In this chapter, some classes of sets, spaces and functions are introduced and investigated:

- Feebly regular separated set
- Feebly regular connected set
- Obtain some results by using these concepts like feebly regular set
  connected function
- Define the notion of connected complement functions in
topological space
- Analyze the characterization of the new concept like
  Quasi – irresolute functions in the semi Uryshon space.

**Definition 4.1:**

Let \((X, \tau)\) be a topological space. Two non empty feebly regular open sets \(A\) and \(B\) of \(X\) are said to be feebly regular separated (abbr. F.reg.separated) if and only if

\[A \cap (\text{F.reg.cl}(B)) \cup (\text{F.reg.cl}(A)) \cap B = \varnothing.\]
Example:

Let $X = \{a, b, c\}$ with $\tau = \{\varnothing, X, \{a\}, \{b, c\}\}$.

$\tau$-complements are $\{X, \varnothing, \{b, c\}, \{a\}\}$.

Now feebly open set =$\{\varnothing.X, \{a\}, \{b, c\}\}$ and

feebly closed = $\{\varnothing.X, \{a\}, \{b, c\}\}$, F.reg.open set =$\{X, \varnothing, \{b, c\}, \{a\}\}$ and

F.reg.closed set =$\{\varnothing, X, \{a\}, \{b, c\}\}$.

Now we take F.reg.open sets $A = \{a\}$ and $B = \{b, c\}$.

Here F.reg.cl(B) =$\{b, c\}$ and F.reg.cl(A) =$\{a\}$.

These sets $A$ and $B$ are satisfied $[A \cap (F.reg.cl(B))] \cup [(F.reg.cl(A)) \cap B] = \varnothing$. Thus $A$ and $B$ are F.reg.separated sets.

Theorem 4.2:

Let $(Y, \tau_Y)$ be a subspace of a topological space $(X, \tau)$ and let $A$ and $B$ be two subsets of $Y$. Then $A$ and $B$ are F.reg.separated on $\tau$ if and only if they are F.reg.separated on $\tau_Y$.

Proof:

Let $A$ and $B$ be F.reg.separated on $(Y, \tau_Y)$.

Now $(F.reg. \text{cl}_Y(A)) = (F.reg. \text{cl}_X(A)) \cap Y$ and

$(F.reg. \text{cl}_Y(B)) = (F.reg. \text{cl}_X(B)) \cap Y$.

Now $[(F.reg. \text{cl}_Y(A)) \cap B] \cup [A \cap (F.reg. \text{cl}_Y(B))]

= [(F.reg. \text{cl}_X(A)) \cap (Y \cap B)] \cup [A \cap [(F.reg. \text{cl}_X(Y \cap B))]

= [(F.reg. \text{cl}_X(A)) \cap B] \cup [A \cap (F.reg. \text{cl}_X(B))].

Hence $[(F.reg. \text{cl}_Y (A)) \cap B] \cup [A \cap (F.reg. \text{cl}_Y(B))] = \varnothing$
\[ \iff (\text{F.reg. cl}_X(A) \cap B) \cup [A \cap (\text{F.reg. cl}_X(B))] = \emptyset. \]

It follows that A and B are F.reg.separated on \( \tau \) if and only if they are F.reg.separated on \( \tau_Y \).

**Theorem 4.3:**
If A and B are F.reg.separated subsets of a space \( X \) and \( C \subseteq A \) and \( D \subseteq B \), then C and D are also F.reg.separated.

**Proof:**
We are given that \( A \cap (\text{F.reg.cl}(B)) = \emptyset \) and \( (\text{F.reg.cl}(A)) \cap B = \emptyset \) \( \rightarrow \)(1).

Also \( C \subseteq A \Rightarrow (\text{F.reg.cl}(C)) \subseteq (\text{F.reg.cl}(A)) \) and

\( D \subseteq B \Rightarrow (\text{F.reg.cl}(D)) \subseteq (\text{F.reg.cl}(B)) \) \( \rightarrow \) (2).

It follows from (1) and (2) that \( C \cap (\text{F.reg.cl}(D)) = \emptyset \) and \( (\text{F.reg.cl}(C)) \cap D = \emptyset \).

Hence C and D are F.reg.separated.

**Theorem 4.4:**
Two F.reg.closed (F.reg.open) subsets A, B of a topological space are F.reg.separated if and only if they are disjoint.

**Proof:**
Since any two F.reg.separated sets are disjoint, we need only prove that two disjoint F.reg.closed (F.reg.open) sets are F.reg.separated.

