Then \( Y-F \) is F.reg.open set in \( Y \).

By definition 5.6 \( f^{-1}(Y-F) \) is \( \delta \)-clopen in \( X \).

That is \( X-f^{-1}(F) \) is \( \delta \)-clopen in \( X \), this implies \( f^{-1}(F) \) is \( \delta \)-clopen in \( X \).

On the other hand, if \( V \) is F.reg.open in \( Y \), then \( Y-V \) is F.reg.closed in \( Y \), by hypothesis, \( f^{-1}(Y-V) = X-f^{-1}(V) \) is \( \delta \)-clopen in \( X \), which implies \( f^{-1}(V) \) is \( \delta \)-clopen in \( X \).
Thus inverse image of every F.reg.open set in Y is δ-clopen in X. Therefore f is totally na-F.reg.continuous function.

Remark 5.8:
(i) Every F.reg.open set is feebly open.
(ii) δ-clopen set is δ-open and δ-closed.
(iii) δ-clopen = regular clopen.

Theorem 5.9:
Every totally na-F.reg.continuous is a na-continuous.

Proof:
Let X and Y be topological space.
Suppose f : X → Y is totally na-F.reg.continuous and U is any F.reg.open subset of Y.
The function f : X → Y is totally na-F.reg.continuous, it follows f⁻¹(U) is δ-clopen in X, by remark 5.8, hence f⁻¹(U) is δ-open in X.

Thus inverse image of every F.reg.open set in Y is δ-open in X. Therefore the function f is na-continuous.

Definition 5.10:
A function f from X to Y is said to be strongly totally na-F.reg.continuous if the inverse image of every F.reg.open set in Y is regular-clopen in X.

Theorem 5.11:
Every strongly totally na-F.reg.continuous function is totally na-F.reg.continuous and vice versa.
Proof:

Suppose a function $f$ from $X$ to $Y$ is strongly totally na-F.reg.continuous and let $S$ be any F.reg.open set in $Y$, by definition 5.10, $f^{-1}(S)$ is regular-clopen in $X$. Thus the inverse image of each F.reg.open set in $Y$ is $\delta$-clopen in $X$.

Therefore $f$ is totally na-F.reg.continuous.

Theorem 5.12:

A function $f$ from $X$ to $Y$ is totally na-F.reg.continuous if and only if for each $p \in X$ and each F.reg.open set $O$ in $Y$ with $f(p) \in O$, there is a $\delta$-clopen set $E$ in $X$ such that $p \in E$ and $f(E) \subseteq O$.

Proof:

Suppose a function $f$ from $X$ to $Y$ is a totally na-F.reg.continuous function where $X$ and $Y$ are topological spaces and F.reg.door symmetrical space. Let $O$ be any F.reg.open set in $Y$ containing $f(p)$ so that $p \in f^{-1}(O)$. Since $f$ is totally na-F.reg.continuous, $f^{-1}(O)$ is $\delta$-clopen in $X$.

Let $E = f^{-1}(O)$. Then $E$ is $\delta$-clopen set in $X$ and $p \in E$.

Also $f(E) = f(f^{-1}(O)) \subseteq O$, this implies $f(E) \subseteq O$.

On the other hand let $O$ be F.reg.open in $Y$, let $p \in f^{-1}(O)$ be arbitrary, this implies $f(p) \in O$, therefore by theorem 5.11, there is a $\delta$-clopen set $f(M_p) \subseteq X$ containing $p$ such that $f(M_p) \subseteq O$, which implies $M_p \subseteq f^{-1}(O)$.

We have $p \in M_p \subseteq f^{-1}(O)$, implies $f^{-1}(O)$ is $\delta$-clopen neighbourhood of $p$, since $p$ is arbitrary, it implies $f^{-1}(O)$ is $\delta$-clopen neighbourhood of its points, hence it is $\delta$-clopen set in $X$.

Therefore $f$ is totally na-F.reg.continuous.
Remark 5.13:

\((X, \tau)\) is F.reg.\(T_1\) space if and only if singleton sets are F.reg.closed sets.

Theorem 5.14:

Every totally na-F.reg.continuous function into a F.reg. \(T_1\) space is strongly totally na-F.reg.continuous function.

Proof:

Suppose a function \(f\) from \(X\) to \(Y\) is a totally na-F.reg.continuous function in a F.reg.\(T_1\) space. Singletons are F.reg.closed sets by remark 5.13. Hence \(f^{-1}(B)\) is \(\delta\)-clopen in \(X\) for every subset \(B\) of \(Y\).

By remark 5.8, \(f^{-1}(B)\) is regular clopen in \(X\).

Therefore \(f\) is strongly totally na-F.reg.continuous function.

Theorem 5.15:

A function \(f : X \to Y\) is totally na-F.reg.continuous and \(P\) is \(\delta\)-clopen subset of \(X\), then the restriction \(f\mid P : X \to Y\) is totally na-F.reg.continuous.

Proof:

Consider the function \(f\mid P : P \to Y\) and \(O\) be any F.reg.open set in \(Y\). Since \(f\) is totally na-F.reg.continuous, \(f^{-1}(O)\) is \(\delta\)-clopen subset of \(X\).

