Chapter 5

Fractional delay segments method on time-delayed recurrent neural networks with impulsive and stochastic effects: An exponential stability approach

5.1 Introduction

Over the past few decades, the analysis of recurrent neural networks in dynamical behaviors has received great attention due to their potential applications in different fields, such as static image treatment, optimization problems, parallel computing, signal processing, etc [14, 102]. In recurrent neural networks the arising of time delays is inevitable, due to the finite switching speed of information storage processing or of amplifiers, communication time and transmission of signals. For sequence, in a neural networks, the appearance of time delay may affect the stability performance generating unstable, divergence and swinging or chaotic behavior. Hence, the study of stability analysis for time-delayed neural
networks has been broad interests from numerous scholars. In accordance, the delay-dependent and delay-independent criteria are two categories of time delays in neural networks, which classified by the existing time delayed results. So far as, one can observe from the available literatures, that the delay-dependent case is less conserved than the delay independent ones. In [18], Cao and Song discussed about the time-delayed criteria for stability performance of BAM neural networks.

On the other hand, we notice that the propagation delays can be distributed over a period of time, because the variety of axon sizes and lengths are too large. Thus, the consideration of distributed delay term is meaningful in neural network system. So, it is significant to inspect both the time delay results in the dynamical behaviors of neural systems, see for instance [8, 116]. Some criteria in terms of LMIs were presented to ascertain the stochastic BAM neural networks with mixed delays to be exponential stable [8]. Meanwhile, to conclude the speed of neural computations, the exponential convergence rate is used. So, it generates much attention from many researchers to studied the property of exponential stability [20, 187]. Furthermore, the exponential stability criteria for neural networks have been investigated by He in [62].

Additionally, several dynamical processes, specifically some biological fields such as bursting rhythm models in pathology, biological neural networks, as well as flying object motions, frequency-modulated signal processing systems and optimal control models in economics are distinguished by abrupt changes of state of the networks at certain moments of time, see [47, 189] and references therein. This changes indicates the impulsive phenomena. As is well known that, the impulses can damage the properties of stability, which possibly leads to oscillation and instability. Therefore, in RNNs, the incorporation of impulsive perturbations is essentially one. Raja et al., in [140], widely discussed about the impulsive behavior of neutral-type NNs and derived its stability criteria by using some inequality techniques.
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On the other hand, the noisy perturbations have received intensive interest in the area of neural networks. Besides, the stochastic phenomenon in neural networks constructed when random variations introducing into the networks, either by giving them stochastic weights, or by giving the network's neurons stochastic transfer functions. Practically, the random fluctuations helps us to escape from instability, so it makes them valuable tool for neural networks. Therefore, the stochastic perturbations in neural networks become a fixate issue for stability analysis of recurrent neural networks, and recently the stability analysis problems for time-delayed stochastic neural networks have attracted broad interest, see for instants [193, 214]. The authors in [171], studied the stochastic effects of state behavior for time-delayed neural networks.

Inspired by the aforementioned discussions, our main motivation of this chapter is to investigate the fractional delay segments on the exponential stability problem for impulsive recurrent neural networks with stochastic effects and mixed time-delays. By employing the stochastic analysis theory, stability theory and some inequality techniques, the exponential stability problem of the concerned neural networks is transformed into the feasibility problem of a set of linear matrix inequalities. Consequently, by constructing an appropriate novel Lyapunov-Krasovskii functional, some new-brand sufficient criteria for exponential stability are formulated in terms of LMIs, which can be justified easily by LMI control toolbox in MATLAB software. Also, two numerical examples with their simulations are provided to demonstrate the superiority and feasibility with less conservatism of the proposed results. The main contributions of this research work are highlighted as follows:

(1) Impulsive effects, stochastic noises, discrete & distributed time delays are taken into account in the stability analysis of proposed recurrent neural networks.

(2) By the implementation, Lyapunov-Krasovskii functionals, stability theory
and some novel inequality techniques, a brand-new sufficient conditions for exponential stability of impulsive RNNs are obtained in terms of LMIs.

(3) By handled both time-delay terms in our stochastic recurrent neural networks with impulsive effects, the allowable upper bounds of discrete time-varying delay is very huge, when compared with the previous results, see Table 5.1, Table 5.2 & Table 5.3 in Example 5.4.1. Also, the comparisons of upper bounds for various \( \mu, \zeta \) are shown in Table 5.4, 5.5 and 5.6 respectively. This explore the approach developed in this chapter is effective and less conservative than some existing literature, which ensures the novelty of this research work.

(4) By separating the delay intervals into fractional segments, we obtain the allowable upper bounds of discrete time-varying delay is very large with accuracy and the average of allowable upper bounds given in Table 5.7.

(5) Two special cases of the main results are derived in Corollary 5.3.4, 5.3.6 and also by using MATLAB LMI control toolbox, we receive that the LMIs of Corollary 5.3.4 are feasible as well as the simulations are given in Examples 5.4.2.

\section{5.2 Problem formulation and preliminaries}

In this chapter, a class of recurrent neural networks with time-varying delays is described by:

\[
\begin{align*}
    \dot{x}(t) &= \left[ -Ax(t) + W_0 f(x(t)) + W_1 g(x(t - \mu(t))) + W_2 \int_{t-\sigma}^{t} h(x(s)) \times ds + f \right] dt + \rho(x(t), x(t - \mu(t)), t) d\omega(t); \quad t > 0, \ t \neq t_k, \\
    x(t_k) &= \mathcal{G}_k x(t_{k-}); \quad t = t_k, \ k \in \mathbb{Z}^+, \\
    x(t) &= \psi(t); \quad t \in [-v, 0],
\end{align*}
\]  

(5.2.1)
where \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in \mathbb{R}^n \) denotes the neuron state vector at time \( t \); The diagonal matrix \( A = \text{diag}\{a_1, a_2, \ldots, a_n\} > 0 \) have positive entries \( a_i > 0 \) \((i = 1, 2, \ldots, n)\); \( f(x(t)) = (f_1(x(t)), f_2(x(t)), \ldots, f_n(x(t)))^T \), \( g(x(t - \mu(t))) = (g_1(x(t - \mu(t))), g_2(x(t - \mu(t))), \ldots, g_n(x(t - \mu(t))))^T \), and \( h(x(t)) = (h_1(x(t)), h_2(x(t)), \ldots, h_n(x(t)))^T \), are the activation functions of the neurons with \( f(0) = g(0) = h(0) = 0 \); \( W_0 \in \mathbb{R}^{n \times n} \) is the connection weight matrix; \( W_1 \) \& \( W_2 \) are the discrete and distributed delayed connection weight matrices respectively; \( J = [J_1, J_2, \ldots, J_n]^T \) represents the external constant input vector; \( \mu(t) \) is the discrete time varying delays which are bounded with \( 0 < \mu(t) < \bar{\mu}, \bar{\mu}(t) \leq \mu < 1 \); \( \sigma \) is distributed constant delay; \( \rho : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^n \) denotes the stochastic disturbance; \( \omega(t) = (\omega_1(t), \omega_2(t), \ldots, \omega_n(t))^T \) is \( n \)-dimensional Brownian motion defined on a complete probability space \((\mathcal{A}, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) with a filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) satisfying the usual conditions (i.e., it is right continuous and \( \mathcal{F}_0 \) contains all \( \mathbb{P} \)-null sets) and \( \mathbb{E}\{d\omega(t)\} = 0, \mathbb{E}\{d\omega^2(t)\} = dt; \mathcal{G}_k \) is impulsive gain matrix; The discrete set \( \{t_k\} \) satisfies \( 0 = t_0 < t_1 < t_2 < \ldots < t_k < \ldots, \lim_{k \to \infty} t_k = \infty; x(t^-_k) \) denote the left-hand limits at \( t_k \); Similarly \( x(t^+_k) \) denote the right-hand limits at time \( t_k \); \( \psi(t) \) is continuous function; \( \nu = \max(\bar{\mu}, \sigma) \). Assume that \( x(t) \) is right-continuous, that is \( x(t^-_k) = x(t_k) \).

On the neuron activation functions, the following assumptions are made to derive the main results.

**Assumption 5.1.** The activation functions \( f_i(t), g_i(t) \) and \( h_i(t) \) in (5.2.1) satisfies the following conditions with bounded property:

\[
\begin{align*}
\alpha_i^- &\leq \frac{f_i(\xi_1) - f_i(\xi_2)}{\xi_1 - \xi_2} \leq \alpha_i^+, \\
\beta_i^- &\leq \frac{g_i(\xi_1) - g_i(\xi_2)}{\xi_1 - \xi_2} \leq \beta_i^+, \\
\gamma_i^- &\leq \frac{h_i(\xi_1) - h_i(\xi_2)}{\xi_1 - \xi_2} \leq \gamma_i^+,
\end{align*}
\]

for all \( \xi_1, \xi_2 \in \mathbb{R}, \xi_1 \neq \xi_2, i = 1, 2, \ldots, n \). The values \( \alpha_i^-, \alpha_i^+, \beta_i^-, \beta_i^+, \gamma_i^-, \gamma_i^+, (i = 1, 2, \ldots, n) \) in this assumption are positive scalars.
Remark 5.2.1. In Assumption 5.1, the scalars $\alpha_i^-, \alpha_i^+, \beta_i^-, \beta_i^+, \gamma_i^-, \gamma_i^+$ are permitted to be greater than zero, less than zero or equal to zero. Thus, the outcomes of the activation functions may be non-monotonic & more general than the Lipschitz conditions and usual sigmoid activation functions as pointed out in [116, 117]. This type of activation functions are very supportive for utilizing an LMI-based approach to lead to be less conservative.

The equilibrium point of (5.2.1) is represented by $x^* = [x^*_1, x^*_2, ..., x^*_n]^T$. By the shifting $u(.) = x(.) - x^*$, one can transform (5.2.1) into the error system as follows:

$$
du(t) = \left[ -Au(t) + W_0 \bar{f}(u(t)) + W_1 \bar{g}(u(t - \mu(t))) + W_2 \int_{t-\sigma}^t \bar{h}(u(s)) \times ds \right] dt + \bar{\rho}(u(t), u(t - \mu(t)), t) d\bar{\omega}(t); \ t > 0, \ t \neq t_k, \ u(t_k) = \mathcal{G}_k u(t_k^-); \ t = t_k, \ k \in \mathbb{Z}^+, \ u(t) = \varphi(t); \ t \in [-v, 0],
$$

where $u(t) = [u_1(t), u_2(t), ..., u_n(t)]^T$, $u(t - \mu(t)) = [u_1(t - \mu(t)), u_2(t - \mu(t)), ..., u_n(t - \mu(t))]^T$, $\bar{f}(u(t)) = [\bar{f}_1(u(t)), \bar{f}_2(u(t)), ..., \bar{f}_n(u(t))]^T$, $\bar{g}(u(t - \mu(t))) = [\bar{g}_1(u(t - \mu(t))), \bar{g}_2(u(t - \mu(t))), ..., \bar{g}_n(u(t - \mu(t)))]^T$, $\bar{h}(u(t)) = [\bar{h}_1(u(t)), \bar{h}_2(u(t)), ..., \bar{h}_n(u(t))]^T$, $\bar{\rho}(u(t), u(t - \mu(t)), t) = [\bar{\rho}_1(u(t), u(t - \mu(t)), t), \bar{\rho}_2(u(t), u(t - \mu(t)), ...) = g(u(t - \mu(t)) + x^*) - g(x^*), \bar{\rho}(u(t), u(t - \mu(t)), t) = \rho(u(t) + x^*, u(t - \mu(t)) + x^*, t) - \rho(x^*, x^*, t)$. 

Now, one can obtain that

$$
\alpha_i^- \leq \frac{\bar{f}_i(\xi_i)}{\xi_i} \leq \alpha_i^+, \quad \bar{f}_i(0) = 0, \quad (5.2.3)
$$

$$
\beta_i^- \leq \frac{\bar{g}_i(\xi_i)}{\xi_i} \leq \beta_i^+, \quad \bar{g}_i(0) = 0, \quad (5.2.4)
$$

$$
\gamma_i^- \leq \frac{\bar{h}_i(\xi_i)}{\xi_i} \leq \gamma_i^+, \quad \bar{h}_i(0) = 0, \quad (5.2.5)
$$

where $i = 1, 2, 3, ..., n$.

Assume that $\bar{\rho}: \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^n$ are Locally Lipschitz continuous and satisfies the following assumption.
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**Assumption 5.2.** There exists positive definite matrices $S_1, S_2$ with appropriate dimensions such that

$$
\text{trace}[\beta^T(u_1(t), u_2(t - \mu(t)), t)\beta(u_1(t), u_2(t - \mu(t)), t)] \\
\leq u_1^T(t)S_1u_1(t) + u_2^T(t - \mu(t))S_2u_2(t - \mu(t)),
$$

for all $u_1(t), u_2(t - \mu(t)) \in \mathbb{R}^n$.

