Chapter 4

Novel results on passivity and exponential passivity for multiple discrete delayed neutral-type neural networks with leakage and distributed time-delays

4.1 Introduction

In current scenario, extensive attention has been paid on neural networks due to their fruitful applications in many fields, such as intelligent robot, signal processing, associative memories, fixed-point computations, automatic control, artificial intelligence, and so on [58, 72]. It is well known that the time delays often occur from finite switching speed of the communication time, amplifiers and faults in the electrical circuits when the implementation of neural networks. Thus, the existence of time delays is inevitable in dynamical systems, which may affects the passiveness generating instability, bad performance, chaotic behavior and swinging or oscillation. Hence, the study on neural networks with time-delays have
been considerable more attention from many researchers. In [159], the authors investigates the passivity criteria for time delayed neural networks with the help of LKF.

In accordance, the delay-dependent and delay-independent criteria are two categories of time delays in neural networks, which classified by the existing time delayed results. So far as, one can observe from available literatures, the delay-dependent case is less conserved than the delay independent ones. Further, time delay can be characterized into two types: a Discrete and Distributed delays. Here, we have taken both time delays, that is multiple discrete time delays and the distributed delays, into account while model our network system, because the length of the axon sizes are too large. For sequence, the passivity of various classes of neural networks with time delay has become an interesting area and different passivity conditions have been established for such NNs for multiple delays [194] and distributed time-delays [215]. As a result, it is noteworthy to inspect both the time-delay results in the passivity behaviors of systems, see for reference [74].

It is pointed out that a special case of time delay in neural networks which incorporated to the negative feedback term of the system, namely forgetting or leakage terms. This type of delays will emerge, owing to some theoretical and technical difficulties. By the result of leakage delays the stability of the dynamical systems may affect and it leads to be unstable. Thus, the consideration of leakage term in neural networks is necessary to investigate the passivity theory [102, 157]. Mala et al., in [127], analyzed the problem of passiveness for neural networks, in which some suitable LKF are handled with leakage terms.

Moreover, neutral time delay is another type of delay which has drawn a lot of attention nowadays. Because, a neutral-type delay phenomenon contains delays both in its derivatives and state variables. Practically, such phenomenon can be established in several fields including mechanics, automatic control, vibrating masses attached to an elastic bar, ecology, robots in contact with rigid environ-
ments, etc. As for as, in [9], H. Bao and J. Cao discussed about the stability performance of neutral type neural networks with time delays.

On the other hand, from the circuit theory, the passivity concept was instigated and plays an essential key role in the design and analysis of linear and nonlinear systems, particularly for high-order systems. Also, the analysis of passivity has received a great attention [10, 121], because the passivity gives a wonderful tool for studying the properties of a nonlinear systems, such as signal processing, synchronization and stability. The core of the passivity theory is that the properties of passiveness for a system can keep the system internally stable. Actually, the product of input and output in the passive neural networks used as the energy provision and embodies the energy attenuation character which is a noteworthy feature of passivity. By the reason of its significance, the results on analysis of passivity for neural networks has been widespread studied in the available literature [40, 87]. In current decades, the problem of passivity in the exponential sense for neural networks has been analyzed in [179], where the new criterion are obtained for the addressed NNs to ensure the exponential passive. Also, Du et al., investigates the existence of time delays in neural networks and exploit the LKF approach, the new sufficient criteria were derived for exponential passivity in [40]. It is noteworthy that, owing to its mathematical complexity, the exponential passivity condition has not been studied in the existing literature for neutral type neural networks with leakage, multiple discrete time-varying delay and distributed delay. So, the aim of this chapter is meaningful.

Motivated by the earlier discussions, the problem for passivity and exponential passivity investigation for the neutral-type neural networks with leakage, multiple discrete delays and distributed time delays has not been fully studied yet, which motivates our current research work. With this aim to fill this gap, in this chapter, we consider the passivity as well as exponential passivity problem for NNNs with leakage delays, multiple discrete delays and distributed delays. By utilizing the Lyapunov-Krasovkii functional, matrix theory and some inequality techniques,
brand-new sufficient conditions for passivity and exponential passivity are formulated in the form of Linear matrix inequalities, which can be substantiated easily by LMI control toolbox in MATLAB software. As well, two numerical examples with their simulations are provided to illustrate the superiority and applicability of the proposed method. The main contributions of this work are highlighted as follows:

(1) The neutral term, leakage time delay, multiple discrete delay & distributed time-delay are taken into account in the passivity and exponential passivity analysis of proposed neural networks.