Since \(P\) is \(\delta\)-clopen subset of \(X\) and \((f\mid P)^{-1}(O) = P \cap f^{-1}(O)\) is \(\delta\)-clopen in \(P\), it follows \((f\mid P)^{-1}(O)\) is \(\delta\)-clopen in \(P\). Hence \(f\mid P\) is totally na-F.reg.continuous.

Definition 5.16:

A function \(f\) from \(X\) to \(Y\) is said to be F.reg.irresolute if the inverse image of every F.reg.open set in \(Y\) is F.reg.open in \(X\).
Theorem 5.17:

If the functions $f$ and $g$ from $X$ to $Y$ and from $Y$ to $Z$ are totally na-F.reg.continuous and F.reg.irresolute respectively then the function $g \circ f$ from $X$ to $Z$ is totally na-F.reg.continuous.

Proof:

Let the functions $f$ and $g$ from $X$ to $Y$ and from $Y$ to $Z$ are totally na-F.reg.continuous and F.reg.irresolute respectively.

Let $O$ be F.reg.open in $Z$.

Since $g$ is F.reg.irresolute, $g^{-1}(O)$ is F.reg.open in $Y$.

Now since $f$ is totally na-F.reg.continuous, $f^{-1}(g^{-1}(O)) = (g \circ f)^{-1}(O)$ is $\delta$-clopen in $X$. Hence $g \circ f : X \to Z$ is totally na-F.reg.continuous.

Definition 5.18:

A function $f$ from $X$ to $Y$ is said to be contra F.reg.irresolute if the inverse image of every F.reg.open set in $Y$ is F.reg.closed in $X$.

Definition 5.19:

A function $f$ from $X$ to $Y$ is said to be almost F.reg.irresolute if $f^{-1}(M)$ is the feebly open set in $X$ for every F.reg.open set $M$ in $Y$.

Definition 5.20:

A function $f : X \to Y$ is said to be contra almost F.reg.irresolute if the inverse image F.reg.open in $Y$ is feebly closed in $X$.

Theorem 5.21:

Every contra F.reg.irresolute function is contra almost F.reg.irresolute function.
Proof:

Suppose \( f : X \rightarrow Y \) is a contra F.reg.irresolute function and \( A \) be any F.reg.open set in \( Y \), by remark 5.8 \( f^{-1}(A) \) is F.reg.closed in \( X \).

Thus the inverse image of each F.reg.open set in \( Y \) is feebly closed in \( X \). Therefore \( f \) is contra almost F.reg. irresolute function.

Definition 5.22:

A space \( X \) is said to be weakly feebly Hausdorff if each elements of \( X \) is an intersection of F.reg.closed sets of \( X \).

Theorem 5.23:

The followings are equivalent for the function \( f \) from \( X \) to \( Y \):

1. \( f \) is contra almost F.reg.irresolute
2. for every F.reg.closed set \( E \) of \( Y \), \( f^{-1}(E) \) is feebly open set of \( X \).
3. for each \( p \in X \) and each F.reg.closed set \( E \) of \( Y \) containing \( f(p) \), there exists feebly open set \( M \) containing \( p \) such that \( f(M) \subset F \).
4. for each \( p \in X \) and each F.reg.open set \( J \) of \( Y \) not containing \( f(p) \), there exists feebly closed set \( K \) not containing \( x \) such that \( f^{-1}(J) \subset K \).

Proof:

(1) \( \Rightarrow \) (2): Let \( E \) be a F.reg.closed set in \( Y \), then \( Y-E \) is a F.reg.open set in \( Y \).

By (1), \( f^{-1}(Y-E) = X - f^{-1}(E) \) is feebly closed set in \( X \). This implies \( f^{-1}(E) \) is feebly open set in \( X \).

(2)\( \Rightarrow \) (1) : Let \( P \) be a F.reg.open set of \( Y \), then \( Y-P \) is a F.reg.closed set in \( Y \).

By (2), \( f^{-1}(Y-P) \) is feebly open set in \( X \).
This implies X-$f^1(P)$ is feebly open set in X, which implies $f^1(P)$ is feebly closed set in X. Therefore, (1) holds.

(2) $\Rightarrow$ (3): Let E be a F.reg.closed set in Y containing f(p), which implies $p \in f^1(E)$.

By (2), $f^1(E)$ is feebly open in X containing p.

Set $M = f^1(E)$, which implies M is feebly open in X containing p and $f(M) = f(f^1(E)) \subseteq E$.

Therefore (3) holds.

(3) $\Rightarrow$ (2) : Let E be a F.reg.closed set in Y containing f(p), which implies $p \in f^1(E)$.

From (3), there exists feebly open $M_p$ in X containing p such that $f(M_p) \subseteq E$. That is $M_p \subseteq f^1(E)$.

Thus $f^1(E) = \bigcup \{M_p : p \in f^1(E)\}$, which is the union of feebly open sets. Therefore, $f^1(E)$ is feebly open set of X.

(3) $\Rightarrow$ (4) : Let J be a F.reg.open set in Y not containing f(p). Then Y-J is a F.reg.closed set in Y containing f(p).

From (3), there exists a feebly open set M in X containing p such that $f(M) \subseteq Y-J$.

This implies $M \subseteq f^1(Y-J) = X-f^1(J)$.

(4) $\Rightarrow$ (3) : Let E be a F.reg.closed set in Y containing f(p).

Then Y-E is a F.reg.open set in Y not containing f(p).