If A and B are both disjoint and F.reg.open, then \( A \cap B = \emptyset \), \( \text{F.reg.cl}(A) = A \) and \( \text{F.reg.cl}(B) = B \) so that \( \text{F.reg.cl}(A) \cap B = \emptyset \) and \( A \cap \text{F.reg.cl}(B) = \emptyset \), showing that A and B are F.reg.separated.
If \( A \) and \( B \) are both disjoint and \( F.\text{reg.open} \), then \( A' \) and \( B' \) are both \( F.\text{reg.closed} \) so that \( F.\text{reg.cl}(A')=A' \) and \( F.\text{reg.cl}(B')=B' \).

Also \( A \cap B = \emptyset \).

Now \( A \subseteq B' \) and \( B \subseteq A' \) \( \Rightarrow \) \( F.\text{reg.cl}(A) \subseteq F.\text{reg.cl}(B') \) \( \Rightarrow \) \( F.\text{reg.cl}(B) \subseteq F.\text{reg.cl}(A') \) \( \Rightarrow \) \( F.\text{reg.cl}(A)) \cap B = \emptyset \) \( \text{and} \) \( F.\text{reg.cl}(B)) \cap A = \emptyset \) \( \Rightarrow \) \( A \) and \( B \) are \( F.\text{reg.separated} \).

**Theorem 4.5:**

Two disjoint sets \( A \) and \( B \) are \( F.\text{reg.separated} \) in a topological space \( (X, \tau) \) if and only if they are both \( F.\text{reg.open} \) and \( F.\text{reg.closed} \) in the subspace \( A \cup B \).

**Proof:**

Let the disjoint sets \( A \) and \( B \) be \( F.\text{reg.separated} \) in \( X \) so that

\[
A \cap F.\text{reg. cl}_X(B) = \emptyset \text{ and } F.\text{reg. cl}_X(A) \cap B = \emptyset.
\]

Let \( E = A \cup B \).

Then \( F.\text{reg.cl}_E(A) = F.\text{reg. cl}_X(A) \cap E \)

\[
= F.\text{reg. cl}_X(A) \cap (A \cup B)
\]

\[
=[F.\text{reg. cl}_X(A) \cap A] \cup [F.\text{reg. cl}_X(A) \cap B]
\]

\[
=A \cup \emptyset = A,
\]

since \( A \subseteq F.\text{reg. cl}_X(A) \) and \( F.\text{reg.cl}_X(A) \cap B = \emptyset \). → (1) Hence \( A \) is \( F.\text{reg.closed} \) in the subspace \( A \cup B \). Similarly \( B \) is \( F.\text{reg.closed} \) in \( A \cup B \).

Again since \( A \cap B = \emptyset \), they are complements of each other in \( E \) and hence they are both \( F.\text{reg.open} \) in \( E \).
Conversely let the disjoint sets A and B be both F.reg.open and F.reg.closed in $A \cup B$.

To show that A and B are F.reg.separated in X, since A is F.reg.closed in E, we have $A = \text{F.reg. cl}_E(A)$

$= \text{F.reg. cl}_X(A) \cap E$

$= \text{F.reg. cl}_X(A) \cap (A \cup B)$

$=[\text{F.reg. cl}_X(A) \cap A] \cup [\text{F.reg. cl}_X(A) \cap B]$  

$= A \cup [\text{F.reg. cl}_X(A) \cap B] \rightarrow (1), \text{since } A \subseteq \text{F.reg. cl}_X(A).$

Since $A \cap B = \emptyset \Rightarrow A \cap [\text{F.reg. cl}_X(A) \cap B] = \emptyset,$

it follows from $(1)$ that $\text{F.reg. cl}_X(A) \cap B = \emptyset.$

Similarly $A \cap [\text{F.reg. cl}_X(B)] = \emptyset.$

Here A and B are F.reg.separated in X.

**Definition 4.6:**

A subset A of X which cannot be expressed as the union of two feebly regular separated sets is said to be feebly regular connected (abbr. F.reg.connected). In another way some discussion about the F.reg.connected set,

(i) if A and B are separated sets then they are F.reg.separated sets

(ii) every F.reg.connected set is a connected set.

(iii) X is F.reg.connected. if and only if X is not the union of two non-empty disjoint F.reg.open sets if and only if $X = A \cup B,$ $A \in \text{F.reg.open}(X),$ $B \in \text{F.reg.open}(X), A \neq \emptyset, B \neq \emptyset$ implies $A \cap B \neq \emptyset.$
**Remark 4.7:**

A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be feebly regular continuous if $f^{-1}(V)$ is F.reg.closed in $X$ for every closed subset $V$ of $Y$.