(2) Based on different novel Lyapunov-Krasovskii functional, some sufficient conditions for passivity and exponential passivity of neutral-type NNs are derived in the form of LMIs. Furthermore, compared to the existing results, the derived conclusions are different and advanced.

(3) By handled the multiple discrete time-varying delay and neutral time-varying delay terms in our addressed neural networks, the allowable upper bounds of multiple discrete time-delay and neutral delay are maximum, when compare with the existing literatures, see Table 4.1, Table 4.2 and Table 4.3 in Example 4.5.1 & 4.5.2. This shows that the approach developed in this chapter is effective and less conservative than some available results.

4.2 Model description and preliminaries

Consider the neutral-type neural networks with multiple discrete time-varying delay, distributed and leakage delays as follows:

$$\dot{u}_i(t) - \sum_{j=1}^{n} W_{2ij}\dot{u}_i(t - \eta_k(t)) = -a_iu_i(t - \delta) + \sum_{j=1}^{n} W_{0ij}g_j(u(t)) + \sum_{j=1}^{n} \sum_{k=1}^{c} W_{1ij}^{(k)}$$

$$\times g_j(u(t - h_k(t))) + \sum_{j=1}^{n} W_{3ij} \int_{t-\sigma}^{t} g_j(u(s)) ds$$

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\begin{equation}
\begin{aligned}
\mathcal{V}_i(t) &= \sum_{j=1}^{n} g_j(u(t)) + \sum_{j=1}^{n} g_j(u(t - h_k(t))) \\
&+ \sum_{j=1}^{n} \int_{t-\sigma}^{t} g_j(u(s)) ds + \mathcal{H}_i(t), \\
&i = 1, 2, 3, ..., n; c \in \mathbb{N}.
\end{aligned}
\end{equation}

It can be remolded as

\begin{equation}
\begin{aligned}
u(t) - W_2 u(t - \eta_k(t)) &= -\Lambda u(t - \delta) + W_0 g(u(t)) + \sum_{k=1}^{c} W_1^{(k)} g(u(t - h_k(t))) \\
&+ W_3 \int_{t-\sigma}^{t} g(u(s)) ds + \mathcal{H}(t), \\
\mathcal{V}(t) &= g(u(t)) + g(u(t - h_k(t))) + \int_{t-\sigma}^{t} g(u(s)) ds + \mathcal{H}(t).
\end{aligned}
\end{equation}

where \( u(t) = [u_1(t), u_2(t), ..., u_n(t)]^T \in \mathbb{R}^n \) be the neurons state vector at time t. \( A = \text{diag}[a_1, a_2, ..., a_n] > 0 \) represents the passive decay rate; \( g(u(t)) = [g_1(u(t)), g_2(u(t)), ..., g_n(u(t))]^T \) is the activation function of neurons, \( W_0 \) denotes the connection weight matrix and \( W_1^{(k)} \), \( W_2 \) & \( W_3 \) are the delayed connection weight matrices; \( \mathcal{H}(t) = [\mathcal{H}_1(t), \mathcal{H}_2(t), ..., \mathcal{H}_n(t)]^T \) indicates the external inputs at time t, the output of the neural networks represent by \( \mathcal{V}(t) \). The function \( \mathcal{V}^T(t) \mathcal{H}(t) \) indicates a supply rate if it is locally integrable; i.e., for all input-output pairs \( \mathcal{H}(t) \in \mathbb{R}^n \) and \( \mathcal{V}(t) \in \mathbb{R}^n \) it satisfies \( \int_{t}^{t+1} |\mathcal{V}^T(\xi) \mathcal{H}(\xi)| d\xi \leq \infty \), for all \( t \geq 0 \). \( h_k(t) \) denotes the discrete transmission delay with \( 0 \leq h_k(t) \leq h_k^* \); \( h_k(t) \leq \tau_k < 1 \), where \( h_k^* = \max\{h_k(t)\} \) and \( \sigma \) is the constant delay, \( \delta \) represent leakage delay, which is a constant. \( \eta_k(t) \) denotes the discrete transmission neutral delay with \( 0 \leq \eta_k(t) \leq \eta_k^* \); \( \eta_k(t) \leq \lambda_k < 1 \), where \( \eta_k^* = \max\{\eta_k(t)\} \).