From (4), there exists feebly closed set K in X not containing p such that $f^1(Y-E) \subseteq K$. This implies X-$f^1(E) \subseteq K$. 

83
Hence, $X-K \subseteq f^{-1}(E)$, that is $f(X-K) \subseteq E$.

Set $M = X - K$, then $M$ is feebly open set containing $p$ in $X$ such that $f(M) \subseteq E$.

**Theorem 5.24:**

The following are equivalent for the function $f : X \to Y$:

1. $f$ is contra almost $F$-regular irresolute continuous.
2. $f^{-1}(F\text{-}\text{reg}\text{-}\text{int}(F\text{-}\text{reg}\text{-}\text{cl}(M)))$ is feebly closed set in $X$ for every $F\text{-}\text{reg}\text{-}\text{open}$ subset $M$ of $Y$.
3. $f^{-1}(F\text{-}\text{reg}\text{-}\text{cl}(F\text{-}\text{reg}\text{-}\text{int}(E)))$ is feebly open set in $X$ for every $F\text{-}\text{reg}\text{-}\text{closed}$ subset $E$ of $Y$.

**Proof:**

(1) $\Rightarrow$ (2): Let $M$ be an $F\text{-}\text{reg}\text{-}\text{open}$ set in $Y$.

Then $F\text{-}\text{reg}\text{-}\text{int}(F\text{-}\text{reg}\text{-}\text{cl}(M))$ is $F\text{-}\text{reg}\text{-}\text{open}$ set in $Y$.

By (1), $f^{-1}(F\text{-}\text{reg}\text{-}\text{int}(F\text{-}\text{reg}\text{-}\text{cl}(M))) \subseteq$ Feebly closed set of $X$. (2) $\Rightarrow$ (1): Obvious.

(1) $\Rightarrow$ (3): Let $E$ be a $F\text{-}\text{reg}\text{-}\text{closed}$ in $Y$.

Then $F\text{-}\text{reg}\text{-}\text{cl}(F\text{-}\text{reg}\text{-}\text{int}(M))$ is $F\text{-}\text{reg}\text{-}\text{closed}$ set in $Y$.

By (1), $f^{-1}(F\text{-}\text{reg}\text{-}\text{cl}(F\text{-}\text{reg}\text{-}\text{int}(M))) \subseteq$ Feebly open set of $X$. (3) $\Rightarrow$ (1): Obvious.

**Theorem 5.25:**

If a function $f$ from $X$ to $Y$ is a contra almost $F\text{-}\text{reg}$ irresolute injection and $Y$ is weakly feebly Hausdorff then $X$ is $F\text{-}\text{reg}\text{-}T_1$.

**Proof:**

Suppose $Y$ is weakly feebly Hausdorff.
For any distinct points \( p \) and \( q \) in \( X \), there exist F.reg.closed sets \( J \) and \( E \) in \( Y \) such that \( f(p) \in J \), \( f(q) \notin J \), \( f(q) \in E \) and \( f(p) \notin E \).

Since \( f \) is contra almost F.reg.continuous, \( f^{-1}(J) \) and \( f^{-1}(E) \) are F.reg.open subsets of \( X \) such that \( p \notin f^{-1}(J) \), \( q \notin f^{-1}(J) \), \( q \in f^{-1}(E) \) and \( p \notin f^{-1}(E) \).

This shows that \( X \) is F.reg.T1.

**Corollary 5.26:**

If a function \( f \) from \( X \) to \( Y \) is a contra F.reg. irresolute injection and \( Y \) is weakly feebly Hausdorff, then \( X \) is F.reg.T1.

**Definition 5.27:**

A topological space \( X \) is called ultra F.reg.Hausdorff space, if for every pair of disjoint points \( p \) and \( q \) in \( X \), there exist disjoint F.reg.clopen sets \( M \) and \( E \) in \( X \) containing \( p \) and \( q \) respectively.

**Theorem 5.28:**

If \( f : X \to Y \) is a contra almost F.reg. irresolute injective function and \( Y \) is ultra F.reg.Hausdorff space then \( X \) is F.reg.T2.

**Proof:**

Let \( p \) and \( q \) be any two distinct points in \( X \).

Since \( f \) is injective \( f(p) \neq f(q) \) and \( Y \) is ultra F.reg.Hausdorff space, there exist disjoint F.reg.clopen sets \( M \) and \( E \) of \( Y \) containing \( f(p) \) and \( f(q) \), respectively.

Then \( p \in f^{-1}(M) \) and \( q \notin f^{-1}(E) \), where \( f^{-1}(M) \) and \( f^{-1}(E) \) are disjoint feebly open sets in \( X \). Therefore \( X \) is F.reg.T2.
Definition 5.29:

\[ F.\text{reg.Fr}(E) = F.\text{reg.cl}(E) - F.\text{reg.int}(E) \]
where \( E \) is a subset of \( X \).

Theorem 5.30:

The set of all points \( p \) of \( X \) at which \( f : X \rightarrow Y \) is not contra almost \( F.\text{reg.} \) irresolute is identical with the union of \( F.\text{reg.Fr} \) of the inverse images of \( F.\text{reg.closed} \) sets of \( Y \) containing \( f(p) \).

Proof:

Assume that \( f \) is not contra almost \( F.\text{reg.} \) irresolute at \( p \in X \).