**Definition 4.8:**

A space which is a union of two disjoint non-empty F.reg.separated sets is called F.reg.disconnected.

**Theorem 4.9:**

A space $X$ is connected if the only subsets of $X$ which are both F.reg.open and F.reg.closed ($= F.reg.clopen$) are $\emptyset$ and $X$.

**Proof:**

If $X = A \cup B$ with $A$ and $B$ are F.reg.open sets and disjoint, then $X-A = B$ and so $B$ is the complement of F.reg.open set and hence is F.reg.closed.

Thus, $B$ is F.reg.clopen. Similarly $A$ is F.reg.clopen.

Conversely, if $A$ is a non-empty proper F.reg.open subset, then $A$ and $X-A$ are F.reg.disconnected set of $X$.

**Theorem 4.10:**

Let $X$ be connected space and let $Y$ be a topological space such that $f : X \rightarrow Y$. Then F.reg.continuous image of a F.reg.connected space is F.reg.connected.

**Proof:**

If $f : X \rightarrow Y$ is F.reg.continuous mapping of a connected space $X$ into an arbitrary topological space $Y$.  

59
We wish to show that \( f(X) \) is F.reg.connected as a subspace of \( Y \). Assume that \( f(X) \) is F.reg.disconnected.

Then there exists \( G_1 \) and \( G_2 \), both are F.reg.open in \( Y \) such that \( G_1 \cap f(X) \neq \emptyset \), \( G_2 \cap f(X) \neq \emptyset \), \( (G_1 \cap f(X)) \cap (G_2 \cap f(X)) = \emptyset \) and \( (G_1 \cap f(X)) \cup (G_2 \cap f(X)) = f(X) \). It follows that \( \phi = f^{-1}(\phi) = f^{-1}[(G_1 \cap f(X)) \cap (G_2 \cap f(X))] \)

\[
= f^{-1}[(G_1 \cap G_2) \cap f(X)] \\
= f^{-1}[G_1] \cap f^{-1}[G_2] \cap f^{-1}[f(X)] \\
= f^{-1}[G_1] \cap f^{-1}[G_2] \cap X \\
= f^{-1}[G_1] \cap f^{-1}[G_2] \text{ and} \\
X = f^{-1}[f(X)] \\
= f^{-1}[(G_1 \cup f(X)) \cup (G_2 \cap f(X))] \\
= f^{-1}[[G_1 \cup G_2] \cap f(X)] \\
= f^{-1}[(G_1 \cup G_2) \cap f(X)] \\
= f^{-1}[G_1 \cup G_2] \cap f^{-1}[f(X)] \\
= \{f^{-1}[G_1] \cup f^{-1}[G_2]\} \cap X = f^{-1}[G_1] \cup f^{-1}[G_2].
\]

Since \( f \) is feebly regular continuous and \( G_1, G_2 \) are F.reg.open in \( Y \) both intersecting \( f(X) \), it follows that \( f^{-1}(G_1) \) and \( f^{-1}(G_2) \) are non-empty F.reg.open subsets of \( X \).

Thus \( X \) has been expressed as a union of two disjoint non-empty F.reg.open subsets of \( X \) and consequently \( X \) is F.reg.disconnected, which is a contradiction.

Hence \( f(X) \) must be F.reg.connected.
Theorem 4.11:

A subset E of a topological space X is F.reg.disconnected if and only if E is the union of two non-empty disjoint sets both F.reg.open (F.reg.closed) in E.

Proof:

Let E be a subset of X and is F.reg.disconnected if and only if there exist non-empty sets G and H both F.reg.open (F.reg.closed) in X such that \( G \cap E \neq \varnothing \), \( H \cap E \neq \varnothing \), \( (G \cap E) \cap (H \cap E) = \varnothing \) and \( (G \cap E) \cup (H \cap E) = E \).

Theorem 4.12:

Let \((X, \tau)\) be a topological space and let E be a subset of X.

Then E is F.reg.disconnected if and only if there exist non-empty sets M and N both F.reg.open (F.reg.closed) in X such that \( M \cap E \neq \varnothing \), \( N \cap E \neq \varnothing \), \( E \subseteq M \cup N \) and \( M \cap N \subseteq X - E \).