Furthermore, the initial condition of neural networks (4.2.2) can be described by

\begin{equation}
\begin{aligned}
u(t) = \phi(t), \quad u(t) = \varphi(t), \quad t \in [-m, 0], \quad m = \max\{h_k^*, \eta_k^*\},
\end{aligned}
\end{equation}

where \( \phi(t) \) & \( \varphi(t) \) are continuous functions on \([-m, 0]\).
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Hence, by the aid of well-known Brower's fixed-point theorem, neural networks (4.2.2) has at least one equilibrium point, say $u^*$.  

**Remark 4.2.1.** With the research motivation of function of the human brain, artificial neural networks are able to learn from experience. These powerful problem solvers are highly effective where traditional, formal analysis would be complexity or impossible. Their strength lies in their ability to make sense out of difficult, noisy, or nonlinear data. Neural networks can provide robust solutions to problems in a wide range of disciplines, particularly areas involving filtering, pattern recognition, classification, prediction, function approximation and optimization. Naturally, the passivity issue is internally stable for neural networks. Time delay is unavoidable in real-world neural networks. In practical implementations by electrical circuits, time-delays occur in the finite switching speed of amplifiers and the signal transmissions among neurons. As is well known that the existence of time-delays in multiple discrete, distributed and neutral terms causes poor performance, oscillation and instability of the concerned neural networks. Thus, by the help of different Lyapunov-Krasovskii functional, the oscillations and instability are controlled while checking the passivity and also the exponential convergence rate increases the speed of system stability while checking the exponential passivity.

Throughout this chapter, we obtain that the sufficient criteria for passivity as well as exponential passivity, we made the following assumptions.

**Assumption 4.1.** The activation function $g$ in (4.2.2) is continuously differentiable, bounded and satisfies

$$0 \leq \frac{g_i(s)}{s} \leq \beta_i, \forall \ s \in \mathbb{R},$$

$$g_i(0) = 0, (j = 1, 2, ..., n), \quad (4.2.4)$$

where $\beta_i > 0$.

**Remark 4.2.2.** It is pointed out that, in Assumption 4.1, the activation functions are less restrictive than the descriptions on sigmoid activation functions. These functions will play a key role in our proposed methods to get necessary conditions for the neural networks
(4.2.2) which leads to passivity as well as exponential passivity. By the Assumption 4.1, we can easily observe that the functions $g_j(.)$ are monotonic increasing and $g_j(s)$, ($j = 1, 2, ..., n$).

**Definition 4.2.3.** The neural networks (4.2.2) with input $H(t)$ & output $v(t)$ is said to be passive if $\exists$ a positive scalar $\mu$ such that

$$2 \int_{t_0}^{t_1} v(t)H(t)dt \geq -\mu \int_{t_0}^{t_1} H^T(t)H(t)dt,$$

for all $t_1 \geq 0$ and for all solutions of (4.2.2) under zero initial condition.

**Definition 4.2.4.** Neural networks (4.2.2) is called passive in the sense of exponential with input $H(t)$ and output $v(t)$, if there exist a Lyapunov function in the sense of exponential (or, called the exponential storage function) $V$ defined on $\mathbb{R}^n$ and a scalar $\gamma > 0$ such that the inequality is satisfied as follows:

$$V(u(t), t) + \gamma V(u(t), t) \leq 2v^T(t)H(t), \; t \geq 0,$$

for all $H(t)$, $t \geq t_0$ and initial conditions $u(0)$, where $V(u(t), t)$ indicates the total derivative of $V(u(t), t)$ along the trajectories $u(t)$ of (4.2.2), $t \geq 0$.