Then there exists \( E \in F.\text{reg.closed} \) \( (Y,f(p)) \) such that \( f(M) \cap (Y-E) \neq \emptyset \) for every \( M \in \text{feebly open} \) \( (X,p) \).

This implies \( M \cap f^{-1}(Y-E) \neq \emptyset \) for every \( M \in \text{feebly open} \) \( (X,p) \).

Therefore, \( p \in F.\text{reg.cl}(f^{-1}(Y-E)) = F.\text{reg.cl}(X-f^{-1}(E)) \) and also

\[ p \subseteq f^{-1}(E) \subseteq F.\text{reg.cl}(f^{-1}(E)). \]

Thus \( p \in F.\text{reg.cl}(f^{-1}(E)) \cap F.\text{reg.cl}(X-f^{-1}(E)). \)

This implies, \( p \in F.\text{reg.cl}(f^{-1}(E)) - F.\text{reg.int}(f^{-1}(E)). \) Therefore, \( p \in F.\text{reg.Fr}(f^{-1}(E)) \).

Conversely, suppose \( p \in F.\text{reg.Fr}(f^{-1}(E)) \) for some \( E \subseteq F.\text{reg.closed}(Y,f(p)) \) and \( f \) is a contra almost \( F.\text{reg.} \) irresolute continuous at \( p \in X \).

Then there exists \( M \in \text{feebly open} \) \( (X,p) \) such that \( f(M) \subseteq E \).

Therefore, \( p \in M \subseteq f^{-1}(E) \) and hence \( p \in F.\text{reg.int}(f^{-1}(E)) \subseteq X - F.\text{reg.Fr}(f^{-1}(E)) \).

This contradicts that \( p \in F.\text{reg.Fr}(f^{-1}(E)) \).

Therefore \( f \) is not contra almost \( F.\text{reg.} \) irresolute.
Theorem 5.31:

If \( f : X \to Y \) is a contra almost F.reg.irresolute surjection and \( X \) is F.reg.connected space, then \( Y \) is connected.

Proof:

Let \( f : X \to Y \) be a contra almost F.reg.irresolute surjective and \( X \) is F.reg.connected space.

Suppose \( Y \) is a not connected space.

Then there exist disjoint F.reg.open sets \( E \) and \( M \) such that \( Y = E \cup M \).

Therefore \( E \) and \( M \) are F.reg.clopen in \( Y \).

Since \( f \) is contra almost F.reg.irresolute, \( f^{-1}(E) \) and \( f^{-1}(M) \) are F.reg.open sets in \( X \). Moreover \( f^{-1}(E) \) and \( f^{-1}(M) \) are non empty disjoint and

\[
X = f^{-1}(E) \cup f^{-1}(M).
\]

This is a contradiction to the fact that \( X \) is F.reg.connected space. Therefore, \( Y \) is connected.

Definition 5.32:

A topological space \( X \) is said to be ultra-F.reg.connected if every two non empty F.reg.closed subsets of \( X \) intersect.

Definition 5.33:

A subset \( Y \) of topological space \( X \) is F.reg.dense in \( X \) if F.reg.cl(\( Y \)) = \( X \).

Definition 5.34:

A topological space \( X \) is said to be hyper-F.reg.connected if every F.reg.open set is F.reg.dense.
Theorem 5.35:

If $X$ is ultra-F.reg.connected and $f : X \rightarrow Y$ is a contra almost F.reg.irresolute surjection, then $Y$ is hyper-F.reg.connected.

Proof:

Let $X$ be an ultra-F.reg.connected and $f : X \rightarrow Y$ is contra almost F.reg.irresolute continuous surjection.

Suppose $Y$ is not hyper-F.reg.connected.

Then there exists a F.reg.open sets $O$, such that $O$ is not F.reg.dense in $Y$. Therefore, there exist non empty F.reg.open subsets $B_1 = \text{F.reg.int}(\text{F.reg.cl}(O))$ and $B_2 = Y - \text{F.reg.cl}(O)$ in $Y$.

Since $f$ is contra almost F.reg. irresolute surjection, $f^{-1}(B_1)$ and $f^{-1}(B_2)$ are disjoint F.reg.closed sets in $X$.

This is contrary to the fact that $X$ is ultra-F.reg.connected. Therefore, $Y$ is hyper-F.reg.connected.

Theorem 5.36:

Let $(X,\tau)$ be F.reg.connected and $(Y,\sigma)$ be F.reg.$T_1$. If $f : (X,\tau) \rightarrow (Y,\sigma)$ is contra almost F.reg.irresolute, then $f$ is constant.

Proof:

We assume that $Y$ is non empty.

Since $Y$ is a F.reg.$T_1$-space, we have $O = \{f^{-1}(p) : p \in Y\}$ is a disjoint F.reg.open partition of $X$. 

88
If \(|O| \geq 2\), then there exists a proper non empty set \(W\) for some \(q \in U\). Since \(X\) is connected, we have \(|q| = 1\).

Hence, \(f\) is constant.

**Theorem 5.37:**

Contra almost F.reg. irresolute images of hyper – F.reg.connected spaces are F.reg.connected.

**Proof:**

Let \(f : (X,\tau) \rightarrow (Y,\sigma)\) be contra almost F.reg.irresolute and \(X\) is hyper- F.reg.connected, that is every F.reg.open subset of \(X\) is F.reg.dense.