Proof:

By theorem 4.11, Y is F.reg.disconnected if and only if there exist non-empty sets M and N both F.reg.open (F.reg.closed) in X such that \( M \cap Y \neq \varnothing \), \( N \cap Y \neq \varnothing \), \( (M \cap Y) \cap (N \cap Y) = \varnothing \) and \( (M \cap Y) \cup (N \cap Y) = Y \).

Now \( (M \cap Y) \cap (N \cap Y) = \varnothing \iff (M \cap N) \cap Y = \varnothing \)

\( \iff M \cap N \subseteq X - Y \) and \( (M \cap Y) \cup (N \cap Y) = Y \)

\( \iff (M \cup N) \cap Y = Y \Rightarrow Y \subseteq M \cup N \).

Theorem 4.13:

Let \((X, \tau)\) be a topological space and let E be a F.reg.connected subset of X such that \( E \subseteq A \cup B \) where A and B are F.reg.separated sets. Then either \( E \subseteq A \) or \( E \subseteq B \).
Proof:

Since A and B are F.reg.separated, \( A \cap (F.\text{reg.cl}(B)) = \varnothing \), \( F.\text{reg.cl}(A) \cap B = \varnothing \).

Now \( E \subseteq A \cup B \Rightarrow E \cap (A \cup B) = (E \cap A) \cup (E \cap B) \)------(1).

We claim that at least one of the sets \( E \cap A \) and \( E \cap B \) is empty.

For, if possible, suppose none of them is empty, that is, suppose that \( E \cap A \neq \varnothing \) and \( E \cap B \neq \varnothing \).

Then \( (E \cap A) \cap (E \cap B) \subseteq (E \cap A) \cap [F.\text{reg.cl}(E) \cap F.\text{reg.cl}(B)] \)

\( = (E \cap E) \cap [A \cap F.\text{reg.cl}(B)] \)

\( = [E \cap F.\text{reg.cl}(E)] \cap \varnothing = \varnothing. \)

Similarly, \( F.\text{reg.cl}(E \cap A) \cap (E \cap B) = \varnothing. \)

Hence \( E \cap A \) and \( E \cap B \) are F.reg.separated sets.

Thus \( E \) has been expressed as the union of two non-empty F.reg.separated sets and consequently \( E \) is F.reg.disconnected. This is a contradiction.

Hence at least one of the sets \( E \cap A \) and \( E \cap B \) is empty. If \( E \cap A = \varnothing \), then (1) gives \( E = E \cap B \) which implies that \( E \subseteq B \).

Similarly if \( E \cap B = \varnothing \), then \( E \subseteq A \).

Hence either \( E \subseteq A \) or \( E \subseteq B \).

Corollary 4.14:

If \( E \) is a F.reg.connected subset of a space \( X \) such that \( E \subseteq A \cup B \) where \( A, B \) are disjoint F.reg.open (F.reg.closed) subsets of \( X \), then \( A \) and \( B \) are F.reg.separated.
Proof:

If $A, B$ are $F$.reg.open with $A \cap B = \varnothing$, then $A \subseteq B' \Rightarrow F\.reg\.cl(A) \subseteq F\.reg\.cl(B') = B'$

$\Rightarrow F\.reg\.cl(A) \cap B = \varnothing$.

Similarly $A \cap F\.reg\.cl(B) = \varnothing$. Hence $A$ and $B$ are $F$.reg.separated.

Definition 4.15:

A function $f : X \rightarrow Y$ is said to be feebly regular set-connected (abbr. $F$.reg.set-connected) if $f^{-1}(V) \in co(X)$ for every $V \in F\.reg\.open(Y)$.

Example:

Consider the function $f$ from the topology $\tau = \{\varnothing, \{a\}, \{b,c\}, X\}$ on $X = \{a,b,c\}$ and the topology $\sigma = \{\varnothing, \{r\}, \{p,q\}, Y\}$ on $Y = \{p, q, r\}$ with $f(a) = r$, $f(b) = q$ and $f(c) = p$.

Clearly $\sigma$ is a $F$.reg.open set.

The inverse image of every $F$.reg.open set in $Y$ is clopen in $X$. Thus $f$ is feebly regular set-connected.

Theorem 4.16:

Let $(X, \tau)$ and $(Y, \sigma)$ be topological spaces. The following statements are equivalent for the function $f : X \rightarrow Y$

(i) $f$ is $F$.reg.set-connected

(ii) $f^{-1}(f\.int(f\.cl(G)))$ is clopen for every $F\.reg\.open$ subset $G$ of $Y$.

Proof:

(i) $\Rightarrow$ (ii) Let $G$ be any $F$.reg.open subset of $Y$.