By using the Leibniz-Newton formula, for any matrices $K_i, A_i (i = 1, 2, 3, ..., 8)$ and $M_j (j = 1, 2, 3)$ with appropriate dimensions, we can get the equations as follows:

$$2[u^T(t-\delta)K_1 + u^T(t)K_2 + u^T(t-h_k(t))K_3 + u^T(t-\sigma)K_4 + u^T(t-\eta_k(t))$$

$$\times K_5 + g^T(u(t))K_6 + g(u(t-h_k(t)))K_7 + \dot{u}^T(t)K_8][u(t)$$

$$- u(t-h_k(t)) - \int_{t-h_k(t)}^t u(s)ds] = 0 \quad (4.2.5)$$

$$2[u^T(t-\delta)A_1 + u^T(t)A_2 + u^T(t-h_k(t))A_3 + u^T(t-\sigma)A_4 + u^T(t-\eta_k(t))$$

$$\times A_5 + g^T(u(t))A_6 + g(u(t-h_k(t)))A_7 + \dot{u}^T(t)A_8][u(t)$$

$$- u(t-\sigma) - \int_{t-\sigma}^t u(s)ds] = 0 \quad (4.2.6)$$

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\[ 2\left[ u^T(t-\delta)M_1 + u^T(t)M_2 + u^T(t)M_3 + H^T(t)B_k^T \right] \left\{ -Au(t-\delta) + W_0 \times g(u(t)) + W_1^k g(u(t)-h_k(t)) + W_3 \int_{t-\sigma}^t g(u(s))ds + H(t) \right\} - \dot{u}(t) + W_2u(t-\eta_k(t)) \right\} = 0. \tag{4.2.7} \]

**Remark 4.2.5.** In [146, 178], the output is fully depends on the state vectors and control inputs of the system. From the aforementioned motivations, the output of NN in our developed method is obtained by the activation function of discrete as well as distributed time-delayed state variable terms, which generates advances of passivity conditions.

**Remark 4.2.6.** The enhanced neural networks proposed with constant value (or time-invariant) of discrete & distributed delays in [76] and the constant time-delay in neutral term is derived in the system of [73] to guarantee the stochastically stable. In this chapter, we contribute to the further improvement of passivity and exponential passivity criteria for neural networks with discrete, neutral and distributed time-varying delays with some bound conditions, which leads to novelty of this designed method.

### 4.3 Passivity results

**Theorem 4.3.1.** Under Assumption 4.1, for given \( \tau_k < 1 \) and \( \lambda_k < 1 \), if there exists matrices \( Q > 0, S > 0, R > 0, Y > 0, L_1, L_2, H_1, H_2 > 0 \) and diagonal matrix \( T > 0 \) and any matrices \( K_i(i=1,2,..,8), B_{k,i}, \) and \( M_j(j=1,2,3) \) with appropriate dimension such that the following LMI is satisfied

\[
\Xi = \begin{bmatrix}
\Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} & \Xi_{16} & \Xi_{17} & \Xi_{18} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & \Xi_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & \Xi_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & \Xi_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & \Xi_{55} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & \Xi_{66} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & \Xi_{77} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & \Xi_{88} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & \Xi_{99} & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & \Xi_{1010} & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & \Xi_{1111} & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & \Xi_{1212} & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & \Xi_{1313} & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & \Xi_{1414} & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & \Xi_{1515} \\
\end{bmatrix} < 0,
\tag{4.3.1}
\]

where

\[
\Xi_{11} = QW_3\Theta_2^{-1} + Y + H_1 + K_2 + A_2, \quad \Xi_{12} = -2QA + 2K_1 + 2A_1 - 2AM_2,
\]