Assume that \(M\) is a proper F.reg.clopen subset of \(Y\).

Then \(O = f^{-1}(M)\) is both feebly open and feebly closed as \(f\) is contra almost F.reg.irresolute.

This shows that \(O\) is feebly regular.

Hence, F.reg.int\((O)\) and F.reg.int\((X\setminus O)\) are disjoint nonempty F.reg.open subset of \(X\). This contradicts the fact that \(X\) is hyper-F.reg.connected. Thus, \(Y\) is F.reg.connected.

**Remark 5.38:**

A subset \(A\) of topological space \(X\) is said to be \(\delta\)-open (briefly \(\delta O\)) if it is the intersection of two regular open sets.

Thus \(\delta\)-feebly open (briefly \(\delta FO\)) is the intersection of two F.reg.open. And also we get the results,

\(\delta O = RO\) (ii) \(F\delta O = F\text{.reg.open}\) (iii) \(\delta FO = F\delta O = F\text{.reg.open}\).
Definition 5.39:

A function \( f : X \to Y \) is called weakly feebly na-continuous if for each \( p \in X \) and each feebly open set \( M \) of \( Y \) containing \( f(p) \), there exists \( E \in \delta FO(X, p) \) such that \( f(E) \subseteq F.reg.cl(M) \).

Theorem 5.40:

If a function \( f : X \to Y \) is F.reg. irresolute then \( f \) is weakly feebly na-continuous function.

Proof:

Let \( p \in X \) and \( M \) be F.reg.open set in \( Y \) containing \( f(p) \). Then F.reg.cl(M) is F.reg.closed in \( Y \) containing \( f(p) \). Since \( f \) is F.reg. irresolute by theorem 5.23 (2), \( f^{-1}(F.reg.(M)) \) is F.reg.open set in \( X \) containing \( x \).

Set \( E = f^{-1}(F.reg.cl(M)) \).

Then \( f(E) \subseteq f(f^{-1}(F.reg.cl(M))) \subseteq F.reg.cl(M) \), by remark 5.8, 5.38. Thus \( f \) is weakly feebly na-continuous function.

Definition 5.41:

A function \( f : X \to Y \) is said to be almost F.reg.na-continuous if \( f^{-1}(M) \) is \( \delta \)-open in \( X \) for each F.reg.open set \( M \) of \( Y \).

Definition 5.42:

A function \( f : X \to Y \) is said to be contra almost F.reg.na-continuous if \( f^{-1}(M) \) is \( \delta \)-closed in \( X \) for each F.reg.open set \( M \) of \( Y \).
**Definition 5.43:**

A space $X$ is F.reg.-extremally disconnected (briefly F.reg.-ED) if the F.reg.closure of every F.reg.open set is F.reg.open in $X$.

**Theorem 5.44:**

Let $Y$ be F.reg.- extremely disconnected (briefly F.reg. E.D). Then a function $f : X \rightarrow Y$ is contra almost F.reg.irresolute then it is almost F.reg.irresolute.

**Proof:**

Let $p \in X$ and $M$ be any F.reg.open set of $Y$ containing $f(p)$. Since $Y$ is F.reg.-ED then $M$ is F.reg.clopen and hence $M$ is F.reg.closed set of $Y$ containing $f(p)$.

Since $f$ is contra almost F.reg.irresolute, by theorem 5.23, there exists feebly open set $E$ in $X$ containing $x$ such that $f(E) \subseteq M$.

Then $f$ is almost F.reg.irresolute.

**Definition 5.45:**

A function $f : X \rightarrow Y$ is said to be biclop.na-continuous if the inverse image of every feebly clopen set $O$ of $Y$ is $\delta$–clopen in $X$.

**Example:**

Let $X = \{l,m,n\}$ be a topological space with topology $\tau = \{\emptyset, X, \{m\}, \{l,n\}\}$

let $Y = \{p,q,r\}$ with topology $\sigma = \{\emptyset, Y, \{q\}, \{p,r\}\}$.

Clearly $\tau$ is a $\delta$-clopen set in $X$ and $\sigma$ is a feebly clopen set in $Y$.

Define $f : X \rightarrow Y$ such that $f(l) = r$, $f(m) = q$, $f(n) = p$.

Clearly the inverse image of every feebly clopen set in $Y$ is $\delta$ - clopen set in $X$.
Result:

(i)  $f$ is an open map and a closed map then $f$ is clopen map. (ii) $f$ is continuous.

(iii) From (i) and (ii) we get $f$ is biclop.na-continuous and $f$ is homeomorphism.

Definition 5.46:

A subset $E$ of $(X, \tau)$ is said to be somewhat nearly open if $\text{int}(\text{cl}(E)) = \emptyset$ and $\text{cl}(\text{int}(E)) = \emptyset$ this implies that $\text{cl}(\text{int}(\text{cl}(E))) = \emptyset$.

Definition 5.47:

A function $f$ from the topological spaces $X$ and $Y$ is somewhat nearly biclop.na-continuous if $f^{-1}(E)$ is somewhat nearly clopen for every feebly clopen set $E$ in $Y$ such that $f^{-1}(E) = \emptyset$.

Remark 5.48:

A set $E$ is somewhat nearly clopen set if and only if $E$ is somewhat nearly open and somewhat nearly closed.