Since $f\.int(f\.cl(G))$ is $F\.reg\.open$, 63
by (i) it follows that $f^{-1}(f.\text{int}(f.\text{cl}(G)))$ is clopen. (ii)$\Rightarrow$(i) Let $V$ be $F.\text{reg.open}$ in $Y$.

By (ii), $f^{-1}(f.\text{int}(f.\text{cl}(V)))$ is clopen in $X$ and hence $f$ is $F.\text{reg.set-connected}$.

**Theorem 4.17:**

If $f : X \rightarrow Y$ is $F.\text{reg.set-connected}$ function and $A$ is any subset of $X$, then the restriction $f/A : A \rightarrow Y$ is $F.\text{reg.set-connected}$ function.

**Proof:**

Let $O$ be a $F.\text{reg.open}$ set in $Y$.

By hypothesis $f^{-1}(O)$ is clopen in $X$.

We have $f^{-1}(O) \cap A = (f/A)^{-1}(O)$ is clopen in $A$. Hence $f/A$ is $F.\text{reg.set-connected}$ function.

**Theorem 4.18:**

Let the functions $f$ and $g$ from $X$ to $Y$ and from $Y$ to $Z$ be set-connected and $F.\text{reg.set-connected}$ respectively.

Then the function $g \circ f$ from $X$ to $Z$ is $F.\text{reg.set-connected}$ function.

**Proof:**

Let $M$ be $F.\text{reg.open}$ in $Z$.

Since $g$ is $F.\text{reg.set-connected}$, $g^{-1}(M)$ is clopen in $Y$. Since $f$ is set-connected, $f^{-1}(g^{-1}(M))$ is clopen in $X$. Hence $g \circ f$ is $F.\text{reg.set-connected}$.
**Definition 4.19:**

A function f from the topological space X into Y is said to be F.reg.open (resp. F.reg.closed) if the image of every open set (closed set) in X is F.reg.open (F.reg.closed) in Y.

**Theorem 4.20:**

If f : X → Y is a surjective F.reg.open and F.reg.closed function and g : Y → Z is a function such that g ∘ f : X → Z is F.reg.set-connected, then g is F.reg.set-connected.

**Proof:**

Let V be F.reg.open in Z.

By definition, (g ∘ f)⁻¹(V) is clopen in X. That is f⁻¹(g⁻¹(V)) is clopen in X.

Since f is surjective, F.reg.open and F.reg.closed, f(f⁻¹(g⁻¹(V))) = g⁻¹(V) is clopen.

Therefore g is F.reg.set-connected.

**Definition 4.21:**

A space X is said to be F.reg.T₁ if for each pair of disjoint points p and q of X, there exist F.reg.open sets E and M containing p and q respectively such that q and p do not belong to E and M.

**Theorem 4.22:**

If a function f from the topological space X into Y is a F.reg.set-connected injection and Y is F.reg.T₁, then X is clopen T₁.
Proof:

Since \( Y \) is \( F \)-regular, for any disjoint points \( p \) and \( q \) in \( X \), there exist \( E, M \subseteq F \)-regular open sets \( Y \) such that \( f(p) \in E \), \( f(q) \notin E \), \( f(p) \notin M \), \( f(q) \in M \).

Since \( f \) is \( F \)-regular set-connected, \( f^{-1}(E) \) and \( f^{-1}(M) \) are clopen in \( X \). Furthermore \( q \notin f^{-1}(E) \) and \( p \notin f^{-1}(M) \).

This shows that \( X \) is clopen \( T_1 \).

Definition 4.23:

A space \( X \) is said to be \( F \)-regular \( T_2 \) or \( F \)-Hausdorff if for each pair of distinct points \( p \) and \( q \) in \( X \), there exist disjoint \( F \)-regular open sets \( E \) and \( M \) in \( X \) such that \( p \) and \( q \) belong to \( E \) and \( M \).

Theorem 4.24:

If the functions \( f \) and \( g \) from the topological space \( X \) into \( Y \) is \( F \)-regular set-connected function and \( Y \) is \( F \)-Hausdorff, then \( E = \{ p \in X : f(p) = g(p) \} \) is \( F \)-regular closed in \( X \).

Proof:

If \( p \in X - E \) then it follows that \( f(p) \neq g(p) \).

Since \( Y \) is \( F \)-Hausdorff, there exist \( F \)-regular open sets \( J \) and \( M \) such that \( f(p) \in J \), \( g(p) \in M \) and \( J \cap M \neq \emptyset \).