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\[ \Xi_{13} = 2W_0Q + 2K_6 + 2A_6 + 2W_0M_2 + 2TY, \quad \Xi_{14} = 2QW_1^{(k)} + 2K_7 + 2A_7 \\
+ 2M_2W_1^{(k)}, \quad \Xi_{16} = 2QW_2 + 2K_5 + 2W_2M_2, \quad \Xi_{17} = 2K_3 - 2K_2 + 2A_3, \]
\[ \Xi_{15} = 2QW_3 + 2W_3M_2, \quad \Xi_{1313} = L_2, \quad \Xi_{110} = 2K_4 + 2A_4 - 2A_2, \]
\[ \Xi_{18} = 2K_8 + 2A_8 - 2M_2, \quad \Xi_{114} = -2A_2, \quad \Xi_{23} = 2W_0M_1, \quad \Xi_{112} = -2K_2, \]
\[ \Xi_{24} = 2W_1^{(k)}M_1, \quad \Xi_{25} = 2M_1W_3, \quad \Xi_{26} = 2W_2M_1, \quad \Xi_{27} = -2K_1, \]
\[ \Xi_{37} = -2K_6, \quad \Xi_{212} = -2K_1, \quad \Xi_{214} = -2A_1, \quad \Xi_{210} = -2A_1, \]
\[ \Xi_{47} = -2K_7, \quad \Xi_{314} = -2A_6, \quad \Xi_{312} = -2K_6, \quad \Xi_{48} = 2W_1^{(k)}M_3, \]
\[ \Xi_{58} = 2W_3M_3, \quad \Xi_{67} = -2K_5, \quad \Xi_{68} = 2W_2M_3, \quad \Xi_{610} = -2A_5, \]
\[ \Xi_{78} = -2K_8, \quad \Xi_{710} = -2K_4 - 2A_3, \quad \Xi_{712} = -2K_4, \quad \Xi_{714} = -2A_3, \]
\[ \Xi_{22} = -2AM_1, \quad \Xi_{1014} = -2A_4, \quad \Xi_{1012} = -2K_4, \quad \Xi_{33} = S + H_2 - 2T, \]
\[ \Xi_{55} = QW_3\Theta_2 + B_1^*W_3\Theta_4, \quad \Xi_{66} = -(1 - \lambda_k)R, \quad \Xi_{77} = -(1 - \tau_k)Y - K_3, \]
\[ \Xi_{88} = -A_8\Theta_1^{-1} - K_8\Theta_3^{-1} - h_k^*L_1 - \eta_k^*L_2 - M_3, \quad \Xi_{99} = -H_2, \]
\[ \Xi_{115} = M_2, \quad \Xi_{215} = M_1 - AB_1^*, \quad \Xi_{315} = W_0B_1^*, \quad \Xi_{1414} = -A_8\Theta_1, \]
\[ \Xi_{815} = M_3 - B_1^*, \quad \Xi_{1515} = B_1^* + B_1^*W_3\Theta_4^{-1}, \quad \Xi_{614} = -2A_5, \]
\[ \Xi_{1212} = -K_8\Theta_3, \quad \Xi_{28} = -2AM_3 - 2M_1, \quad \Xi_{310} = -2A_6, \quad \Xi_{414} = -2A_7, \]
\[ \Xi_{44} = -(1 - \tau_k)S, \quad \Xi_{814} = -2A_8, \quad \Xi_{1111} = L_1, \quad \Xi_{615} = B_1^*W_2, \]
\[ \Xi_{415} = W_1^{(k)}B_1^*, \quad \Xi_{38} = 2W_0M_3, \quad \Xi_{412} = -2K_7, \quad \Xi_{612} = -2K_5, \]
\[ \Xi_{1010} = -A_4, \quad \Xi_{810} = -2A_8, \quad \Xi_{812} = -2K_8, \]

then the addressed neural networks (4.2.2) is passive.

**Proof.** Consider the following Lyapunov-Krasovskii functional for NNs (4.2.2) as

\[ V(u(t), t) = \sum_{p=1}^{8} V_p(u(t), t), \quad (4.3.2) \]

where

\[ V_1(u(t), t) = u^T(t)Qu(t), \]
\[ V_2(u(t), t) = \int_{t-h_k(t)}^{t} u^T(s)Yu(s)ds, \]