Definition 5.49:

A function $f$ from the topological spaces $X$ and $Y$ is slightly biclop.na-continuous if for each $p \in X$ and each feebly clopen set $M$ of $Y$ containing $f(p)$ there exist a $\delta$-open set $E$ containing $p$ such that $f(E) \subseteq M$.

Definition 5.50:

A function $f$ from the topological spaces $X$ and $Y$ is almost biclop.na-continuous if for each $p \in X$ and each feebly clopen set $M$ of $Y$ containing $f(p)$ there exist a $\delta$-clopen set $E$ containing $x$ such that $f(E) \subseteq \text{int}(\text{cl}(M))$. 
Definition 5.51:
A function $f$ from the topological spaces $X$ and $Y$ is $\delta$-clopen irresolute if the inverse image of every $\delta$-clopen subset in $Y$ is $\delta$-clopen in $X$.

Theorem 5.52:
Every biclop-na-continuous function is na-continuous.

Proof:
Let $f : X \to Y$ be biclop-na-continuous function. To show that $f$ is na-continuous, let $E$ be any feebly clopen subset of $Y$.

Since $f$ is biclop-na-continuous, then $f^{-1}(E)$ is $\delta$-clopen in $X$. $f^{-1}(E)$ is $\delta$-open in $X$.

Hence $f$ is na-continuous map.

Theorem 5.53:
For the function $f$ from the topological space $X$ into $Y$, the following statements are equivalent.

(a) $f$ is biclop-na-continuous

(b) $f : X \to Y$ is biclop-na-continuous if for every feebly clopen set $M$ of $Y$ containing $f(p)$ there exist $\delta$-clopen set $E$ containing $p$ such that $f(E) \subseteq M$.

Proof:

(a) $\implies$ (b) : Let $p \in X$ and let $M$ be a feebly clopen set in $Y$ containing $f(p)$.

Then, by (b), $f^{-1}(M)$ is $\delta$-clopen in $X$ containing $p$.

Let $E = f^{-1}(M)$. Then, $f(E) \subseteq M$. 

93
(b)$\Rightarrow$(a) : Let $M$ be a feebly clopen set of $Y$, and let $p \in f^{-1}(M)$. Since $f(p) \subseteq M$, there exists $E$ containing $p$ such that $f(E) \subseteq M$. It then follows that $p \in E \subseteq f^{-1}(M)$. Hence $f^{-1}(M)$ is $\delta$-clopen.

**Theorem 5.54:**

If a function $f$ from the topological spaces $X$ and $Y$ is biclopa-continuous, then it is slightly biclopa-continuous.

**Proof:**

Let $p \in X$, and let $M$ is feebly clopen. Since $f$ is biclopa-continuous, there exist $\delta$-clopen set $E$ containing $x$ such that $f(E) \subseteq M$.

Since $\delta$-clopen set is $\delta$-open, we have $E$ is $\delta$-open and so $f$ is slightly biclopa-continuous. Thus $f$ is slightly biclopa-continuous.

**Theorem 5.55:**

If $f$ is a function from the topological space $X$ into $Y$ and $X = M_1 \cup M_2$ where $M_1$ and $M_2$ and $\delta$-copen set and $f/M_1$ and $f/M_2$ are biclopa-continuous, then $f$ is biclopa-continuous.

**Proof:**

Let $E$ be a feebly clopen subset of $Y$.

Then, since $(f/M_1)$ and $(f/M_2)$ are both biclopa-continuous, therefore $(f/M_1)^{-1}(E)$ and $(f/M_2)^{-1}(E)$ are both $\delta$-clopen set in $M_1$ and $M_2$ respectively. Since $M_1$ and $M_2$ are $\delta$-clopen subsets of $X$, therefore $(f/M_1)^{-1}(E)$ and $(f/M_2)^{-1}(E)$ are both $\delta$-clopen subsets of $X$.

Also, $f^{-1}(E) = (f/M_1)^{-1}(E) \cup (f/M_2)^{-1}(E)$. 

94
Thus $f^{-1}(E)$ is the union of two $\delta$-clopen sets and is therefore $\delta$-clopen.

Hence $f$ is biclop.na-continuous.

**Theorem 5.56:**

If $f$ is a function from the topological space $X$ into $Y$ and $X = M_1 \cup M_2$ and if $(f/M_1)$ and $(f/M_2)$ are both biclop.na-continuous at a point $p$ belongs to $M_1 \cap M_2$, then $f$ is biclop.na-continuous at $p$.

**Proof:**

Let $E$ be any feebly clopen set containing $f(p)$.

Since $p \in M_1 \cap M_2$ and $(f/M_1)$, $(f/M_2)$ are both biclop.na-continuous at $p$, therefore there exist $\delta$-clopen sets $T_1$ and $T_2$ such that $p \in M_1 \cap T_1$ and $f(M_1 \cap V_1) \subset E$, and $p \in (M_2 \cap T_2)$ and $f(M_2 \cap T_2) \subset E$.

Now since $X = M_1 \cup M_2$, therefore $f(T_1 \cap T_2) = f(M_1 \cap T_1 \cap T_2) \cup f(M_2 \cap T_1 \cap T_2) \subset f(M_1 \cap T_1) \cup f(M_2 \cap T_2) \subset E$.