Since \( f \) and \( g \) are \( F \)-regular set-connected, \( f^{-1}(f(\text{int}(f(\text{cl}(J)))) \) and \( g^{-1}(f(\text{int}(f(\text{cl}(M)))) \) are clopen in \( X \) with \( p \in f^{-1}(f(\text{int}(f(\text{cl}(J)))) \) and \( p \in g^{-1}(f(\text{int}(f(\text{cl}(M)))) \).
Definition 4.25:

A function $f$ from the topological space $X$ into $Y$ is said to be quasi ultra $F$-regular open if $f(M)$ is open in $Y$ with $F$-regular connected complement of every $F$-regular open set $M$ in $X$.

Theorem 4.26:

A function $f$ from the topological spaces $X$ and $Y$ is quasi ultra $F$-regular open if and only if for every subset $M$ of $X$, $f(F$-regular int$(M)) \subseteq \text{int}(f(M))$.

Proof:

Let $f$ be a quasi ultra $F$-regular open set.

Now, we have $\text{int}(M) \subseteq M$ and $F$-regular int$(M)$ is $F$-regular open set. Hence, we obtain that $f(F$-regular int$(M)) \subseteq f(M)$.

As $f(F$-regular int$(M))$ is open, $f(F$-regular int$(M)) \subseteq \text{int}(f(M))$.

Conversely, assume that $M$ is $F$-regular open set in $X$.

Then, $f(M) = f(F$-regular int$(M)) \subseteq \text{int } f(M))$. But $\text{int}(f(M)) \subseteq f(M)$. Consequently $f(M) = \text{int } f(M)$ and hence $f$ is quasi ultra $F$-regular open.

Lemma 4.27:

If a function $f$ from the topological spaces $X$ and $Y$ is quasi ultra $F$-regular, then $F$-regular int$(f^{-1}(E)) \subseteq f^{-1}(\text{int}(E))$ for every subset $E$ of $Y$ with $F$-regular connected complement.
**Proof:**

Let M be any open subset of Y with F.reg.connected complement.

Then, F.reg.int(f⁻¹(M)) is a F.reg.open set in X and f is quasi ultra F.reg.open, then f(F.reg.int(f⁻¹(M))⊂int(f(f⁻¹(M))⊂int(M).

Thus, F.reg.int(f⁻¹(M))⊂f⁻¹(int(M)).

**Theorem 4.28:**

If f is quasi ultra F.reg.open then for each subset E of X, f(F.reg.int(E))⊂int(f(E)) where the function f from the topological space X into Y.

**Proof:**

It follows from theorem 4.26.

**Theorem 4.29:**

A function f from the topological space X into Y is quasi ultra F.reg.open if and only if for any subset B of Y and for any F.reg.closed set F of X containing f⁻¹(B), there exist a closed set G of Y containing B with F.reg.connected complement such that f⁻¹(G)⊂F.

**Proof:**

Suppose f is quasi ultra F.reg.open.

Let B⊂Y and F be F.reg.closed set of X containing f⁻¹(B). Now, put G=Y-f(X-F).

It is clear that f⁻¹(B)⊂F implies B⊂G.

Since f is quasi ultra F.reg.open, we obtain G as a closed set of Y with F.reg.connected complement. Moreover, we have f⁻¹(G)⊂F.
Conversely, let $U$ be a F.reg.open set of $X$ and put $B=Y\setminus f(U)$. Then $X\setminus U$ is a F.reg.closed set in $X$ containing $f^{-1}(B)$.

By hypothesis, there exists a closed set $F$ of $Y$ with F.reg.connected complement such that $B\subseteq F$ and $f^{-1}(F)\subseteq X\setminus U$.

Hence $f(U)\subseteq Y\setminus F$ with F.reg.connected complement.

On the other hand, it follows that $B\subseteq F$, $Y\setminus F\subseteq Y\setminus B=f(U)$.

Thus, $f(U)\subseteq Y\setminus F$ with F.reg.connected complement which is open and hence $f$ is a quasi ultraF.reg.open function.

**Theorem 4.30:**

A function $f : X \to Y$ is quasi ultra F.reg.open if and only if $f^{-1}(\text{cl}(B)) \subseteq \text{F.reg.cl}(f^{-1}(B))$ for every subset $B$ of $Y$ with F.reg.connected complement.

**Proof:**

Suppose that $f$ is quasi ultra F.reg.open.

For any subset $B$ of $Y$, $f^{-1}(B) \subseteq \text{F.reg.cl}(f^{-1}(B))$.