Consider a general stochastic system $dz(t) = f(z(t), t)dt + g(z(t), t)\,d\omega(t)$, $t \geq 0$ with initial condition $z(0) = z_0 \in \mathbb{R}^n$, where $f : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$. Let $C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+; \mathbb{R}^+)$ denote a collection of all non-negative functions $V$ on $\mathbb{R}^n \times \mathbb{R}^+$ which are twice continuously differentiable in $z$ and once differentiable in $t$. For any $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+; \mathbb{R}^+)$ define $\mathcal{L}V : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$ by

$$
\mathcal{L}V(z(t), t) = V_t(z(t), t) + V_z(z(t), t)f(z(t), t) \\
+ \frac{1}{2}\text{trace}(g^T(z(t), t)V_{zz}(z(t), t)g(z(t), t)),
$$

where

$$
V_t(z(t), t) = \frac{\partial V(z(t), t)}{\partial t}, \\
V_z(z(t), t) = \left(\frac{\partial V(z(t), t)}{\partial z_1}, \frac{\partial V(z(t), t)}{\partial z_2}, ..., \frac{\partial V(z(t), t)}{\partial z_n}\right), \\
V_{zz}(z(t), t) = \frac{\partial^2 V(z(t), t)}{\partial z \partial z}.
$$

By the generalized Itô’s formula, we get that

$$
\mathbb{E}V(u(t), t) = \mathbb{E}V(u(0), 0) + \mathbb{E}\int_0^t \mathcal{L}V(u(s), s)ds.
$$

Let $u(t)$ denote the state trajectory from the initial data value $u(\theta) = \varphi(\theta)$ on $-\nu \leq \theta \leq 0$ in $L^2_{\mathcal{F}_0}([-\nu, 0]; \mathbb{R}^n)$. Clearly, the neural networks (5.2.2) admits a trivial solution $u(t, 0) \equiv 0$ corresponding to the initial data $\varphi = 0$. For simplicity, we can write $u(t; \varphi) = u(t)$.

**Remark 5.2.2.** As specified in [25], the existence of an equilibrium point for neural networks (5.2.2) can be proved by Brower’s fixed-point theorem & Assumption 5.1. In the
consequence, we shall investigate the exponential stability of the trivial solution, which implies that the equilibrium point is unique.

**Definition 5.2.3.** The trivial solution of neural networks (5.2.2) is said to be exponentially stable in the mean square sense if for any \( \varphi \in L_{\mathcal{F}_c}^{2}([-\nu,0];\mathbb{R}^n) \) there exist constants \( \eta \geq 0 \) and \( \chi > 0 \) such that

\[
\mathbb{E}\{u(t)\} \leq \chi e^{-\eta t} \sup_{-\nu \leq s \leq 0} \mathbb{E}\{\|\varphi(s)\|^2\}, \quad \forall \ t > 0.
\]

**Definition 5.2.4.** We introduce the stochastic Lyapunov-Krasovskii functional \( V \in C^{2,1}(\mathbb{R}^+ \times \mathbb{R}^n;\mathbb{R}^+) \) of the system (5.2.2), the weak infinitesimal generator of random process \( \mathcal{L}V \) from \( \mathbb{R}^+ \times \mathbb{R}^n \) to \( \mathbb{R}^+ \) is defined by

\[
\mathcal{L}V(t,u(t)) = \lim_{\Delta t \to 0^+} \frac{1}{\Delta t} [\mathbb{E}\{V((t+\Delta t),u(t+\Delta t))|u(t)\} - V(t,u(t))].
\]

**Remark 5.2.5.** To establish the time-delayed exponential stability criteria for recurrent neural networks (5.2.2), a different LKF is handled in the following Theorem 5.3.1. Further, the fractional segments in time-varying delays are analyzed, which helps to obtain the maximum allowable upper bounds is large when compared with the previous results.

**Remark 5.2.6.** Li & Feng in [86] analyzed the impact of delay interval on recurrent neural networks for ensuring the stability performance. In [181], Wu et al., discussed about the exponential convergence rate of RNNs when both time-delays arised. Moreover, the comparisons for maximum allowable upper bounds of discrete time-varying delays has been listed. The authors in [95] investigates the stability behavior in the sense of exponential for recurrent neural networks with mixed time delays. Further, in [95], the contribution of activation functions is very important to find the stability criteria with less conservatism. The impulsive effects on RNNs and its stability criteria is conversed by Zhou and Zhang in [211] via Lyapunov-Krasovskii functionals.

In all the above mentioned references, the stability problem for recurrent neural networks is considered only with discrete time-varying delays or mixed time delays or impulses, but both time-delays, stochastic noise and impulsive effects has not been taken into...
account and also no investigates about exponential stability, via delays in fractional segments or intervals at a time. So, consider the above facts are very challenged and advanced in this research work.

5.3 Exponential stability results for deterministic systems

In this part, we will develop some novel exponential stability criterion for the time-delayed recurrent neural networks (5.2.2).

Theorem 5.3.1. Suppose that Assumptions 5.1 and 5.2 holds, for given scalars $Y_1 = \text{diag}\{\alpha_1^-, \alpha_2^-, \ldots, \alpha_n^-\}$, $Y_2 = \text{diag}\{\alpha_1^+, \alpha_2^+, \ldots, \alpha_n^+\}$, $Y_3 = \text{diag}\{\beta_1^-, \beta_2^-, \ldots, \beta_n^-\}$, $Y_4 = \text{diag}\{\beta_1^+, \beta_2^+, \ldots, \beta_n^+\}$, $Y_5 = \text{diag}\{\gamma_1^-, \gamma_2^-, \ldots, \gamma_n^-\}$, $Y_6 = \text{diag}\{\gamma_1^+, \gamma_2^+, \ldots, \gamma_n^+\}$, $\mu \leq 1$, if there exists symmetric positive definite matrices $K, L_i (i = 1, 2, 3, 4), \hat{L}_j (j = 1, 2)$, $M_1, M_2, M_4, M_5, M_6$ and $M_3 = \begin{bmatrix} M_{311} & M_{312} & M_{313} \\ * & M_{322} & M_{323} \\ * & * & M_{333} \end{bmatrix}$, positive diagonal matrices $\Lambda_p = \text{diag}\{\lambda_{p1}, \lambda_{p2}, \ldots, \lambda_{pn}\}$, $(p = 1, 2, 3)$, $\Theta_l = \text{diag}\{\theta_{l1}, \theta_{l2}, \ldots, \theta_{ln}\}$, $(l = 1, 2, 3)$, $N_q$, $(q = 1, 2, 3)$, such that the following LMIs holds:

$$K \leq \delta I,$$  

$$G_k^T K G_k - K \preceq 0,$$  

Case (i) When $0 \leq \mu(t) \leq \frac{\mu}{3}$

$$\Sigma_1 = \begin{bmatrix} \Gamma_{1,1} & \Gamma_{1,2} & \Gamma_{1,3} & \Gamma_{1,4} & \Gamma_{1,5} & 0 & 0 & 0 & 0 & 0 & \Gamma_{1,10} & \Gamma_{1,11} & 0 & 0 \\ * & * & \Gamma_{2,3} & \Gamma_{2,4} & \Gamma_{2,5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \Gamma_{3,4} & 0 & 0 & 0 & \Gamma_{3,6} & \Gamma_{3,7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Gamma_{4,5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \Gamma_{5,6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Gamma_{6,7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \Gamma_{7,8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \Gamma_{8,9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \end{bmatrix} \preceq 0,$$  

$$\Sigma_2 = \begin{bmatrix} \Gamma_{1,11} & \Gamma_{1,12} & \Gamma_{1,13} & \Gamma_{2,1} & \Gamma_{2,2} & \Gamma_{2,3} & \Gamma_{2,4} & \Gamma_{2,5} & \Gamma_{2,6} & \Gamma_{2,7} & \Gamma_{2,8} & \Gamma_{2,9} & \Gamma_{2,10} & \Gamma_{2,11} & \Gamma_{2,12} & \Gamma_{2,13} \\ * & * & * & \Gamma_{3,1} & \Gamma_{3,2} & \Gamma_{3,3} & \Gamma_{3,4} & \Gamma_{3,5} & \Gamma_{3,6} & \Gamma_{3,7} & \Gamma_{3,8} & \Gamma_{3,9} & \Gamma_{3,10} & \Gamma_{3,11} & \Gamma_{3,12} & \Gamma_{3,13} \\ * & * & * & * & \Gamma_{4,1} & \Gamma_{4,2} & \Gamma_{4,3} & \Gamma_{4,4} & \Gamma_{4,5} & \Gamma_{4,6} & \Gamma_{4,7} & \Gamma_{4,8} & \Gamma_{4,9} & \Gamma_{4,10} & \Gamma_{4,11} & \Gamma_{4,12} \\ * & * & * & * & * & \Gamma_{5,1} & \Gamma_{5,2} & \Gamma_{5,3} & \Gamma_{5,4} & \Gamma_{5,5} & \Gamma_{5,6} & \Gamma_{5,7} & \Gamma_{5,8} & \Gamma_{5,9} & \Gamma_{5,10} & \Gamma_{5,11} \\ * & * & * & * & * & * & \Gamma_{6,1} & \Gamma_{6,2} & \Gamma_{6,3} & \Gamma_{6,4} & \Gamma_{6,5} & \Gamma_{6,6} & \Gamma_{6,7} & \Gamma_{6,8} & \Gamma_{6,9} & \Gamma_{6,10} \\ * & * & * & * & * & * & * & \Gamma_{7,1} & \Gamma_{7,2} & \Gamma_{7,3} & \Gamma_{7,4} & \Gamma_{7,5} & \Gamma_{7,6} & \Gamma_{7,7} & \Gamma_{7,8} & \Gamma_{7,9} & \Gamma_{7,10} \\ * & * & * & * & * & * & * & * & \Gamma_{8,1} & \Gamma_{8,2} & \Gamma_{8,3} & \Gamma_{8,4} & \Gamma_{8,5} & \Gamma_{8,6} & \Gamma_{8,7} & \Gamma_{8,8} & \Gamma_{8,9} & \Gamma_{8,10} \\ * & * & * & * & * & * & * & * & * & \Gamma_{9,1} & \Gamma_{9,2} & \Gamma_{9,3} & \Gamma_{9,4} & \Gamma_{9,5} & \Gamma_{9,6} & \Gamma_{9,7} & \Gamma_{9,8} & \Gamma_{9,9} & \Gamma_{9,10} \\ * & * & * & * & * & * & * & * & * & * & \Gamma_{10,1} & \Gamma_{10,2} & \Gamma_{10,3} & \Gamma_{10,4} & \Gamma_{10,5} & \Gamma_{10,6} & \Gamma_{10,7} & \Gamma_{10,8} & \Gamma_{10,9} & \Gamma_{10,10} \\ * & * & * & * & * & * & * & * & * & * & * & \Gamma_{11,1} & \Gamma_{11,2} & \Gamma_{11,3} & \Gamma_{11,4} & \Gamma_{11,5} & \Gamma_{11,6} & \Gamma_{11,7} & \Gamma_{11,8} & \Gamma_{11,9} & \Gamma_{11,10} \\ * & * & * & * & * & * & * & * & * & * & * & * & \Gamma_{12,1} & \Gamma_{12,2} & \Gamma_{12,3} & \Gamma_{12,4} & \Gamma_{12,5} & \Gamma_{12,6} & \Gamma_{12,7} & \Gamma_{12,8} & \Gamma_{12,9} & \Gamma_{12,10} \\ * & * & * & * & * & * & * & * & * & * & * & * & * & \Gamma_{13,1} & \Gamma_{13,2} & \Gamma_{13,3} & \Gamma_{13,4} & \Gamma_{13,5} & \Gamma_{13,6} & \Gamma_{13,7} & \Gamma_{13,8} & \Gamma_{13,9} & \Gamma_{13,10} \\ \end{bmatrix} \preceq 0, \quad (5.3.3)$$
Case (ii) When $\frac{\mu}{3} \leq \mu(t) \leq \frac{2}{3} \mu$

\[
\varepsilon_2 = \begin{bmatrix}
\Gamma_{1,1} & \Gamma_{1,2} & \Gamma_{1,3} & \Gamma_{1,4} & \Gamma_{1,5} & 0 & 0 & 0 & 0 & \gamma_{1,10} & \Gamma_{1,11} & 0 & 0 \\
0 & \Gamma_{2,2} & \Gamma_{2,3} & \Gamma_{2,4} & \Gamma_{2,5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\ast & \ast & \gamma_{3,3} & \Gamma_{3,4} & 0 & 0 & 0 & \gamma_{3,10} & \Gamma_{3,11} & 0 & 0 & 0 & 0 \\
\ast & \ast & \ast & \Gamma_{4,4} & 0 & 0 & 0 & 0 & \gamma_{4,10} & \Gamma_{4,11} & 0 & 0 & 0 \\
\ast & \ast & \ast & \ast & \Gamma_{5,5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \Gamma_{6,6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \Gamma_{7,7} & 0 & 0 & 0 & 0 & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \Gamma_{8,8} & 0 & 0 & 0 & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \Gamma_{9,9} & 0 & 0 & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \Gamma_{10,10} & \Gamma_{10,11} & \Gamma_{10,12} & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \Gamma_{11,11} & \Gamma_{11,12} & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \Gamma_{12,12} & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \Gamma_{13,13} \\
\end{bmatrix} < 0, \quad (5.3.4)
\]