Thus, $T_1 \cap T_2 = T$ is a $\delta$-clopen set containing $p$ such that $f(T) \subset E$ and hence $f$ is biclop.na-continuous at $p$.

**Theorem 5.57:**

Every restriction of a biclop.na-continuous function is biclop.na-continuous.

**Proof:**

Let $f$ be a biclop.na-continuous function of $X$ into $Y$ and let $E$ be any subset of $X$.

For any feebly clopen subset $J$ of $Y$, $(f/E)^{-1}(M) = E \cap f^{-1}(M)$.

95
But \( f \) being biclop.na-continuous, \( f^{-1}(J) \) is \( \delta \)-clopen and hence \( E \cap f^{-1}(M) \) is a relatively \( \delta \)-clopen subset of \( E \), that is \( (f/E)^{-1}(M) \) is a \( \delta \)-clopen subset of \( E \).

Hence \( f/E \) is biclop.na-continuous.

**Theorem 5.58:**

Let \( f \) be a function from the topological space \( X \) into \( Y \) and let \( p \) be a point of \( X \). If there exist a \( \delta \)-clopen set \( M \) of \( p \) such that the restriction of \( f \) to \( M \) is biclop.na-continuous at \( p \), then \( f \) is biclop.na-continuous at \( p \).

**Proof:**

Let \( E \) be any feebly clopen set containing \( f(p) \).

Since \( f/M \) is biclop.na-continuous at \( p \), therefore there is an \( \delta \)-clopen set \( J_1 \) such that \( p \in M \cap J_1 \) and \( f(M \cap J_1) \subset E \). Thus \( M \cap J_1 \) is \( \delta \)-clopen set of \( p \).

**Theorem 5.59:**

Let \( X = J_1 \cup J_2 \), where \( J_1 \) and \( J_2 \) are \( \delta \)-clopen sets in \( X \).

Let the functions \( f \) and \( g \) from \( J_1 \) to \( Y \) and from \( J_2 \) to \( Y \) be biclop.na-continuous. If \( f(p) = g(p) \) for each \( p \in J_1 \cap J_2 \).

Then the function \( h \) from \( J_1 \cap J_2 \) to \( Y \) such that \( h(p) = f(p) \) for \( p \in J_1 \) and \( h(p) = g(p) \) for \( p \in J_2 \) is biclop.na-continuous.

**Proof:**

Let \( E \) be a feebly clopen set of \( Y \). Now \( h^{-1}(E) = f^{-1}(E) \cup g^{-1}(E) \).

Since \( f \) and \( g \) are biclop.na-continuous, \( f^{-1}(E) \) and \( g^{-1}(E) \) are \( \delta \)-clopen set in \( J_1 \) and \( J_2 \) respectively. But \( J_1 \) and \( J_2 \) are both \( \delta \)-clopen sets in \( X \).
Since union of two \( \delta \)-clopen sets is \( \delta \)-clopen, so \( h^{-1}(E) \) is a \( \delta \)-clopen set in \( X \).

Hence \( h \) is biclop.na-continuous.

**Theorem 5.60:**

Let a function \( f \) from the topological space \( X \) into \( Y \) be biclop.na-continuous surjection and \( E \) be \( \delta \)-clopen subset of \( X \). If \( f \) is feebly clopen function, then the function \( g \) from \( E \) to \( f(E) \), defined by \( g(p) = f(p) \) for each \( p \in E \) is biclop.na-continuous.

**Proof:**

Suppose that \( J = f(E) \).

Let \( p \in E \) and \( M \) be any feebly clopen set in \( J \) containing \( g(p) \). Since \( J \) is feebly clopen set in \( Y \) and \( M \) is feebly clopen in \( J \).

Since \( f \) is biclop.na-continuous, there exist a \( \delta \)-clopen set \( K \) in \( X \) containing \( p \).

Taking \( W = K \cap E \), we have \( g \) is a biclop.na-continuous function.

**Theorem 5.61:**

Let a function \( f \) from the topological space \( X \) to \( Y \) be biclop.na-continuous.

If \( Y \) is a semi clopen subset of \( Z \), then the function \( f \) from the topological space \( X \) to \( Z \) is biclop.na-continuous.

**Proof:**

Let \( E \) be any feebly clopen set of \( Z \).

Since \( Y \) is semi clopen, \( E \cap Y \) is a semi clopen set in \( Y \). Since \( f \) is biclop.na-continuous, \( f^{-1}(E \cap Y) \) is \( \delta \)-clopen in \( X \). But \( f(p) \in Y \) for each \( p \in X \).

Thus \( f^{-1}(E) = f^{-1}(E \cap Y) \) is a \( \delta \)-clopen set of \( X \).

Therefore \( f : X \to Z \) is biclop.na-continuous.
**Theorem 5.62:**

Let \( f : X \to Y \) and \( g : Y \to Z \) be functions.

(i) Then \( g \circ f \) is \( \delta \)-irresolute if \( f \) is na-continuous and \( g \) is almost open.

(ii) Then \( g \circ f \) is \( \delta \)-clopen irresolute if \( f \) is biclop.na-continuous and \( g \) is almost biclop.na-continuous.

**Proof:**

(i) To prove: \( g \circ f \) is \( \delta \)-irresolute. Given \( f \) is na-continuous and \( g \) is almost open. Let \( K \) be any \( \delta \)-open subset of \( Z \).