Therefore by theorem 4.28 and 4.29, there exists a closed set in $Y$ with F.reg.connected complement such that $B\subseteq F$ and $f^{-1}(F)\subseteq \text{F.reg.cl}(f^{-1}(B))$.

Therefore, we obtain $f^{-1}(\text{cl}(B)) \subseteq f^{-1}(F) \subseteq \text{F.reg.cl}(f^{-1}(B))$.

Conversely, let $B \subseteq Y$ and $F$ be a F.reg.closed of $X$ containing $f^{-1}(B)$.

Put $W=\text{cl}_Y(B)$. Then we have $B \subseteq W$ and $W$ is closed and $f^{-1}(W) \subseteq \text{F.reg.cl}(f^{-1}(B))$. Thus $f$ is quasi ultra F.reg.open.
**Definition 4.31:**

A function $f : X \rightarrow Y$ is said to be quasi ultra F.reg.closed if the image of each F.reg.closed set in $X$ is closed in $Y$ with F.reg.connected complement.

**Theorem 4.32:**

If a function $f : X \rightarrow Y$ is quasi ultra F.reg.closed, then $f^{-1}(\text{int}(B)) \subset \text{F.reg.int}(f^{-1}(B))$ for every subset $B$ of $Y$ with F.reg.connected complement.

**Proof:**

Similar to the proof of lemma 4.27.

**Theorem 4.33:**

A function $f : X \rightarrow Y$ is quasi ultra F.reg.closed if and only if for any subset $B$ of $Y$ and for any F.reg.open set $G$ of $X$ containing $f^{-1}(B)$, there exists an open set $U$ of $Y$ with F.reg.connected complement containing $B$ such that $f^{-1}(U) \subset G$.

**Proof:**

Similar to that of theorem 4.29.

**Definition 4.34:**

A function $f : X \rightarrow Y$ is contra-irresolute if the inverse image of every semi open set of $Y$ is semi-closed in $X$.

**Example:**

Let $X = \{l,m,n\}$ and $Y = \{u,v,w\}$ be two non-empty sets with topologies as $\tau = \{\emptyset, X, \{l\}, \{m,n\}\}$ and $\sigma = \{\emptyset, Y, \{w\}, \{u,v\}\}$.
Define a function $f : X \rightarrow Y$ such that

$f(l) = w$

$f(m) = v$ $f(n) = u$

Then $f^{-1}(\phi) = \emptyset$

$f^{-1}(Y) = f^{-1}\{u, v, w\} = \{l, m, n\} = X$

$f^{-1}\{w\} = \{l\}$

$f^{-1}\{u, v\} = \{m, n\}$.

**Remark 4.35:**

Every mildly Hausdorff strogly $s$-closed space is locally indiscrete.

**Theorem 4.36:**

Let a function $f$ from $X$ to $Y$ be continuous and let $X$ be locally indiscrete. Then $f$ is contra-continuous.

**Proof:**

It follows from the remark 4.35 and the definition.

**Theorem 4.37:**

Let $X, Y$ and $Z$ be the topological spaces and let the functions $f$ and $g$ from $X$ to $Y$ and from $Y$ to $Z$ be contra irresolute and R.C-continuous function and let $Y$ be extremally disconnected space then the function $g \circ f$ from $X$ to $Z$ is contra semicontinuous.

**Proof:**

Let $O$ be open in $Z$ then $g^{-1}(O)$ is regular closed in $Y$.

Since $Y$ is extremally disconnected, $g^{-1}(O)$ is semi-open in $Y$. Then $f^{-1}(g^{-1}(O))$ is semi-closed in $X$. 71
That is \((g \circ f)^{-1}(O) = f^{-1}(g^{-1}(O))\) is semi-closed in \(X\). That implies that \(g \circ f\) is contra semi continuous.

**Theorem 4.38 :**

Let a function \(f\) from the topological spaces \(X\) to \(Y\) be quasi irresolute injective function. If \(Y\) is semi-Uryshon, then \(X\) is semi-Uryshon.

**Proof :**

Let \(p_1\) and \(p_2\) be any two distinct points of \(X\). Then \(f(p_1) \neq f(p_2)\).

Since \(Y\) is semi-Uryshon, there exist semi open sets \(O_1\) and \(O_2\) of \(Y\) containing \(f(p_1)\) and \(f(p_2)\) respectively such that, \(s.cl(O_1) \cap s.cl(2) = \emptyset\).