Case (iii) When $\frac{2}{3} \mu \leq \mu(t) \leq \mu$

\[
\varepsilon_3 = \begin{bmatrix}
\Gamma_{1,1} & \Gamma_{1,2} & \Gamma_{1,3} & \Gamma_{1,4} & \Gamma_{1,5} & 0 & 0 & 0 & 0 & \delta_{1,10} & \Gamma_{1,11} & 0 & 0 \\
0 & \Gamma_{2,2} & \Gamma_{2,3} & \Gamma_{2,4} & \Gamma_{2,5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\ast & \ast & \ast & \ast & \Gamma_{3,3} & 0 & 0 & 0 & \delta_{3,10} & \Gamma_{3,11} & \Gamma_{3,12} & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \Gamma_{4,4} & 0 & 0 & 0 & \delta_{4,10} & \Gamma_{4,11} & \Gamma_{4,12} & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \Gamma_{5,5} & 0 & 0 & 0 & \delta_{5,10} & \Gamma_{5,11} & \Gamma_{5,12} & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \Gamma_{6,6} & 0 & 0 & 0 & \delta_{6,10} & \Gamma_{6,11} & \Gamma_{6,12} & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \Gamma_{7,7} & 0 & 0 & 0 & \delta_{7,10} & \Gamma_{7,11} & \Gamma_{7,12} & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \Gamma_{8,8} & 0 & 0 & 0 & \delta_{8,10} & \Gamma_{8,11} & \Gamma_{8,12} & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \Gamma_{9,9} & 0 & 0 & 0 & \delta_{9,10} & \Gamma_{9,11} & \Gamma_{9,12} & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \Gamma_{10,10} & \Gamma_{10,11} & \Gamma_{10,12} & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \Gamma_{11,11} & \Gamma_{11,12} & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \Gamma_{12,12} & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \Gamma_{13,13} \\
\end{bmatrix} < 0, \quad (5.3.5)
\]

where

\[
\Gamma_{1,1} = 2cK - KA - KA + \delta S_1 + 4cY_1A\Theta_1 - 4cY_2\Lambda_1A - A(Y_2\Lambda_1 - Y_1\Theta_1)
\]

\[
-(Y_2\Lambda_1 - Y_1\Theta_1)A + \eta_1,
\]

\[
\eta_1 = \sum_{i=1}^{4} L_i + \sum_{j=1}^{2} L_j + M_1 + M_{3,1} + (\frac{\mu}{3})^2 A
\]

\[
[M_4 + M_5 + M_6]A - 2Y_1N_1Y_2 - e^{-2c_2^2}M_6,
\]

\[
\delta_{1,10} = M_{3,12} + e^{-2c_2^2}M_6,
\]

\[
\Gamma_{1,2} = KW_0 - 2(A - W_0 Y_1)\Theta_1 + 2c(Y_2 W_0 + A)\Lambda_1 + (\Theta_1 - \Lambda_1) + (Y_2\Lambda_1
\]

\[
- Y_1\Theta_1)W_0 + \eta_2 \Gamma_{1,11} = M_{3,13}, \quad \Gamma_{2,2} = 4c\Theta_1W_0 - 4c\Lambda_1W_0 + \eta_3
\]

\[
\eta_2 = -AW_0(\sum_{j=1}^{2} \tilde{L}_j) - (\frac{\mu}{3})^2[M_4 + M_5 + M_6]AW_0 + N_1(Y_1 + Y_2),
\]

\[
\Gamma_{1,4} = KW_1 - 2cY_1W_1 + 2cY_2W_1\Lambda_1 + (Y_2\Lambda_1 - Y_1\Theta_1)W_1 - AW_1(\sum_{j=1}^{2} \tilde{L}_j)
\]

\[
-(\frac{\mu}{3})^2[M_4 + M_5 + M_6]AW_1, \quad \Gamma_{6,6} = -e^{-2c_2^2}M_1 - 2Y_5N_3Y_{6,1}
\]

\[
\Gamma_{10,12} = -e^{-2c_2^2}M_{3,13}, \quad \gamma_{10,11} = M_{3,33} - e^{-2c_2^2}M_{3,12}, \quad \Gamma_{1,3} = \frac{3}{2}e^{-2c_2^2}M_6,
\]

\[
\eta_3 = 2(\Theta_1 - \Lambda_1)W_0 + W_0(\sum_{j=1}^{2} \tilde{L}_j)W_0 + (\frac{\mu}{3})^2W_0(M_4 + M_5 + M_6)W_0 - 2N_1,
\]

\[
\Gamma_{3,3} = \delta S_2 - e^{-2c_2^2}(1 - \mu)L_1 - e^{-2c_2^2}L_4 - 3e^{-2c_2^2}M_6 - 2Y_3N_2Y_4,
\]

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\( \Gamma_{5,5} = \left( \frac{\rho}{3} \right)^2 W_2 [M_4 + M_5 + M_6] W_2 - \frac{1}{\sigma} M_2 + W_2 \left( \sum_{j=1}^{2} \bar{L}_j \right) W_2, \quad \Gamma_{4,4} = -2N_2, \)

\( \Gamma_{7,7} = -2N_3, \quad \Gamma_{8,8} = -e^{-2c\delta} \bar{L}_1 (1 - \mu), \quad \Gamma_{9,9} = -e^{-2c\delta} \bar{L}_2, \)

\( \Gamma_{10,10} = -e^{-2c\delta} \delta_4 M_6 - e^{-2c\delta} \delta_5 M_5 + \eta_4, \quad \eta_4 = M_{322} - e^{-2c\delta} \delta_3 M_{311} - e^{-2c\delta} \delta_2 L_2, \)

\( \Gamma_{13,13} = \sigma M_2, \quad \Gamma_{3,4} = N_2 (Y_3 + Y_4), \quad \Gamma_{10,11} = M_{322} - e^{-2c\delta} \delta_3 M_{312} + e^{-4c\delta} \delta_5 M_5, \)

\( \Gamma_{2,4} = 2z \Theta_1 W_1 - 2z \Lambda_1 W_1 + (\Theta_1 - \Lambda_1) W_1 + W_0 (\frac{\rho}{3})^2 [M_4 + M_5 + M_6] W_1, \)

\( \Gamma_{3,10} = -e^{-2c\delta} \delta_5 M_6, \quad \Gamma_{11,12} = -e^{-2c\delta} \delta_3 M_{322} + e^{-2c\delta} \delta_4 M_4, \quad \Gamma_{6,7} = N_3 (Y_5 + Y_6), \)

\( \Gamma_{11,11} = -e^{-4c\delta} \delta_3 L_3 + M_{333} - e^{-2c\delta} \delta_5 M_{322} - e^{-2c\delta} \delta_4 M_4 - e^{-4c\delta} \delta_5 M_5, \quad \Gamma_{1,10} = M_{312}, \)

\( \Gamma_{2,5} = 2z \Theta_1 W_2 - 2z \Lambda_1 W_2 + (\Theta_1 - \Lambda_1) W_2 + W_0 (\frac{\rho}{3})^2 [M_4 + M_5 + M_6] W_2, \)

\( \Gamma_{3,8} = -2z Y_3 \Theta_2 + 2z Y_4 + (Y_4 \Lambda_2 - Y_3 \Theta_2), \quad \Gamma_{4,8} = 2z \Theta_2 - 2z \Lambda_2 + (\Theta_2 - \Lambda_2), \)

\( \delta_{3,12} = e^{-2c\delta} \delta_4 M_4, \quad \delta_{6,9} = -2z Y_5 \Theta_3 + 2z Y_6 + (Y_6 \Lambda_3 - Y_5 \Theta_3), \quad \gamma_{7,9} = 2z \Theta_3 - 2z \Lambda_3 + (\Theta_3 - \Lambda_3), \quad \delta_{3,11} = e^{-2c\delta} \delta_3 M_4, \quad \delta_{11,12} = -e^{-2c\delta} \delta_3 M_{322}, \)

\( \gamma_{3,3} = -e^{-2c\delta} (1 - \mu) L_1 + \delta S_2 - e^{-2c\delta} \delta_4 L_4 - \frac{3}{2} e^{-4c\delta} \delta_5 M_5, \quad \gamma_{3,10} = e^{-4c\delta} \delta_5 M_5, \)

\( \gamma_{3,11} = e^{-4c\delta} \delta_3 M_5, \quad \gamma_{1,10} = M_{312} + e^{-2c\delta} \delta_5 M_6, \quad \delta_{3,3} = -e^{-2c\delta} (1 - \mu) L_1 + \delta S_2 - e^{-2c\delta} \delta_4 L_4 - 3 M_4, \quad \gamma_{11,12} = -e^{-2c\delta} \delta_5 M_4 - e^{-2c\delta} \delta_3 M_{322} - e^{-2c\delta} \delta_2 L_2, \)

\( \Gamma_{1,5} = K W_2 - 2z Y_1 \Theta_1 W_2 + 2z Y_2 \Lambda_1 W_2 + (Y_2 \Lambda_1 - Y_1 \Theta_1) W_2 + \left( \sum_{j=1}^{2} \bar{L}_j \right) W_2. \)

Then the equilibrium point of recurrent neural networks (5.2.2) is exponentially stable in the mean square sense with the exponential convergence rate \( \varsigma \).

**Proof.** Consider the following Lyapunov-Krasovskii functional:

\[
V(u(t)) = \sum_{i=1}^{n} V_i(u(t)),
\]

where

\[
V_1(u(t)) = e^{2cT} u^T(t) K u(t),
\]
\[
V_2(u(t)) = 2 e^{2cT} \sum_{i=1}^{n} \left[ \left( \int_{0}^{u_{i}(t)} \vartheta(f_{i}(s) - \alpha_{i} s) ds + \int_{0}^{u_{i}(t)} \lambda_{i}(\alpha_{i} s - f_{i}(s)) ds \right) \right. \\
\left. + \left( \int_{0}^{u_{i}(t)} \vartheta(\tilde{g}_{i}(s) - \tilde{\beta}_{i} s) ds + \int_{0}^{u_{i}(t)} \lambda_{i}(\tilde{\beta}_{i} s - \tilde{g}_{i}(s)) ds \right) \right]
\]
\[ V_3(u(t)) = \int_{t-\mu(t)}^t e^{2c_s u^T(s)} L_1 u(s) ds + \int_{t-\frac{\mu}{3}}^t e^{2c_s u^T(s)} L_2 u(s) ds + \int_{t-\frac{2\mu}{3}}^t e^{2c_s u^T(s)} L_3 u(s) ds + \int_{t-\frac{2\mu}{3}}^t e^{2c_s u^T(s)} L_4 u(s) ds + \int_{t-\mu(t)}^t e^{2c_s u^T(s)} L_5 u(s) ds + \int_{t-\mu(t)}^t e^{2c_s u^T(s)} L_6 u(s) ds, \]

\[ V_4(u(t)) = \int_{t-\sigma}^t e^{2c_s u^T(s)} M_1 u(s) ds, \]

\[ V_5(u(t)) = \int_{-\sigma}^0 \int_{t+s}^t e^{2c_s h^T(u(s))} M_2 h(u(s)) ds, \]

\[ V_6(u(t)) = \int_{t-\frac{\mu}{3}}^t e^{2c_s u^T(s)} \begin{bmatrix} u(s) \\ u(s-\frac{\mu}{3}) \\ u(s-\frac{2\mu}{3}) \end{bmatrix}^T \begin{bmatrix} u(s) \\ u(s-\frac{\mu}{3}) \\ u(s-\frac{2\mu}{3}) \end{bmatrix} ds, \]

\[ V_7(u(t)) = \frac{\mu}{3} \int_{-\frac{2\mu}{3}}^t \int_{t+\theta}^t e^{2c_s u^T(s)} M_4 u(s) ds d\theta + \frac{\mu}{3} \int_{-\frac{2\mu}{3}}^t \int_{t+\theta}^t e^{2c_s u^T(s)} M_5 u(s) ds d\theta + \frac{\mu}{3} \int_{-\frac{2\mu}{3}}^t \int_{t+\theta}^t e^{2c_s u^T(s)} M_6 u(s) ds d\theta. \]

For \( t \neq t_k, t > 0 \), for all \( k \in \mathbb{Z}^+ \). Then, by Assumption 5.2 and the condition (5.3.1), we have

\[ \text{trace}[\rho^T(u_1, u_2, t) K \rho(u_1, u_2, t)] \leq \delta [u_1^T S_1 u_1 + u_2^T S_2 u_2]. \]

Hence, by the aid of Lemma 1.10.7, 1.10.8 and the conditions (5.2.3)-(5.2.5) it follows from (5.2.2) and the definition of weak infinitesimal operator \((LV)\) that

\[ LV(u(t)) = \sum_{i=1}^7 LV_i(u(t)), \]

where

\[ LV_1(u(t)) = 2c e^{2c t} u^T(t) Ku(t) + 2c e^{2c t} u^T(t) K \left[ -Au(t) + W_0 \tilde{f}(u(t)) \right] + \text{trace} \left[ \tilde{h}(u(t)) \right] + e^{2c t} \]

\[ \times \text{trace} \left[ \rho^T(u(t), u(t-\mu(t)), t) K \rho(u(t), u(t-\mu(t)), t) \right] \leq e^{2c t} u^T(t) [2cK - 2KA + \delta S_1] u(t) + e^{2c t} u^T(t) [2KW_0] \]