Since \( g \) is almost open, so \( g^{-1}(K) \) is open subset of \( Y \).

Since \( f \) is na-continuous, by, \( (g \circ f)^{-1}(K) = f^{-1}(g^{-1}(K)) \) is \( \delta \)-open subset in \( X \). Thus \( g \circ f \) is \( \delta \)-irresolute function.

(ii) To prove: \( g \circ f \) is \( \delta \)-clopen irresolute.

Given \( f \) is biclop.na-continuous and \( g \) is almost biclop.na-continuous. Let \( E \) be any \( \delta \)-clopen subset of \( Z \).

Since \( g \) is almost biclop.na-continuous, so \( g^{-1}(E) \) is clopen subset of \( Y \).

Since \( f \) is biclop.na-continuous.

Now \( (g \circ f)^{-1}(E) = f^{-1}(g^{-1}(E)) \) is \( \delta \)-clopen subset in \( X \). Thus \( g \circ f \) is \( \delta \)-clopen irresolute.

**Theorem 5.63:**

Let \( f \) and \( g \) be the functions from the topological spaces \( X \) into \( Y \) and \( Y \) into \( Z \) such that their composition \( g \circ f \) from \( X \) to \( Z \) be biclop.na-continuous then the following statement true. If \( f \) is \( \delta \)-clopen irresolute and surjective, then \( g \) is biclop.na-continuous.
Proof:

Let E be $\delta$-clopen set of Y.

Since f is $\delta$-clopen irresolute, $f^{-1}(E)$ is $\delta$-clopen in X. Since $(g\circ f)(f^{-1}(E))$ is feebly-clopen in Z.

That is $g(E)$ is feebly-clopen in Z, since f is surjective.

Therefore g is biclopen-continuous.

Theorem 5.64:

If a function f from the topological spaces X to Y is na-continuous injection and Y is feebly-$T_2$, then X is $\delta$-$T_2$.

Proof:

Let p and q be two distinct points of X.

Then $f(p)$ and $f(q)$ are two distinct points of Y.

Thus, there exist two disjoint feebly open E and M containing p and q respectively. Then by remark 1.33, $f^{-1}(E)$ and $f^{-1}(M)$ and two $\delta$-open sets in X.

Clearly, $p \in f^{-1}(E)$, $q \in f^{-1}(M)$, and $f^{-1}(E) \cap f^{-1}(M) = \emptyset$. Thus X is $\delta$-$T_2$.

Theorem 5.65:

If a function f from X to Y is na-continuous, bijection and X is $\delta$-regular, then Y is feebly-regular.

Proof:

Let E be a feebly closed subset of Y and let $q \in E$. Let $q = f(p)$.

Since f is na-continuous, by remark 1.33, $f^{-1}(E)$ is $\delta$-closed in X so that $f^{-1}(Y) = p \in f^{-1}(E)$.
Let $G = f^{-1}(E)$. Then $p \in G$.

Thus, by $\delta$-regularity of $X$, there exist two disjoint $\delta$-open sets $M$ and $J$ such that $K \subseteq M$ and $p \in J$.

Thus, we have $E = f(K) \subseteq f(M)$ and $Y = f(p) \cup f(J)$ and $f(M) \cap f(J) = \emptyset$. As $f$ is na-continuous, $f(M)$ and $f(J)$ are feebly open in $Y$.

Thus $f(M) \cap f(J) = \emptyset$, $Y = f(p) \cup f(J)$ and $E \subseteq f(M)$. Hence $Y$ is feebly-regular.

**Theorem 5.66:**

If a function $f$ from the topological space $X$ into $Y$ is na-continuous, surjection, and $X$ is $\delta$-normal then $Y$ is feebly-normal.

**Proof:**

Let $E_1$ and $E_2$ be two disjoint feebly closed subsets of $Y$.

Since $f$ is na-continuous, by remark 1.33, $f^{-1}(E_1)$ and $f^{-1}(E_2)$ are two $\delta$-closed subsets in $X$.

Let $K_1 = f^{-1}(E_1)$, and let $K_2 = f^{-1}(E_2)$.

Then $K_1$ and $K_2$ are two disjoint $\delta$-closed subsets of $(X, \tau)$.

Since $X$ is $\delta$-normal, there exists two disjoint $\delta$-open sets $J$ and $M$ such that $K_1 \subseteq J$ and $K_2 \subseteq M$.

We thus have $E_1 = f(K_1) \subseteq f(J)$ and $E_2 = f(K_2) \subseteq f(M)$.

Also $f(J)$ and $f(M)$ are two disjoint feebly open sets in $Y$.

Hence $Y$ is feebly-normal.
Theorem 5.67:

If a function $f$ from $X$ to $Y$ is na-continuous surjection and $X$ is $\delta$-connected, then $Y$ is connected.

Proof:

Suppose $Y$ is not connected. Then, there exist non empty disjoint feebly open sets $J_1$ and $J_2$ such that $Y=J_1 \cup J_2$. Hence, we have $f^{-1}(J_1) \cup f^{-1}(J_2) = X$ and $f^{-1}(J_1) \cap f^{-1}(J_2) = \emptyset$. $f^{-1}(J_1)$ and $f^{-1}(J_2)$ are $\delta$-open sets.

Therefore $X$ is not $\delta$-connected. This is a contradiction and so the result follows.