Again since \(f\) is quasi irresolute there exist \(E_1, E_2 \in s.o.(X)\) containing \(p_1\) and \(p_2\) such that, \(f(s.cl(E_1)) \subseteq s.cl(f(E_1)) = s.cl(O_1)\) and \(f(s.cl(E_2)) \subset s.cl(f(E_2)) = s.cl(O_2)\). Hence, \(s.cl(E_1) \cap s.cl(E_2) = \emptyset\), by (1) and (2).

Thus \(X\) is semi-Uryshon.

**Theorem 4.39 :**

Let a function \(f\) from the topological space \(X\) to \(Y\) be quasi irresolute function. If \(Y\) is semi-Uryshon then the set \(S = \{(p_1, p_2) : f(p_1) = f(p_2)\}\) is a semi \(0\)-closed subset of \(X \times X\).

**Proof :**

Let \((p_1, p_2) \not\in A\). That is \((p_1, p_2) \in X \times X - S\).

Then \(f(p_1) \neq f(p_2)\).

Since \(Y\) is semi Uryshon, there exist \(O_1, O_2 \in s.o\) \((Y)\) contains \(f(p_1)\)
And $f(p_2)$ respectively such that $\text{s.cl } (O_1) \cap \text{s.cl } (O_2) = \emptyset$.

Quasi irresoluteness of $f$ implies that, there exist $E_1, E_2 \in \text{s.o}(X)$ containing $p_1$ and $p_2$ such that $f(\text{s.cl}(E_2)) \subset \text{s.cl}(O_2)$, where $i = 1,2$.

Put $E = E_1 \times E_2$. Then $(p_1, p_2) \in E \in \text{s.o } (X \times X)$ and $S \cap \text{s.cl}(E) = \emptyset$.

Hence $(p_1, p_2) \not\in \text{s.cl}(S)$. Thus, $S$ is semi $\theta$-closed in $X \times X$.

**Theorem 4.40**:

Let a function $f$ from $X$ to $Y$ be quasi irresolute and if $Y$ is semi-Uryshon, then the graph $G(f)$ is semi closed in $X \times Y$ where $X$ and $Y$ are the topological spaces.

**Proof**:

Let $(p, q) \not\in G(f)$. Then $f(p) \neq q$ and hence there exist $E, M \in \text{s.o}(Y)$ such that $f(p) \in E$, $q \in M$ and $\text{s.cl}(E) \cap \text{s.cl}(M) = \emptyset$.

Since $f$ is quasi irresolute, there exists $J \in \text{s.o}(X)$ containing $p$ such that $f(\text{s.cl}(J)) \subset \text{s.cl}(f(J)) = \text{s.cl}(E)$.

Therefore, $\text{s.cl}(M) \cap f(\text{s.cl}(J)) = \emptyset$.

Hence, $\text{s.cl}(J \times M) \cap G(f) = \emptyset$.

This shows that, $(p, q) \not\in \text{s.cl}(G(f))$ and $G(f)$ is semi $\theta$-closed in $X \times Y$.

**Theorem 4.41**:

Let a function $f$ from the topological spaces $X$ to $Y$ be quasi irresolute surjection and if $X$ is s-closed, then $Y$ is s-closed.

**Proof**:

Let $\{M_\alpha : \alpha \in \Lambda\}$ be any semi-open cover of $Y$. For each $p \in X$, there exists $\alpha(p) \in \Lambda$ such that $f(p) \in M_{\alpha(p)}$.
Quasi irresoluteness of f gives, there exists $E_p \in \text{s.o}(X)$ containing p such that $f(\text{s.cl } E(p)) \subset \text{s.cl}(f(E(p))) = \text{s.cl}M(p)$. Since $\{E_p : p \in X\}$ is semi-open cover of X, there exist a finite points say $p_1, p_2, \ldots, p_n$ of X such that $X = \bigcup_{i=1}^{n} \{\text{s.cl}(E_{p_i}) : i = 1, 2, \ldots, n\}$, since f is surjective.

This implies that Y is s-closed.

**Example:**

Let $X = Y = \{l, m, n\}$, $\tau = \{\emptyset, X, \{n\}\}$ and $\sigma = \{\emptyset, Y, \{l\}, \{m\}, \{l, m\}\}$.

Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be the identity function and let $g = f^{-1}$. Then f is not $\theta$ - irresolute.

The function $g: (Y, \sigma) \rightarrow (X, \tau)$ is $\theta$-irresolute but not irresolute. In addition, g is neither pre-irresolute nor $\alpha$-irresolute.