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\[ \mathcal{L}V_2(u(t)) = 4\xi e^{2st} \sum_{i=1}^{n} \left[ \int_0^{u_i(t)} \theta_{1i}(\tilde{f}_i(s) - \alpha_i^- s)ds + \int_0^{u_i(t)} \lambda_{1i}(\alpha_i^+ s - \tilde{f}_i(s))ds \right] + 2e^{2st} \left[ (f^T(u(t)) - u^T(t)Y_1)\Theta_1 \hat{u}(t) + (u^T(t)Y_2 - f^T(u(t)))\Lambda_1 \hat{u}(t) \right] + 4\xi e^{2st} \\
\times \sum_{i=1}^{n} \left[ \int_0^{u_i(t)} \theta_{2i}(\tilde{g}_i(s) - \beta_i^- s)ds + \int_0^{u_i(t)} \lambda_{2i}(\beta_i^+ s - \tilde{g}_i(s))ds \right] + 2e^{2st} \left[ (\tilde{g}^T(u(t) - \mu(t))) - u^T(t - \sigma) \right] \\
\times (Y_4 - \tilde{g}^T(u(t) - \mu(t)))\Lambda_2 \hat{u}(t - \mu(t)) - u^T(t - \mu(t))Y_3 - \tilde{h}^T(u(t - \sigma)) - u^T(t - \sigma))Y_5 \\
\times \Theta_3 \hat{u}(t - \sigma) + (u^T(t - \sigma)Y_6 - \tilde{h}^T(u(t - \sigma)))\Lambda_3 \hat{u}(t - \sigma) \right] \leq 4\xi e^{2st} \left[ (f^T(u(t)) - u^T(t)Y_1)\Theta_1 \hat{u}(t) + (u^T(t)Y_2 - f^T(u(t))) \right] \\
\times \Lambda_1 \hat{u}(t) + 2e^{2st} \left[ f^T(u(t)))(\Theta_1 - \Lambda_1) + u^T(t)(Y_2\Lambda_1 - Y_1) \right. \\
\times \Theta_1] \hat{u}(t) + 4\xi e^{2st} \left[ (\tilde{g}^T(u(t) - \mu(t))) - u^T(t - \mu(t))Y_3)\Theta_2 \right. \\
\times \theta_1(u(t - \mu(t)) + (u^T(t - \mu(t))Y_4 - \tilde{g}^T(u(t - \mu(t))))\Lambda_2 \\
\times u(t - \mu(t))] + 2e^{2st} \left[ \tilde{g}^T(u(t) - \mu(t)))\Theta_2 - \Lambda_2 \right. \\
\times \hat{u}(t - \mu(t)))(Y_4\Lambda_2 - Y_3\Theta_2) \hat{u}(t - \mu(t)) + 4\xi e^{2st} \\
\times [\tilde{h}^T(u(t - \sigma)) - u^T(t - \sigma)Y_5)\Theta_3 \hat{u}(t - \sigma) + (u^T(t - \sigma) \\
\times Y_6 - \tilde{h}^T(u(t - \sigma)))\Lambda_3 \hat{u}(t - \sigma)] + 2e^{2st}[\tilde{h}^T(u(t - \sigma)) \\
\times (\Theta_3 - \Lambda_3) + u^T(t - \sigma)(Y_6\Lambda_3 - Y_5\Theta_3) \hat{u}(t - \sigma), \quad (5.3.8) \]

\[ \mathcal{L}V_3(u(t)) \leq e^{2st} u^T(t) \left( \sum_{i=1}^{4} L_i \right) u(t) + e^{2st} \hat{u}^T(t) \left( \sum_{j=1}^{2} L_j \right) \hat{u}(t) - e^{2st} \left( t - \frac{\sigma}{2} \right) \]

\times (1 - \mu)u^T(t - \mu(t))L_1u(t - \mu(t)) - e^{2st} \left( t - \frac{\sigma}{2} \right)
\begin{align*}
\times u^T(t - \frac{\bar{\mu}}{3})L_2 u(t - \frac{\bar{\mu}}{3}) - e^{2\zeta(t - \frac{\bar{\mu}}{3})}\frac{2\bar{\mu}}{3}L_3 \\
\times u(t - \frac{2\bar{\mu}}{3}) - e^{2\zeta(t - \frac{\bar{\mu}}{3})}u^T(t - \frac{2\bar{\mu}}{3})L_4 u(t - \frac{\bar{\mu}}{3}) - e^{2\zeta(t - \frac{\bar{\mu}}{3})} \\
\times u^T(t - \mu(t))L_1 \hat{u}(t - \mu(t))(1 - \mu) - e^{2\zeta(t - \sigma)} \\
\times u^T(t - \sigma)L_2 \hat{u}(t - \sigma),
\end{align*}

(5.3.9)

\begin{align*}
\mathcal{L}V_4(u(t)) &= e^{2\zeta t}u^T(t)M_1 u(t) - e^{2\zeta(t - \sigma)}u^T(t - \sigma)M_1 u(t - \sigma),
\end{align*}

(5.3.10)

\begin{align*}
\mathcal{L}V_5(u(t)) &\leq \sigma e^{2\zeta t}h^T(u(t))M_2 h(u(t)) - \int_{t - \sigma}^{t} e^{2\zeta \eta}h^T(u(s))M_2 h(u(s))ds \\
&\leq \sigma e^{2\zeta t}h^T(u(t))M_2 h(u(t)) - \frac{1}{\sigma} \sigma e^{2\zeta t} \left( \int_{t - \sigma}^{t} h^T(u(s))ds \right)^T \\
&\times M_2 \left( \int_{t - \sigma}^{t} h^T(u(s))ds \right),
\end{align*}

(5.3.11)

\begin{align*}
\mathcal{L}V_6(u(t)) &= e^{2\zeta t} \begin{bmatrix}
    u(t) \\
    u(t - \frac{\bar{\mu}}{3}) \\
    u(t - \frac{\bar{\mu}}{3})
\end{bmatrix}^T M_3 \begin{bmatrix}
    u(t) \\
    u(t - \frac{\bar{\mu}}{3}) \\
    u(t - \frac{\bar{\mu}}{3})
\end{bmatrix} - e^{2\zeta(t - \frac{\bar{\mu}}{3})} \\
&\times \begin{bmatrix}
    u(t - \frac{\bar{\mu}}{3}) \\
    u(t - \frac{\bar{\mu}}{3}) \\
    u(t - \frac{\bar{\mu}}{3})
\end{bmatrix}^T M_3 \begin{bmatrix}
    u(t - \frac{\bar{\mu}}{3}) \\
    u(t - \frac{\bar{\mu}}{3}) \\
    u(t - \frac{\bar{\mu}}{3})
\end{bmatrix},
\end{align*}

(5.3.12)

\begin{align*}
\mathcal{L}V_7(u(t)) &= \left( \frac{\bar{\mu}}{3} \right)^2 e^{2\zeta t}u^T(t)M_4 + M_5 + M_6 \hat{u}(t) - \frac{\bar{\mu}}{3} \int_{t - \frac{\bar{\mu}}{3}}^{t} e^{2\zeta s} \\
&\times u^T(s)M_5 \hat{u}(s)ds - \frac{\bar{\mu}}{3} \int_{t - \frac{\bar{\mu}}{3}}^{t} e^{2\zeta s} \hat{u}^T(s)M_5 \hat{u}(s) \\
&\times ds - \frac{\bar{\mu}}{3} \int_{t - \frac{\bar{\mu}}{3}}^{t} e^{2\zeta s} \hat{u}^T(s)M_4 \hat{u}(s)ds.
\end{align*}

(5.3.13)

**Case (i):**

When $0 \leq \mu(t) \leq \frac{\bar{\mu}}{3}$. By utilizing the bounds lemma of [137], we can get that

\[ -\frac{\bar{\mu}}{3} \int_{t - \frac{\bar{\mu}}{3}}^{t} e^{2\zeta s} \hat{u}^T(s)M_5 \hat{u}(s)ds \]

\[ \leq e^{2\zeta(t - \frac{\bar{\mu}}{3})} \begin{bmatrix}
    u(t) \\
    u(t - \frac{\bar{\mu}}{3}) \\
    u(t - \frac{\bar{\mu}}{3})
\end{bmatrix}^T \begin{bmatrix}
    -M_6 & 2M_6 & 0 \\
    M_6 & -3M_6 & 2M_6 \\
    0 & 2M_6 & -M_6
\end{bmatrix} \begin{bmatrix}
    u(t) \\
    u(t - \frac{\bar{\mu}}{3}) \\
    u(t - \frac{\bar{\mu}}{3})
\end{bmatrix} \]

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\[
\begin{align*}
\times & \begin{bmatrix} u(t) \\ u(t - \mu(t)) \\ u(t - \frac{2}{3} \bar{\mu}) \end{bmatrix} \\
\frac{\bar{\mu}}{3} \int_{t-\bar{\mu}}^{t-\frac{2}{3} \bar{\mu}} e^{2 \zeta(s)} \hat{u}^T(s) M_4 \hat{u}(s) ds \\
& \leq -e^{2 \zeta(t-\bar{\mu})} \begin{bmatrix} u(t - \frac{2}{3} \bar{\mu}) - u(t - \bar{\mu}) \\ -u(t - \bar{\mu}) \end{bmatrix}^T M_4 \begin{bmatrix} u(t - \frac{2}{3} \bar{\mu}) \\ -u(t - \bar{\mu}) \end{bmatrix} \\
-\frac{\bar{\mu}}{3} \int_{t-\frac{2}{3} \bar{\mu}}^{t-\frac{2}{3} \bar{\mu}} e^{2 \zeta(s)} \hat{u}^T(s) M_5 \hat{u}(s) ds \\
& \leq -e^{2 \zeta(t-\frac{2}{3} \bar{\mu})} \begin{bmatrix} u(t - \frac{2}{3} \bar{\mu}) - u(t - \frac{2}{3} \bar{\mu}) \\ -u(t - \frac{2}{3} \bar{\mu}) \end{bmatrix}^T M_5 \begin{bmatrix} u(t - \frac{2}{3} \bar{\mu}) \\ -u(t - \frac{2}{3} \bar{\mu}) \end{bmatrix} \\
\end{align*}
\]

**Case (ii):**

When $\frac{\rho}{3} \leq \mu(t) \leq \frac{2}{3} \bar{\mu}$. Then, we obtain

\[
\begin{align*}
\frac{\bar{\mu}}{3} \int_{t-\frac{2}{3} \bar{\mu}}^{t-\frac{2}{3} \bar{\mu}} e^{2 \zeta(s)} \hat{u}^T(s) M_5 \hat{u}(s) ds \\
& \leq e^{2 \zeta(t-\frac{2}{3} \bar{\mu})} \begin{bmatrix} u(t - \frac{2}{3} \bar{\mu}) \\ u(t - \mu(t)) \\ u(t - \frac{2}{3} \bar{\mu}) \end{bmatrix}^T \begin{bmatrix} -M_5 & 2M_5 & 0 \\ M_5 & -3M_5 & 2M_5 \\ 0 & 2M_5 & -M_5 \end{bmatrix} \begin{bmatrix} u(t - \frac{2}{3} \bar{\mu}) \\ u(t - \mu(t)) \\ u(t - \frac{2}{3} \bar{\mu}) \end{bmatrix} \\
-\frac{\bar{\mu}}{3} \int_{t-\frac{2}{3} \bar{\mu}}^{t-\frac{2}{3} \bar{\mu}} e^{2 \zeta(s)} \hat{u}^T(s) M_4 \hat{u}(s) ds \\
& \leq -e^{2 \zeta(t-\bar{\mu})} \begin{bmatrix} u(t - \frac{2}{3} \bar{\mu}) - u(t - \bar{\mu}) \\ -u(t - \bar{\mu}) \end{bmatrix}^T M_4 \begin{bmatrix} u(t - \frac{2}{3} \bar{\mu}) \\ -u(t - \bar{\mu}) \end{bmatrix} \\
-\frac{\bar{\mu}}{3} \int_{t-\frac{2}{3} \bar{\mu}}^{t-\frac{2}{3} \bar{\mu}} e^{2 \zeta(s)} \hat{u}^T(s) M_6 \hat{u}(s) ds \\
& \leq -e^{2 \zeta(t-\frac{2}{3} \bar{\mu})} \begin{bmatrix} u(t) - u(t - \frac{2}{3} \bar{\mu}) \\ u(t) - u(t - \frac{2}{3} \bar{\mu}) \end{bmatrix}^T M_6 \begin{bmatrix} u(t) - u(t - \frac{2}{3} \bar{\mu}) \\ u(t) - u(t - \frac{2}{3} \bar{\mu}) \end{bmatrix}.
\end{align*}
\]
Case (iii): 

When \( \frac{2}{3} \bar{\mu} \leq \mu(t) \leq \bar{\mu} \). From the bounds lemma of [137], one can obtain that

\[
-\frac{\bar{\mu}}{3} \int_{t-\bar{\mu}}^{t-\frac{2}{3}\bar{\mu}} e^{2csT(s)}M_4\hat{u}(s)ds \\
\leq e^{2c(t-\bar{\mu})} \left[ \begin{array}{c} u(t-\frac{2}{3}\bar{\mu}) \\ u(t-\bar{\mu}) \end{array} \right]^T \left[ \begin{array}{ccc} -M_4 & M_4 & 0 \\ M_4 & -3M_4 & 2M_4 \\ 0 & 2M_4 & -M_4 \end{array} \right] \\
\times \left[ \begin{array}{c} u(t-\frac{2}{3}\bar{\mu}) \\ u(t-\bar{\mu}) \end{array} \right] \\
(5.3.20)
\]

\[
-\frac{\bar{\mu}}{3} \int_{t-\frac{\bar{\mu}}{3}}^{t} e^{2csT(s)}M_6\hat{u}(s)ds \\
\leq -e^{2c(t-\frac{\bar{\mu}}{3})} \left[ u(t) - u(t-\frac{\bar{\mu}}{3}) \right]^T M_6 \left[ u(t) - u(t-\frac{\bar{\mu}}{3}) \right] \\
(5.3.21)
\]

\[
-\frac{\bar{\mu}}{3} \int_{t-\frac{2}{3}\bar{\mu}}^{t-\frac{\bar{\mu}}{3}} e^{2csT(s)}M_5\hat{u}(s)ds \\
\leq -e^{2c(t-\frac{2}{3}\bar{\mu})} \left[ u(t-\frac{\bar{\mu}}{3}) - u(t-\frac{2}{3}\bar{\mu}) \right]^T M_5 \left[ u(t-\frac{\bar{\mu}}{3}) - u(t-\frac{2}{3}\bar{\mu}) \right] \\
\quad - u(t-\frac{2}{3}\bar{\mu}) \right]. \\
(5.3.22)
\]