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\[ V_3(u(t), t) = \int_{t-h_k(t)}^{t} g^T(u(s))Sg(u(s))ds, \]
\[ V_4(u(t), t) = \int_{t-\eta_k(t)}^{t} \dot{u}^T(s)R\dot{u}(s)ds, \]
\[ V_5(u(t), t) = \int_{-\eta_k}^{0} \int_{t+\theta}^{t} \dot{u}^T(s)L_1\dot{u}(s)ds \, d\theta, \]
\[ V_6(u(t), t) = \int_{-\eta_k}^{0} \int_{t+\theta}^{t} \dot{u}^T(s)L_2u(s)ds \, d\theta, \]
\[ V_7(u(t), t) = \int_{t-\sigma}^{t} u^T(s)H_1u(s)ds, \]
\[ V_8(u(t), t) = \int_{t-\sigma}^{t} g^T(u(s))H_2g(u(s))ds. \]

Taking the derivative of \( V(u(t), t) \) along the trajectories of (4.2.2), we have

\[
V(u(t), t) = 2u^T(t)Q\dot{u}(t) + u^T(t)Yu(t) - (1 - h_k(t))u^T(t - h_k(t))Y \\
\times u(t - h_k(t)) + g^T(u(t))Sg(u(t)) - (1 - h_k(t)) \\
\times g^T(u(t - h_k(t)))Sg(u(t - h_k(t))) + \dot{u}^T(t)R\dot{u}(t) \\
- (1 - \eta_k(t))\dot{u}^T(t - \eta_k(t))R\dot{u}(t - \eta_k(t)) + h_k^{\ast}\dot{u}^T(t) \\
\times L_1\dot{u}(t) - \int_{t-h_k^{\ast}}^{t} \dot{u}^T(s)L_1\dot{u}(s)ds + \eta_k^{\ast}\dot{u}^T(t)L_2\dot{u}(t) \\
- \int_{t-\eta_k^{\ast}}^{t} \dot{u}^T(s)L_2\dot{u}(s)ds + u^T(t)H_1u(t) - u^T(t - \sigma) \\
\times H_1u(t - \sigma) + g^T(u(t))H_2g(u(t)) - g^T(u(t - \sigma)) \\
\times H_2g(u(t - \sigma)). \tag{4.3.3}
\]

Then, from (4.2.5) to (4.2.7), one can obtain that

\[
V(u(t), t) \leq 2u^T(t)Q\dot{u}(t) + u^T(t)Yu(t) - (1 - \tau_k)u^T(t - h_k(t))Yu(t - h_k(t)) \\
+ g^T(u(t))Sg(u(t)) - (1 - \tau_k)g^T(u(t - h_k(t)))Sg(u(t - h_k(t))) \\
- h_k(t)) + \dot{u}^T(t)R\dot{u}(t) - (1 - \lambda_k)\dot{u}^T(t - \eta_k(t))R\dot{u}(t - \eta_k(t)) \\
u^T(t)H_1u(t) - u^T(t - \sigma)H_1u(t - \sigma) + g^T(u(t))H_2g(u(t)) \\
g^T(u(t - \sigma))H_2g(u(t - \sigma)) + 2u^T(t - \delta)K_1 + u^T(t)K_2 \\
u^T(t - h_k(t))K_3 + u^T(t - \sigma)K_4 + \dot{u}^T(t - \eta_k(t))K_5 + g^T(u(t))
\]

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\[\times K_6 + g^T(u(t - h_k(t)))K_7 + \dot{u}^T(t)K_8 \left[ u(t) - u(t - h_k(t)) \right] \]
\[- \int_{t-h_k(t)}^t \dot{u}(s)ds \right] + 2[u^T(t - \delta)K_1 + \dot{u}^T(t)K_2 + \dot{u}^T(t - h_k(t))K_3 + \dot{u}^T(t - \eta_k(t))K_4 + g^T(u(t))K_5 \]
\[+ g^T(u(t - h_k(t)))K_7 + \dot{u}^T(t)K_8 \left[ u(t) - u(t - \sigma) - \int_{t-\sigma}^t \dot{u}(s)ds \right] + 2[u^T(t - \delta)M_1 + \dot{u}^T(t)M_2 + \dot{u}^T(t)M_3 + \mathcal{H}^T(t)B_*^\top] \]
\[\times \left\{ - Au(t - \delta) + W_0g(u(t)) + W_1^{(k)}g(u(t - h_k(t))) + W_3 \right\} \]
\[\times \int_{t-\sigma}^t g(u(s))ds + \mathcal{H}^T(t) - \dot{u}(t) + W_2\dot{u}(t - \eta_k(t)) - 2 \]
\[\times g^T(u(t))Tg(u(t)) + 2g^T(u(t))Tg(u(t)) + h_k^* \dot{u}^T(t)\dot{u}(t) \]
\[- \int_{t-h_k(t)}^t \dot{u}(s)L_1\dot{u}(s)ds + \eta_k^* \dot{u}^T(t)L_2\dot{u}(t) - \int_{t-\eta_k^*}^t \dot{u}^T(s)L_2 \dot{u}(s)ds, \] (4.3.4)