Additionally, to derive the main results with reduced conservatism, as a result of (5.2.3), (5.2.4) & (5.2.5), we insert the inequalities with positive diagonal matrices \( N_1, N_2 & N_3 \), as follows:

\[
e^{2cT\left[ -2f^T(u(t))N_1f(u(t)) + 2u^T(t)N_1(Y_1 + Y_2)f(u(t)) \\
\quad - 2u^T(t)Y_1N_1Y_2u(t) \right] } \geq 0, \\
(5.3.23)
\]

\[
e^{2cT\left[ -2g^T(u(t-\mu(t)))N_2g(u(t-\mu(t))) + 2u^T(t-\mu(t))N_2(Y_3 + Y_4) \\
\times g(u(t-\mu(t))) - 2u^T(t-\mu(t))Y_3N_2Y_4 \\
\times u(t-\mu(t)) \right] } \geq 0, \\
(5.3.24)
\]

\[
e^{2cT\left[ -2h^T(u(t-\sigma))N_3h(u(t-\sigma)) + 2u^T(t-\sigma)N_3(Y_5 + Y_6) \\
\times h(u(t-\sigma)) - 2u^T(t-\sigma)Y_5N_3Y_6 \\
\times u(t-\sigma) \right] } \geq 0. \\
(5.3.25)
\]
Case (i) : When $0 \leq \mu(t) \leq \frac{p}{4}$, in accordance to (5.3.3), from the equations (5.3.7)-(5.3.16), (5.3.23)-(5.3.25), we have

$$\mathcal{L}V(u(t)) \leq e^{2\xi t} \xi^T(t) \Sigma_1 \xi(t) \leq 0,$$  \hfill (5.3.26)

where

\[
\xi^T(t) = \begin{bmatrix} u^T(t) \bar{f}^T(u(t)) u^T(t - \mu(t)) \bar{g}^T(u(t - \mu(t))) \left( \int_{t-\sigma}^{t} \bar{h}(u(s)) ds \right)^T \\
\quad u^T(t - \sigma) \bar{h}^T(u(t - \sigma)) \bar{u}^T(t - \mu(t)) u^T(t - \sigma) u^T(t - \frac{2}{3}) \\
\quad u^T(t - \frac{2}{3} \bar{\mu}) u^T(t - \bar{\mu}) \bar{h}^T(u(t)) \end{bmatrix}.
\]

Case (ii) : When $\frac{p}{4} \leq \mu(t) \leq \frac{2}{3} \bar{\mu}$. In accordance to (5.3.4), from the equations (5.3.7)-(5.3.13), (5.3.17)-(5.3.19), (5.3.23)-(5.3.25), we get

$$\mathcal{L}V(u(t)) \leq e^{2\xi t} \xi^T(t) \Sigma_2 \xi(t) \leq 0.$$  \hfill (5.3.27)

Case (iii) : When $\frac{2}{3} \bar{\mu} \leq \mu(t) \leq \bar{\mu}$. In accordance to (5.3.5), by the equations (5.3.7)-(5.3.13), (5.3.20)-(5.3.25), we obtain that

$$\mathcal{L}V(u(t)) \leq e^{2\xi t} \xi^T(t) \Sigma_3 \xi(t) \leq 0.$$  \hfill (5.3.28)

Now, for $t = t_k$, by the condition (5.3.2), it can be easily get that

\[
V_1(u(t_k)) - V_1(u(t_{k^-})) = e^{2\xi t_k} u^T(t_k) K u(t_k) - e^{2\xi t_k} u^T(t_{k^-}) K u(t_{k^-}) \\
= e^{2\xi t_k} u^T(t_k) \bar{G}_k^T \bar{G}_k K \bar{G}_k u(t_k) - e^{2\xi t_k} u^T(t_{k^-}) \bar{G}_k^T \bar{G}_k K \bar{G}_k u(t_{k^-}) \\
\quad \times K u(t_{k^-}) \\
= e^{2\xi t_k} u^T(t_{k^-}) [ \bar{G}_k^T \bar{G}_k - K ] u(t_{k^-}) \\
\leq 0.
\]

Thus,

$$V_1(u(t_k)) \leq V_1(u(t_{k^-})), \ k \in \mathbb{Z}^+.$$  \hfill (5.3.29)
It follows that

\[ V(u(t_k)) \leq V(u(t_{k-})), \quad k \in \mathbb{Z}^+. \quad (5.3.30) \]

From (5.3.26)-(5.3.28) & (5.3.30), we can easily obtain

\[ V(u(t)) \leq V(u(0)). \quad (5.3.31) \]

Taking expectation on both sides of (5.3.31), then

\[ \mathbb{E}\{V(u(t))\} \leq \mathbb{E}\{V(u(0))\}. \quad (5.3.32) \]

Alternatively,

\[
\begin{align*}
V_1(u(0)) & \leq \lambda_{\text{max}}(K) \|u(0)\|^2 \\
& \leq \lambda_{\text{max}}(K) \sup_{-\nu \leq s \leq 0} \{ \|\varphi(s)\|^2 \} \\
V_2(u(0)) & \leq 2[\tilde{f}(u(0)) - Y_1u(0)]^T \Theta_1 u(0) + 2[u(0)Y_2 - \tilde{f}(u(0))]^T \\
& \times \lambda_1 u(0) + 2[\tilde{g}(u(-\mu(0))) - Y_3u(-\mu(0))]^T \Theta_2 \\
& \times u(-\mu(0)) + 2[u(-\mu(0))Y_4 - \tilde{g}(u(-\mu(0)))]^T \Lambda_2 \\
& \times u(-\mu(0)) + 2[\tilde{h}(u(-\sigma)) - Y_5u(-\sigma)]^T \Theta_3 u(-\sigma) \\
& + 2[Y_6u(-\sigma) - \tilde{h}(u(-\sigma))]^T \Lambda_3 u(-\sigma) \\
& \leq 2\lambda_{\text{max}}(Y_2 - Y_1)[\lambda_{\text{max}}(\Theta_1) + \lambda_{\text{max}}(\Lambda_1)] \sup_{-\nu \leq s \leq 0} \{ \|\varphi(s)\|^2 \} \\
& + 2\lambda_{\text{max}}(Y_4 - Y_3)[\lambda_{\text{max}}(\Theta_2) + \lambda_{\text{max}}(\Lambda_2)] \\
& \times \sup_{-\nu \leq s \leq 0} \{ \|\varphi(s)\|^2 \} + 2\lambda_{\text{max}}(Y_6 - Y_5)[\lambda_{\text{max}}(\Theta_3) \\
& + \lambda_{\text{max}}(\Lambda_3)] \sup_{-\nu \leq s \leq 0} \{ \|\varphi(s)\|^2 \} \quad (5.3.34) \\
V_3(u(0)) & \leq \left( \tilde{\mu} \lambda_{\text{max}}(L_1) + \frac{\bar{\mu}}{3} \lambda_{\text{max}}(L_2) + \frac{2\bar{\mu}}{3} \lambda_{\text{max}}(L_3) + \tilde{\mu} \lambda_{\text{max}}(L_4) \\
& + \frac{\bar{\mu}}{3} \lambda_{\text{max}}(L_1) + \sigma\lambda_{\text{max}}(L_2) \right) \sup_{-\nu \leq s \leq 0} \{ \|\varphi(s)\|^2 \} \quad (5.3.35) \\
V_4(u(0)) & \leq \sigma\lambda_{\text{max}}(M_1) \sup_{-\nu \leq s \leq 0} \{ \|\varphi(s)\|^2 \} \quad (5.3.36) \\
V_5(u(0)) & \leq \sigma^2 \gamma\lambda_{\text{max}}(M_2) \sup_{-\nu \leq s \leq 0} \{ \|\varphi(s)\|^2 \} \quad (5.3.37)
\end{align*}
\]
\[ V_6(u(0)) \leq \int_{-\frac{\tilde{\mu}}{3}}^{0} e^{2s\frac{\mu}{\lambda}} \begin{bmatrix} u(s) \\ u(s - \frac{\tilde{\mu}}{3}) \\ u(s - \frac{2\tilde{\mu}}{3}) \end{bmatrix}^T M_3 \begin{bmatrix} u(s) \\ u(s - \frac{\tilde{\mu}}{3}) \\ u(s - \frac{2\tilde{\mu}}{3}) \end{bmatrix} ds \\
\leq \int_{-\frac{\tilde{\mu}}{3}}^{0} [u^T(s)M_{311}u(s) + 2u^T(s)M_{312}u(s - \frac{\tilde{\mu}}{3}) + 2u^T(s) \\
\times M_{313}u(s - \frac{2\tilde{\mu}}{3}) + u^T(s - \frac{\tilde{\mu}}{3})M_{323}u(s - \frac{2\tilde{\mu}}{3})M_{333} \\
\times u(s - \frac{2\tilde{\mu}}{3})] ds. \]

By Lemma 1.10.16, we have

\[ V_6(u(0)) \leq \frac{\tilde{\mu}}{3} \left[ \lambda_{\max}(M_{311}) + 2\lambda_{\max}(M_{312}) + 2\lambda_{\max}(M_{313}) + \lambda_{\max}(M_{322}) \right] \\
+ 2\lambda_{\max}(M_{323}) + \lambda_{\max}(M_{333}) \sup_{-\nu \leq s \leq 0} \{ \| \varphi(s) \|^2 \} \tag{5.3.38} \]

\[ V_7(u(0)) \leq \frac{\tilde{\mu}}{3} \lambda_{\max}(M_4) \int_{-\frac{\tilde{\mu}}{3}}^{0} \int_{-\frac{\tilde{\mu}}{3}}^{0} u^T(s) \dot{u}(s) ds \ d\theta + \frac{\tilde{\mu}}{3} \lambda_{\max}(M_5) \\
\times \int_{-\frac{\tilde{\mu}}{3}}^{0} \int_{-\frac{\tilde{\mu}}{3}}^{0} u^T(s) \dot{u}(s) ds \ d\theta + \frac{\tilde{\mu}}{3} \lambda_{\max}(M_6) \int_{-\frac{\tilde{\mu}}{3}}^{0} \int_{-\frac{\tilde{\mu}}{3}}^{0} u^T(s) \\
\times \dot{u}(s) ds \ d\theta \\
\leq \left\{ \frac{\tilde{\mu}}{3} \left[ \lambda_{\max}(M_4) \left( \frac{5\tilde{\mu}^2}{18} \right) \right] + \frac{\tilde{\mu}}{3} \left[ \lambda_{\max}(M_5) \left( \frac{3\tilde{\mu}^2}{18} \right) \right] + \frac{\tilde{\mu}}{3} \\
\times \left[ \lambda_{\max}(M_6) \frac{\tilde{\mu}^2}{18} \right] \right\} \left[ \lambda_{\max}(AA^T) + \lambda_{\max}(W_0W_0^T) + \beta^2 \\
\times \lambda_{\max}(W_1W_1^T) + \gamma^2 \lambda_{\max}(W_2W_2^T) \right] \sup_{-\nu \leq s \leq 0} \{ \| \varphi(s) \|^2 \}. \tag{5.3.39} \]

From the equations (5.3.33)-(5.3.39), we get the following inequalities:

\[ \mathbb{E}\{V(u(0))\} \leq \mathcal{M}^* \sup_{-\nu \leq s \leq 0} \mathbb{E}\{\| \varphi(s) \|^2 \}, \tag{5.3.40} \]

where

\[ \mathcal{M}^* = \lambda_{\max}(K) + 2\lambda_{\max}(Y_2 - Y_1)(\lambda_{\max}(\Theta_1) + \lambda_{\max}(\Lambda_1)) + 2 \\
\times \lambda_{\max}(Y_4 - Y_3)\left[ \lambda_{\max}(\Theta_2) + \lambda_{\max}(\Lambda_2) \right] + 2\lambda_{\max}(Y_6 \\
- Y_5)\left[ \lambda_{\max}(\Theta_3) + \lambda_{\max}(\Lambda_3) \right] + \frac{\tilde{\mu}}{3} \lambda_{\max}(L_1) + \frac{\tilde{\mu}}{3} \lambda_{\max}(L_2) \]

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\[+ \frac{2\bar{\mu}}{3} \lambda_{\max}(L_3) + \bar{\mu} \lambda_{\max}(L_4) + \frac{\bar{\mu}}{3} \lambda_{\max}(L_1) + \sigma \lambda_{\max}(L_2)\]
\[+ \sigma \lambda_{\max}(M_1) + \sigma^2 \gamma \lambda_{\max}(M_2) + \frac{\bar{\mu}}{3} [\lambda_{\max}(M_{31}) + 2\lambda_{\max}(M_{32}) + 2\lambda_{\max}(M_{33}) + \lambda_{\max}(M_{34})]
+ \frac{\bar{\mu}^3}{54} \lambda_{\max}(M_4) + \frac{3\bar{\mu}^3}{54} \lambda_{\max}(M_5) + \frac{\bar{\mu}^3}{54} \]
\[\times \lambda_{\max}(M_6) \right\} \{ \lambda_{\max}(AA^T) + \alpha^2 \lambda_{\max}(W_0W_0^T) + \beta^2
\times \lambda_{\max}(W_1W_1^T) + \gamma^2 \lambda_{\max}(W_2W_2^T) \}.\]

On the other hand, we can conclude that
\[\mathbb{E} \{ V(u(t)) \} \geq e^{2ct} \lambda_{\min}(K) \mathbb{E} \{ ||u(t)||^2 \}.\]  \hspace{1cm} (5.3.41)

Hence,
\[\mathbb{E} \{ ||u(t)|| \} \leq \sqrt{\frac{\mathcal{M}^*}{\lambda_{\min}(K)}} e^{-ct} \sup_{-\nu \leq s \leq 0} \mathbb{E} \{ ||\phi(s)|| \}.\]  \hspace{1cm} (5.3.42)

Therefore, by (5.3.42) and Definition 5.2.3, the neural networks (5.2.2) is exponentially stable in mean square sense. \hfill \Box

**Remark 5.3.2.** The integral terms \(2e^{2cs} \sum_{i=1}^{n} \int_{0}^{u_i(t)} \tilde{\varphi}(\tilde{f}_i(s) - a_i^- s) ds, 2e^{2cs}
\times \sum_{i=1}^{n} \int_{0}^{u_i(t)} \lambda_i (a_i^+(s) - \tilde{f}_i s) ds, 2e^{2cs} \sum_{i=1}^{n} \int_{0}^{u_i(t)} \varphi (\tilde{g}_i(s) - \beta_i^- s) ds, 2e^{2cs}
\times \sum_{i=1}^{n} \int_{0}^{u_i(t)} \lambda_i (\beta_i^+(s) - \tilde{g}_i s) ds, 2e^{2cs} \sum_{i=1}^{n} \int_{0}^{u_i(t)} \varphi (\tilde{h}_i(s) - \gamma_i^- s) ds\) and \(2e^{2cs}
\times \sum_{i=1}^{n} \int_{0}^{u_i(t)} \lambda_i (\gamma_i^+(s) - \tilde{h}_i s) ds\), in Lyapunov-Krasovskii functional \(V_2(u(t))\) are involved and may lead to attain enhanced feasibility for exponential stability criteria.

**Remark 5.3.3.** Suppose the stochastic disturbances have not appeared in neural networks (5.2.2), then the system can be remodelled as:
\[du(t) = \left[ -Au(t) + W_0 \tilde{f}(u(t)) + W_1 \tilde{g}(u(t - \mu(t))) + W_2
\times \int_{t - \sigma}^{t} \tilde{h}(u(s)) ds \right] dt; \quad t > 0, \quad t \neq t_k,\]
\[u(t_k) = \tilde{g}_k u(t_k^-); \quad t = t_k, \quad k \in \mathbb{Z}^+,\]
\[u(t) = \varphi(t); \quad t \in [-\nu, 0].\]  \hspace{1cm} (5.3.43)

Then by Theorem 5.3.1, it is easy to have the following Corollary 5.3.4.
Corollary 5.3.4. Suppose that Assumptions 5.1 and 5.2 holds, for given scalars $Y_1 = \text{diag}\{\alpha_1^{-}, \alpha_2^{-}, ..., \alpha_n^{-}\}, Y_2 = \text{diag}\{\alpha_1^{+}, \alpha_2^{+}, ..., \alpha_n^{+}\}, Y_3 = \text{diag}\{\beta_1^{-}, \beta_2^{-}, ..., \beta_n^{-}\}, Y_4 = \text{diag}\{\beta_1^{+}, \beta_2^{+}, ..., \beta_n^{+}\}, Y_5 = \text{diag}\{\gamma_1^{-}, \gamma_2^{-}, ..., \gamma_n^{-}\}, Y_6 = \text{diag}\{\gamma_1^{+}, \gamma_2^{+}, ..., \gamma_n^{+}\}, \mu \leq 1$,

if there exists symmetric positive definite matrices $K, L_i (i = 1, 2, 3, 4), \tilde{L}_j (j = 1, 2)$, $M_1, M_2, M_4, M_5, M_6$ and $M_3 = \begin{bmatrix} M_{311} & M_{312} & M_{313} \\ M_{321} & M_{322} & M_{323} \\ M_{331} & M_{332} & M_{333} \end{bmatrix}$, positive diagonal matrices $\Lambda_p = \text{diag}\{\lambda_p^1, \lambda_p^2, ..., \lambda_p^n\}, (p = 1, 2, 3), \Theta_l = \text{diag}\{\theta_{l1}, \theta_{l2}, ..., \theta_{ln}\}, (l = 1, 2, 3), N_q, (q = 1, 2, 3),$ such that the following LMI hold:

$$\tilde{G}_k^T K \tilde{G}_k - K \leq 0,$$

(5.3.44)

Case (i) : When $0 \leq \mu(t) \leq \frac{p}{3}$

$$\tilde{G}_k^T K \tilde{G}_k - K \leq 0,$$

(5.3.45)

Case (ii) : When $\frac{p}{3} \leq \mu(t) \leq \frac{2}{3} \bar{\mu}$

$$\tilde{G}_k^T K \tilde{G}_k - K \leq 0,$$

(5.3.46)

Case (iii) : When $\frac{2}{3} \bar{\mu} \leq \mu(t) \leq \bar{\mu}$

$$\tilde{G}_k^T K \tilde{G}_k - K \leq 0,$$

(5.3.47)
where

\[
\Gamma^*_{1,1} = 2\varsigma K - KA - KA + 4\varsigma Y_1 A \Theta_1 - 4\varsigma Y_2 A \Lambda_1 A - A (Y_2 \Lambda_1 - Y_1 \Theta_1) \\
- (Y_2 \Lambda_1 - Y_1 \Theta_1) A + \eta_1, \quad \eta_1 = \sum_{i=1}^{4} L_i + \frac{1}{2} L_j + M_1 + M_{311} + \left( \frac{\mu}{3} \right)^2 \\
\times A [M_4 + M_5 + M_6] A - 2Y_1 N_1 Y_2 - e^{-2\xi_1} M_6, \quad \gamma^*_{3,3} = -e^{-2\varsigma \rho} (1 - \mu)L_1 \\
- e^{-2\varsigma \rho} L_4 - \frac{3}{2} e^{-4\xi_1} M_5, \quad \delta^*_{3,3} = -e^{-2\varsigma \rho} (1 - \mu)L_1 - e^{-2\varsigma \rho} L_4 - 3M_4, \\
\Gamma^*_{3,3} = -e^{-2\varsigma \rho} (1 - \mu)L_1 - e^{-2\varsigma \rho} L_4 - 3e^{-2\xi_1} M_6 - 2Y_3 N_2 Y_4.
\]

The remaining values of \( \Xi^*_1, \Xi^*_2 \) and \( \Xi^*_3 \) are same as in Theorem 5.3.1. Then the recurrent neural networks (5.3.43) is exponentially stable with the exponential convergence rate \( \varsigma \).

**Proof.** Similar to the proof of Theorem 5.3.1, let \( \tilde{\rho}(u(t), u(t - \mu(t)), t) \ d\tilde{\omega}(t) = 0 \), we can easily derive this Corollary. Its proof is straightforward and hence omitted. \( \square \)

**Remark 5.3.5.** In case, the impulsive effect & distributed delay term are absent in system (5.2.2), then we get the following neural networks

\[
du(t) = \left[-Au(t) + W_0f(u(t)) + W_1g(u(t - \mu(t)))\right] dt \\
+ \tilde{\rho}(u(t), u(t - \mu(t)), t) d\tilde{\omega}(t); \ t > 0,
\]

\[
u(t) = \varphi(t); \ t \in [-v, 0]. \ (5.3.48)
\]

Then by Theorem 5.3.1, it is easy to have the following Corollary 5.3.6.

**Corollary 5.3.6.** Under the Assumptions 5.1 & 5.2, for given scalars \( Y_1 = \text{diag}\{\alpha_1^-, \alpha_1^+, ..., \alpha_n^-, \alpha_n^+\} \), \( Y_2 = \text{diag}\{\alpha_1^+, \alpha_2^-, ..., \alpha_n^+, \alpha_n^-\} \), \( Y_3 = \text{diag}\{\beta_1^-, \beta_2^-, ..., \beta_n^-, \beta_n^+\} \), \( Y_4 = \text{diag}\{\beta_1^+, \beta_2^+, ..., \beta_n^+, \beta_n^-\} \), \( Y_5 = \text{diag}\{\gamma_1^-, \gamma_2^-, ..., \gamma_n^-, \gamma_n^+\} \), \( Y_6 = \text{diag}\{\gamma_1^+, \gamma_2^+, ..., \gamma_n^+, \gamma_n^-\} \), \( \mu \leq 1 \), if there exists symmetric positive definite matrices \( K, L_i (i = 1, 2, 3, 4), L_j (j = 1, 2), M_1, M_2, M_4, M_5, M_6 \) and \( M_3 = \begin{bmatrix}
M_{311} & M_{312} & M_{313} \\
* & M_{322} & M_{323} \\
* & * & M_{333}
\end{bmatrix} \), positive diagonal matrices \( \Lambda_p = \text{diag}\{\lambda_{p1}, \lambda_{p2}, ..., \lambda_{pn}\}, (p = 1, 2, 3) \), \( \Theta_i = \text{diag}\{\theta_{i1}, \theta_{i2}, ..., \theta_{in}\}, (i = 1, 2, 3) \), \( N_q, (q = 1, 2, 3) \), such that the following LMIs hold:

\[
K \leq \delta I, \quad (5.3.49)
\]

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\textbf{Case (i) : When }0 \leq \mu(t) \leq \frac{\mu}{3} \text{ }

\begin{equation}
\Xi_1 = \begin{bmatrix}
\Gamma_{1,1} & \Gamma_{1,2} & \Gamma_{1,3} & \Gamma_{1,4} & 0 & \Omega_{1,6}^* & \Omega_{1,7} & 0 \\
* & \Gamma_{2,2} & 0 & \Gamma_{2,4} & 0 & 0 & 0 & 0 \\
* & * & \Gamma_{3,3} & \Gamma_{3,4} & \Omega_{3,5} & \Omega_{3,6} & 0 & 0 \\
* & * & * & \Gamma_{4,4} & \Omega_{4,5} & 0 & 0 & 0 \\
* & * & * & * & \Omega_{5,5} & 0 & 0 & 0 \\
* & * & * & * & * & \Omega_{6,6} & \Omega_{6,7} & \Omega_{6,8} \\
* & * & * & * & * & * & \Omega_{7,7} & \Omega_{7,8} \\
* & * & * & * & * & * & * & \Omega_{8,8}
\end{bmatrix} < 0, \quad (5.3.50)
\end{equation}

\textbf{Case (ii) : When }\frac{\mu}{3} \leq \mu(t) \leq \frac{\mu}{3} \bar{\mu} \text{ }

\begin{equation}
\Xi_2 = \begin{bmatrix}
\Gamma_{1,1} & \Gamma_{1,2} & \Gamma_{1,3} & \Gamma_{1,4} & 0 & \gamma_{1,6}^* & \Omega_{1,7} & 0 \\
* & \Gamma_{2,2} & 0 & \Gamma_{2,4} & 0 & 0 & 0 & 0 \\
* & * & \gamma_{3,3} & \Gamma_{3,4} & \Omega_{3,5} & \gamma_{3,6}^* & \gamma_{3,7} & 0 \\
* & * & * & \Gamma_{4,4} & \Omega_{4,5} & 0 & 0 & 0 \\
* & * & * & * & \Omega_{5,5} & 0 & 0 & 0 \\
* & * & * & * & * & \gamma_{6,6}^* & \gamma_{6,7} & \Omega_{6,8} \\
* & * & * & * & * & * & \Omega_{7,7} & \Omega_{7,8} \\
* & * & * & * & * & * & * & \Omega_{8,8}
\end{bmatrix} < 0, \quad (5.3.51)
\end{equation}

\textbf{Case (iii) : When }\frac{\mu}{3} \bar{\mu} \leq \mu(t) \leq \bar{\mu} \text{ }

\begin{equation}
\Xi_3 = \begin{bmatrix}
\Gamma_{1,1} & \Gamma_{1,2} & \Gamma_{1,3} & \Gamma_{1,4} & 0 & \delta_{1,6}^* & \Omega_{1,7} & 0 \\
* & \Gamma_{2,2} & 0 & \Gamma_{2,4} & 0 & 0 & 0 & 0 \\
* & * & \delta_{3,3} & \Gamma_{3,4} & \Omega_{3,5} & \delta_{3,6}^* & \delta_{3,7}^* & \delta_{3,8} \\
* & * & * & \Gamma_{4,4} & \Omega_{4,5} & 0 & 0 & 0 \\
* & * & * & * & \Omega_{5,5} & 0 & 0 & 0 \\
* & * & * & * & * & \Omega_{6,6} & \Omega_{6,7} & \Omega_{6,8} \\
* & * & * & * & * & * & \Omega_{7,7} \delta_{7,8} & \Omega_{8,8} \\
* & * & * & * & * & * & * & \Omega_{8,8}
\end{bmatrix} < 0, \quad (5.3.52)
\end{equation}

where

\[ \Gamma_{1,1} = 2\epsilon K - KA - KA + \delta S_1 + 4\epsilon Y_1 A \Theta_1 - 4\epsilon Y_2 A_1 A - A(Y_2 A_1 - Y_1 \Theta_1) \]

\[ - (Y_2 A_1 - Y_1 \Theta_1) A + \eta_1, \quad \eta_1 = \sum_{i=1}^{4} L_i + \sum_{j=1}^{2} L_j + M_i + M_{3i1} + \left( \frac{\mu}{3} \right)^2 A \]

\[ \times [M_4 + M_5 + M_6] A - 2Y_1 N_1 Y_2 - e^{-2\frac{\mu}{3} M_{6,6}}, \quad \Omega_{6,8} = -e^{-2\frac{\mu}{3} M_{6,6}} \]

\[ \Gamma_{1,2} = KW_0 - 2(A - W_0 Y_1) \Theta_1 + 2\epsilon (Y_2 W_0 + A) A_1 + (\Theta_1 - \Lambda_1) + (Y_2 A_1 - Y_1 \Theta_1) W_0 + \eta_2, \quad \eta_2 = -AW_0 \left( \sum_{j=1}^{2} L_j \right) - \left( \frac{\mu}{3} \right)^2 \left[ M_4 + M_5 + M_6 \right] AW_0 \]

\[ + N_1 (Y_1 + Y_2), \quad \Gamma_{1,3} = \frac{3}{2} e^{-2\frac{\mu}{3} M_{6,6}}, \quad \Gamma_{1,4} = KW_1 - 2\epsilon Y_1 W_1 + 2\epsilon Y_2 \]

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\[ \times W_1 \Lambda_1 + (Y_2 \Lambda_1 - Y_1 \Theta_1)W_1 - AW_1(\sum_{j=1}^{2} \tilde{L}_j) - \left(\frac{\bar{\rho}}{3}\right)^2[M_4 + M_5 + M_6] \]

\[ \times AW_1, \quad \Omega_{1,6} = M_{312}, \quad \Omega_{6,7} = M_{323} - e^{-2\varsigma \tilde{\rho}} M_{312} + e^{-4\varsigma \tilde{\rho}} M_{5} , \]

\[ \Gamma_{3,4} = N_2 (Y_3 + Y_4), \quad \Gamma_{2,2} = 4\varsigma \Theta_1 W_0 - 4\varsigma \Lambda_1 W_0 + \eta_3 + 2 (\Theta_1 - \Lambda_1) W_0 \]

\[ + W_0(\sum_{j=1}^{2} \tilde{L}_j) W_0 + \left(\frac{\bar{\rho}}{3}\right)^2 W_0 (M_4 + M_5 + M_6) W_0 - 2 N_1, \quad \Omega_{3,6} = e^{-2\varsigma \tilde{\rho}} \]

\[ \times M_6, \quad \delta_{1,6} = M_{312} + e^{-2\varsigma \tilde{\rho}} M_6, \quad \delta_{3,7} = e^{-2\varsigma \tilde{\rho}} M_4, \quad \delta_{7,8} = -e^{-2\varsigma \tilde{\rho}} M_{323}, \]

\[ \gamma_{1,6} = M_{312} + e^{-2\varsigma \tilde{\rho}} M_6, \quad \gamma_{6,7} = M_{323} - e^{-2\varsigma \tilde{\rho}} M_{312}, \quad \Omega_{1,7} = M_{313}, \]

\[ \Gamma_{3,3} = \delta S_2 - e^{-2\varsigma \tilde{\rho}} (1 - \mu) L_1 - e^{-2\varsigma \tilde{\rho}} L_4 - 3 e^{-2\varsigma \tilde{\rho}} L_5 - 2 Y_3 N_2 Y_4, \]

\[ \Omega_{5,5} = -e^{-2\varsigma \tilde{\rho}} L_1 (1 - \mu), \quad \Omega_{6,6} = -e^{-2\varsigma \tilde{\rho}} M_6 - e^{-2\varsigma \tilde{\rho}} M_5 + \eta_4, \]

\[ \Gamma_{4,4} = -2 N_2, \quad \eta_4 = M_{322} - e^{-2\varsigma \tilde{\rho}} M_{311} - e^{-2\varsigma \tilde{\rho}} L_2, \quad \Omega_{8,8} = -e^{-2\varsigma \tilde{\rho}} M_4 \]

\[ - e^{-2\varsigma \tilde{\rho}} M_{333} - e^{-2\varsigma \tilde{\rho}} L_4, \quad \Omega_{7,8} = -e^{-2\varsigma \tilde{\rho}} M_{322} + e^{-2\varsigma \tilde{\rho}} M_4, \]

\[ \Gamma_{2,4} = 2 \varsigma \Theta_1 W_1 - 2 \varsigma \Lambda_1 W_1 + (\Theta_1 - \Lambda_1) W_1 + W_0\left(\frac{\bar{\rho}}{3}\right)^2[M_4 + M_5 + M_6] W_1, \]

\[ \Omega_{7,7} = -e^{-4\varsigma \tilde{\rho}} L_3 + M_{333} - e^{-2\varsigma \tilde{\rho}} M_{322} - e^{-2\varsigma \tilde{\rho}} M_4 - e^{-4\varsigma \tilde{\rho}} M_5, \quad \delta_{3,8} = e^{-2\varsigma \tilde{\rho}} \]

\[ \times M_4, \quad \Omega_{3,5} = -2 \varsigma Y_3 \Theta_2 + 2 \varsigma Y_4 + (Y_4 \Lambda_2 - Y_3 \Theta_2), \quad \Omega_{4,5} = 2 \varsigma \Theta_2 \]

\[ - 2 \varsigma \Lambda_2 + (\Theta_2 - \Lambda_2). \]

Then system (5.3.48) is exponentially stable in the mean square sense with the exponential convergence rate index \(\varsigma\).

**Remark 5.3.7.** In this chapter, the discrete delay interval \([0, b]\) divided into \([0, \frac{b}{3}], [\frac{b}{3}, \frac{2b}{3}]\) and \([\frac{2b}{3}, b]\). For a sequence, each segments handled in different Lyapunov - Krasovskii functionals to obtain variety of LMIs, which yields novelty of this chapter.

**Remark 5.3.8.** The method enhanced in this chapter, that the relationship between time-varying delays and their upper bounds are fully considered. In addition, some inequality techniques and fractional segments in LKF, are used to obtain the maximum upper bounds for discrete time-varying delays, see Example 5.4.1 & 5.4.2. And also the state responses of (5.3.48) are depicted in Figure 5.4. Hence this approaches and methods may reduces the conservative criterions.
5.4 Numerical examples with comparisons

In this section, we provide two numerical examples with their simulations to testify the superiority and advantage of the above findings.

Example 5.4.1. Consider a two-neuron time-delayed recurrent neural networks (5.2.2) with stochastic disturbances, where

\[ A = \begin{bmatrix} 1 & 0 \\ 0 & 1.5 \end{bmatrix}, \quad W_0 = \begin{bmatrix} -1 & 0.5 \\ 0.5 & -1 \end{bmatrix}, \quad W_1 = \begin{bmatrix} -0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix}, \]

\[ Y_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Y_3 = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}, \quad Y_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \]

\[ Y_5 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Y_6 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 0.3 & 0.7 \\ 0.7 & 0.4 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0.2 & 0.8 \\ 0.5 & 0.3 \end{bmatrix}, \]

\[ G_k = \begin{bmatrix} 0.3 & 0.5 \\ 0.7 & 0.2 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad W_2 = \begin{bmatrix} -1.5 & 0.3 \\ 0.3 & -1.5 \end{bmatrix}, \]

\[ \bar{\rho}(u(t), u(t - \mu(t)), t) = \begin{bmatrix} 0.3 * u(t) \\ 0.1 * u(t - \mu(t)) \\ 0.2 * u(t - \mu(t)) \\ 0.4 * (u(t) + u(t - \mu(t))) \end{bmatrix}, \]

\[ \sigma = 0.2, \mu = 0.3, \bar{\mu} = 1.5, \zeta = 1, \mu(t) = 0.05 * \cos t + 0.08. \]

The following activation functions in neural networks (5.2.2) is expressed by

\[ \tilde{f}(u(.)) = \tilde{g}(u(.)) = \tilde{h}(u(.)) = 0.01 * \tan(.). \]

By utilizing the MATLAB LMI control toolbox, we solve the LMIs (5.3.1)-(5.3.5) in Theorem 5.3.1, one can obtain the following feasible solutions

Case (i): When \( 0 \leq \mu(t) \leq \frac{\bar{\mu}}{3} \)

\[ K = \begin{bmatrix} 297.4327 & 43.1135 \\ 43.1135 & 248.2842 \end{bmatrix}, \quad L_1 = \begin{bmatrix} 129.6604 & 12.5487 \\ 12.5487 & 145.4912 \end{bmatrix}, \]

\[ L_2 = \begin{bmatrix} 241.2601 & -15.2993 \\ -15.2993 & 239.7575 \end{bmatrix}, \quad L_3 = \begin{bmatrix} 195.2024 & -1.1016 \\ -1.1016 & 200.5533 \end{bmatrix}, \]

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\[ L_4 = \begin{bmatrix} 149.0607 & 19.9117 \\ 11.9117 & 161.2766 \end{bmatrix}, \quad \bar{L}_1 = \begin{bmatrix} 126.2069 & -2.8498 \\ -2.8498 & 183.4237 \end{bmatrix}, \]

\[ \bar{L}_2 = \begin{bmatrix} 101.3926 & -2.4908 \\ -2.4908 & 178.8969 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 133.6347 & 6.6002 \\ 6.6002 & 144.7483 \end{bmatrix}, \]

\[ M_2 = \begin{bmatrix} 166.6933 & -52.3475 \\ -52.3475 & 296.2633 \end{bmatrix}, \quad M_{311} = \begin{bmatrix} 241.2601 & -15.2993 \\ -15.2993 & 239.7575 \end{bmatrix}. \]

**Figure 5.1:** State trajectories of concerned neural networks (5.2.2) with stochastic noises

**Figure 5.2:** Impulsive effects of concerned neural networks (5.2.2)
Table 5.1: MAUB $\bar{\mu}$ when $0 \leq \mu(t) \leq \tfrac{2}{3} \bar{\mu}$

<table>
<thead>
<tr>
<th>Methods</th>
<th>$\bar{\mu} &gt; 0$</th>
<th>Status</th>
</tr>
</thead>
<tbody>
<tr>
<td>In [2]</td>
<td>0.8411</td>
<td>feasible</td>
</tr>
<tr>
<td>In [36]</td>
<td>1.7685</td>
<td>feasible</td>
</tr>
<tr>
<td>In [138]</td>
<td>2.882</td>
<td>feasible</td>
</tr>
<tr>
<td>In [203]</td>
<td>3.3574</td>
<td>feasible</td>
</tr>
<tr>
<td>In [201]</td>
<td>3.8739</td>
<td>feasible</td>
</tr>
<tr>
<td>In [174]</td>
<td>4.6141</td>
<td>feasible</td>
</tr>
<tr>
<td>In [180]</td>
<td>11.9128</td>
<td>feasible</td>
</tr>
<tr>
<td>In [113]</td>
<td>12.9895</td>
<td>feasible</td>
</tr>
</tbody>
</table>

Theorem 5.3.1: $24.9731$ feasible

\[
\Lambda_3 = \begin{bmatrix} 0.2222 & 0 \\ 0 & 0.2222 \end{bmatrix}, \quad \delta = 457.8393.
\]

Due to page restriction, some of the feasible solutions are ignored here.

Case (ii): When $\frac{2}{3} \leq \mu(t) \leq \frac{3}{2} \bar{\mu}$

\[
K = \begin{bmatrix} 184.3284 & 2.1622 \\ 2.1622 & 175.5190 \end{bmatrix}, \quad L_1 = \begin{bmatrix} 195.5362 & -30.6835 \\ -30.6835 & 232.2150 \end{bmatrix},
\]

\[
L_2 = \begin{bmatrix} 360.1819 & -113.5741 \\ -113.5741 & 395.5414 \end{bmatrix}, \quad L_3 = \begin{bmatrix} 273.4868 & -97.8512 \\ -97.8512 & 310.3006 \end{bmatrix},
\]

\[
L_4 = \begin{bmatrix} 249.6840 & -38.0942 \\ -38.0942 & 275.7278 \end{bmatrix}, \quad L_1 = \begin{bmatrix} 176.5499 & -47.0087 \\ -47.0087 & 320.3038 \end{bmatrix},
\]

\[
L_2 = \begin{bmatrix} 158.3050 & -38.5343 \\ -38.5343 & 314.5785 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 182.6822 & -30.6795 \\ -30.6795 & 213.2553 \end{bmatrix},
\]

\[
M_2 = \begin{bmatrix} 310.6333 & -76.8329 \\ -76.8329 & 534.4909 \end{bmatrix}, \quad M_{31} = \begin{bmatrix} 360.1819 & -113.5741 \\ -113.5741 & 395.5414 \end{bmatrix},
\]

\[
\Lambda_1 = \begin{bmatrix} 312.4878 & 0 \\ 0 & 695.7133 \end{bmatrix}, \quad \Lambda_2 = \begin{bmatrix} 0.1429 & 0 \\ 0 & 0.1429 \end{bmatrix},
\]

\[
\Lambda_3 = \begin{bmatrix} 0.2222 & 0 \\ 0 & 0.2222 \end{bmatrix}, \quad \delta = 252.3612.
\]

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Table 5.2: MAUB $\bar{\mu}$ when $\frac{2}{5} \leq \mu(t) \leq \frac{3}{5} \bar{\mu}$

<table>
<thead>
<tr>
<th>Methods</th>
<th>$\bar{\mu} &gt; 0$</th>
<th>Status</th>
</tr>
</thead>
<tbody>
<tr>
<td>In [123]</td>
<td>1.88</td>
<td>feasible</td>
</tr>
<tr>
<td>In [35]</td>
<td>2.5680</td>
<td>feasible</td>
</tr>
<tr>
<td>In [62]</td>
<td>6.247</td>
<td>feasible</td>
</tr>
<tr>
<td>In [184]</td>
<td>21.1573</td>
<td>feasible</td>
</tr>
</tbody>
</table>

Theorem 5.3.1: 26.425 feasible

Due to page limitation, some of the feasible solutions are ignored here.

Case (iii): When $\frac{2}{5} \bar{\mu} \leq \mu(t) \leq \bar{\mu}$

\[
K = \begin{bmatrix} 286.1838 & 35.9863 \\ 35.9863 & 271.6308 \end{bmatrix}, \quad L_1 = \begin{bmatrix} 84.8003 & 13.5585 \\ 13.5585 & 132.1982 \end{bmatrix},
\]

\[
L_2 = \begin{bmatrix} 160.6892 & 8.0090 \\ 8.0090 & 206.1634 \end{bmatrix}, \quad L_3 = \begin{bmatrix} 133.9874 & 13.6528 \\ 13.6528 & 200.7249 \end{bmatrix},
\]

\[
L_4 = \begin{bmatrix} 100.0023 & 13.7172 \\ 13.7172 & 152.7700 \end{bmatrix}, \quad \hat{L}_1 = \begin{bmatrix} 58.8333 & 1.5770 \\ 1.5770 & 50.3468 \end{bmatrix},
\]

\[
\hat{L}_2 = \begin{bmatrix} 47.1206 & 3.7692 \\ 3.7692 & 50.6134 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 77.6690 & 5.7908 \\ 5.7908 & 99.0707 \end{bmatrix},
\]

\[
M_2 = \begin{bmatrix} 72.4023 & -13.2490 \\ -13.2490 & 36.1766 \end{bmatrix}, \quad M_{311} = \begin{bmatrix} 160.6892 & 8.0090 \\ 8.0090 & 206.1634 \end{bmatrix},
\]

\[
\Lambda_1 = \begin{bmatrix} 121.6330 & 0 & 0 \\ 0 & 85.0721 & 0 \end{bmatrix}, \quad \Lambda_2 = \begin{bmatrix} 0.4000 & 0 \\ 0 & 0.2500 \end{bmatrix},
\]

\[
\Lambda_3 = \begin{bmatrix} 0.222 & 0 \\ 0 & 0.222 \end{bmatrix}, \quad \delta = 423.0894.
\]

Due to page limitation, some of the feasible solutions are ignored here.

Therefore, via MATLAB LMI control toolbox, the state trajectories of proposed neural networks (5.2.2) are obtained, which are illustrated in Figure 5.1 and also the impulsive effects of NNs (5.2.2) is shown in Figure 5.2. Furthermore, by solving the LMIs (5.3.1)-(5.3.5) in Theorem 5.3.1 the feasible solutions can be easily obtained. From the above three
Table 5.3: MAUB $\bar{\mu}$ when $\frac{2}{3}\bar{\mu} \leq \mu(t) \leq \bar{\mu}$

<table>
<thead>
<tr>
<th>Methods</th>
<th>$\bar{\mu} &gt; 0$</th>
<th>Status</th>
</tr>
</thead>
<tbody>
<tr>
<td>In [196]</td>
<td>0.2642</td>
<td>feasible</td>
</tr>
<tr>
<td>In [167]</td>
<td>1.96</td>
<td>feasible</td>
</tr>
<tr>
<td>In [202]</td>
<td>3.81</td>
<td>feasible</td>
</tr>
<tr>
<td>In [82]</td>
<td>4.17</td>
<td>feasible</td>
</tr>
<tr>
<td>Theorem 5.3.1</td>
<td>26.742</td>
<td>feasible</td>
</tr>
</tbody>
</table>

cases, the achieved discrete time-delay upper bounds $\bar{\mu}$ of neural networks (5.2.2) are very large, which are listed in Table 5.1, Table 5.2 and Table 5.3. According to Theorem 5.3.1, we can conclude that the recurrent neural networks (5.2.2) is exponentially stable in the mean square sense for the MAUB $\bar{\mu} = 26.0467$. Hence, this experiment demonstrate the contributions of this chapter is more extensive and less conservative than some available literatures.

Example 5.4.2. Consider a two-neuron time-delayed recurrent neural networks (5.3.43) with stochastic disturbances, where

$$A = \begin{bmatrix} 2.5 & 0 \\ 0 & 3 \end{bmatrix}, \quad W_0 = \begin{bmatrix} -0.5 & 0.4 \\ 0.3 & -1.2 \end{bmatrix}, \quad W_1 = \begin{bmatrix} -0.8 & 0.2 \\ -0.6 & 1 \end{bmatrix},$$

$$Y_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad Y_3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad Y_4 = \begin{bmatrix} 1.2 \\ 0 \end{bmatrix},$$

$$Y_5 = \begin{bmatrix} 0.4 \\ 0 \end{bmatrix}, \quad Y_6 = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 0.2 & 0.4 \\ 0.6 & 0.2 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.1 \end{bmatrix},$$

$$G_k = \begin{bmatrix} 0.1 & 0.6 \\ 0.3 & 0.3 \end{bmatrix}, \quad I = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad W_2 = \begin{bmatrix} -0.9 & 0.7 \\ 0.1 & -0.6 \end{bmatrix},$$

$\sigma = 0.2, \mu = 0.3, \bar{\mu} = 1.5, \zeta = 1, \mu(t) = 0.05 \times cost + 0.08$. The following activation functions in neural networks (5.3.43) is expressed by

$$\tilde{f}(u(.)) = \tilde{g}(u(.)) = \tilde{h}(u(.)) = 0.3 \times sin(.).$$
Table 5.4: MAUB $\tilde{\mu}$ for various $\mu, \zeta$ when $\frac{\zeta}{3} \leq \mu(t) \leq \frac{\tilde{\mu}}{3}$

<table>
<thead>
<tr>
<th>$\zeta &gt; 0$</th>
<th>$0 &lt; \mu &lt; 1$</th>
<th>$\tilde{\mu} &gt; 0$ [Corollary 5.3.4]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6</td>
<td>0.3</td>
<td>30.36</td>
</tr>
<tr>
<td>0.4</td>
<td>0.5</td>
<td>46.32</td>
</tr>
<tr>
<td>1.4</td>
<td>0.6</td>
<td>13.22</td>
</tr>
<tr>
<td>1.7</td>
<td>0.8</td>
<td>11.08</td>
</tr>
<tr>
<td>2.5</td>
<td>0.9</td>
<td>7.53</td>
</tr>
</tbody>
</table>

Table 5.5: MAUB $\tilde{\mu}$ for various $\mu, \zeta$ when $\frac{\tilde{\mu}}{3} \leq \mu(t) \leq \tilde{\mu}$

<table>
<thead>
<tr>
<th>$\zeta &gt; 0$</th>
<th>$0 &lt; \mu &lt; 1$</th>
<th>$\tilde{\mu} &gt; 0$ [Corollary 5.3.4]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6</td>
<td>0.3</td>
<td>45.6</td>
</tr>
<tr>
<td>0.3</td>
<td>0.4</td>
<td>59.75</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>60.43</td>
</tr>
<tr>
<td>0.4</td>
<td>0.7</td>
<td>64.91</td>
</tr>
<tr>
<td>3.8</td>
<td>0.9</td>
<td>5.14</td>
</tr>
</tbody>
</table>

By utilizing the MATLAB LMI control toolbox, we solve the LMIs (5.3.44)-(5.3.47) in Corollary 5.3.4, one can obtain the following feasible solutions.

Case (i): When $0 \leq \mu(t) \leq \frac{\tilde{\mu}}{3}$

\[
K = \begin{bmatrix} 93.5135 & 19.2460 \\ 19.2460 & 103.9725 \end{bmatrix}, \quad L_1 = \begin{bmatrix} 26.8410 & 4.8582 \\ 4.8582 & 26.5179 \end{bmatrix},
\]

\[
L_2 = \begin{bmatrix} 53.9558 & 6.3805 \\ 6.3805 & 55.2295 \end{bmatrix}, \quad L_3 = \begin{bmatrix} 46.0929 & 8.1690 \\ 8.1690 & 46.3342 \end{bmatrix},
\]

\[
L_4 = \begin{bmatrix} 32.3370 & 5.6483 \\ 5.6483 & 32.4489 \end{bmatrix}, \quad L_1 = \begin{bmatrix} 10.6419 & 2.8659 \\ 2.8659 & 8.9497 \end{bmatrix},
\]

\[
\tilde{L}_2 = \begin{bmatrix} 10.4774 & 2.7760 \\ 2.7760 & 10.2738 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 25.6914 & 2.4699 \\ 2.4699 & 24.0643 \end{bmatrix},
\]

Due to page limitation, some of the feasible solutions are ignored here.

Case (ii): When $\frac{\tilde{\mu}}{3} \leq \mu(t) \leq \frac{3}{2} \tilde{\mu}$

\[
K = 10^3 \times \begin{bmatrix} 1.9530 & 0.3595 \\ 0.3595 & 1.7720 \end{bmatrix}, \quad L_1 = \begin{bmatrix} 525.0336 & 154.0803 \\ 154.0803 & 657.0035 \end{bmatrix},
\]
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\[
L_2 = \begin{bmatrix} 808.2075 & 179.6285 \\ 179.6285 & 916.2269 \end{bmatrix}, \quad L_3 = \begin{bmatrix} 685.8293 & 218.8894 \\ 218.8894 & 794.3283 \end{bmatrix},
\]

\[
L_4 = 10^3 \times \begin{bmatrix} 0.8122 & 0.1932 \\ 0.1932 & 1.1322 \end{bmatrix}, \quad \tilde{L}_1 = \begin{bmatrix} 207.5894 & 51.8928 \\ 51.8928 & 169.6937 \end{bmatrix},
\]

\[
\tilde{L}_2 = \begin{bmatrix} 201.4030 & 49.7873 \\ 49.7873 & 186.6060 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 378.3844 & 62.7872 \\ 62.7872 & 383.7466 \end{bmatrix},
\]

Due to page limitation, some of the feasible solutions are ignored here.

**Case (iii): When \( \frac{3}{2} \bar{\mu} \leq \mu(t) \leq \bar{\mu} \)**

**Table 5.6: Average value of MAUB \( \bar{\mu} \) in Theorem 5.3.1**

<table>
<thead>
<tr>
<th>Case (i)</th>
<th>Case (ii)</th>
<th>Case (iii)</th>
<th>Average</th>
</tr>
</thead>
</table>

\[
K = \begin{bmatrix} 117.6003 & 11.7177 \\ 11.7177 & 127.0201 \end{bmatrix}, \quad L_1 = \begin{bmatrix} 47.7436 & 8.1934 \\ 8.1934 & 48.7013 \end{bmatrix},
\]

\[
L_2 = \begin{bmatrix} 91.4727 & 9.7156 \\ 9.7156 & 94.6644 \end{bmatrix}, \quad L_3 = \begin{bmatrix} 84.0982 & 14.0082 \\ 14.0082 & 87.2643 \end{bmatrix},
\]

\[
L_4 = \begin{bmatrix} 58.8446 & 9.8299 \\ 9.8299 & 60.5210 \end{bmatrix}, \quad \tilde{L}_1 = \begin{bmatrix} 12.7105 & 3.6301 \\ 3.6301 & 8.8938 \end{bmatrix},
\]

\[
\tilde{L}_2 = \begin{bmatrix} 12.3936 & 3.8481 \\ 3.8481 & 9.4193 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 41.2564 & 3.3948 \\ 3.3948 & 39.5715 \end{bmatrix},
\]

Due to page limitation, some of the feasible solutions are ignored here.

Thus, the state responses of neural networks without stochastic noises (5.3.43) are depicted in Figure 5.3 by the aid of control toolbox in MATLAB software. By solving the LMIs (5.3.44)-(5.3.47) in Corollary 5.3.4 the solutions with feasibility can be attained in three cases. In addition, the discrete time delayed upper bounds \( \bar{\mu} \) for different \( \mu \) and \( \zeta \) are classified in Table 5.4, Table 5.5 & Table 5.6. Therefore, under Corollary 5.3.4, one can conclude that the remodified neural networks (5.3.43) is exponentially stable in the mean square sense. This indicates the effectiveness and validity of our research work when compared with some existing literatures.

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Table 5.7: MAUB $\beta$ for various $\mu, \zeta$ when $0 \leq \mu(t) \leq \frac{\mu}{3}$

<table>
<thead>
<tr>
<th>$\zeta &gt; 0$</th>
<th>$0 &lt; \mu &lt; 1$</th>
<th>$\bar{\mu} &gt; 0$ [Corollary 5.3.4]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8</td>
<td>0.5</td>
<td>26.43</td>
</tr>
<tr>
<td>0.5</td>
<td>0.7</td>
<td>28.17</td>
</tr>
<tr>
<td>1</td>
<td>0.4</td>
<td>23.72</td>
</tr>
<tr>
<td>1.4</td>
<td>0.8</td>
<td>18.13</td>
</tr>
<tr>
<td>2.7</td>
<td>0.3</td>
<td>8.072</td>
</tr>
<tr>
<td>0.6</td>
<td>0.3</td>
<td>43.21</td>
</tr>
</tbody>
</table>

Figure 5.3: State responses of delayed neural networks (5.3.43) without stochastic noises

5.5 Conclusions

In this chapter, via fractional delay intervals, the problem of exponential stability criteria for stochastic recurrent neural networks with impulsive effects and both time-delays have been investigated. By means of Lyapunov-Krasovskii functionals and M-matrix theory, several novel sufficient criteria in terms of LMIs have been derived to ensure the exponential stability of the considered neural networks. Further, depends on the system parameters, the exponential convergence rate is estimated. At last, the proposed methods can be easily validated by using the MAT-
Figure 5.4: The state trajectory of NNs (5.3.48) without distributed delay term

LAB LMI control toolbox in example section which are less conservative than some existing literature, also the numerical examples demonstrate the effectiveness and merits of the proposed results.

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