where the diagonal matrix $T > 0$. By Assumption 4.1, the condition (4.2.4) implies that

\[g^T(u(t))Tg(u(t)) \leq u^T(t)TYg(u(t)), \] (4.3.5)

where $Y = \text{diag}(\beta_1, \beta_2, ..., \beta_n)$, $\beta_j > 0, j = 1, 2, ..., n$ are given in Assumption 4.1.

Then it follows from (4.2.5) to (4.2.7) that

\[2[u^T(t - \delta)K_1 + \dot{u}^T(t)K_2 + \dot{u}^T(t - h_k(t))K_3 + \dot{u}^T(t - \sigma)K_4 + \dot{u}^T(t - \eta_k(t))K_5 + \dot{u}^T(t)K_8] \]
\[\times \left[ u(t) - u(t - h_k(t)) - \int_{t-h_k(t)}^t \dot{u}(s)ds \right] \]
\[= 2\left\{ [u^T(t - \delta)K_1u(t) + \dot{u}^T(t)K_2u(t) + \dot{u}^T(t - h_k(t))K_3u(t) \right. \]
\[+ \dot{u}^T(t - \sigma_1)K_4u(t) + \dot{u}(t - \eta_k(t))K_5u(t) + g^T(u(t))K_6u(t) \]
\[+ g^T(u(t - h_k(t)))K_7u(t) + \dot{u}^T(t)K_8u(t) - \dot{u}^T(t - \delta)K_1 \]
\[\times u(t - h_k(t)) - u^T(t - h_k(t))K_2u(t - h_k(t)) - \dot{u}^T(t - h_k(t))K_3 \]
\[\times u(t - h_k(t)) - \dot{u}^T(t - \sigma)K_4u(t - h_k(t)) - \dot{u}^T(t - \eta_k(t))K_5 \]
\[\times u(t - h_k(t)) - g^T(u(t))K_6u(t - h_k(t)) - g^T(u(t - h_k(t)) \]

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\begin{align*}
\times K_7 u(t - h_k(t)) - u^T(t)K_8 u(t - h_k(t))] \right) - 2[u^T(t - \delta) \\
\times K_1 + u^T(t)K_2 + u^T(t - h_k(t))K_3 + u^T(t - \sigma)K_4 + \hat{u}^T(t - \eta_k(t)) \\
\times K_5 + g^T(u(t))K_6 + g^T(u(t - h_k(t)))K_7 + \hat{u}^T(t)K_8 \\
\times \int_{t-h_k(t)}^{t} \hat{u}(\alpha)d\alpha + h_k^* \hat{u}^T(t)L_1 \hat{u}(t) - \int_{t-h_k(t)}^{t} \hat{u}^T(s)L_1 \hat{u}(s)ds, \quad (4.3.6) \\
2[u^T(t - \delta)A_1 + u^T(t)A_2 + u^T(t - h_k(t))A_3 + u^T(t - \sigma)A_4 + \hat{u}^T(t - \eta_k(t)) \\
\times A_5 + g^T(u(t))A_6 + g^T(u(t - h_k(t)))A_7 + \hat{u}^T(t)A_8 \\
\times [u(t) - u(t - \sigma) - \int_{t-\sigma}^{t} \hat{u}(s)ds] \\
= 2[u^T(t - \delta)A_1 u(t) + u^T(t)A_2 u(t) + u^T(t - h_k(t))A_3 u(t) \\
+ u^T(t - \sigma)A_4 u(t) + \hat{u}^T(t - \eta_k(t))A_5 u(t) + g^T(u(t))A_6 u(t) \\
+ g^T(u(t - h_k(t)))A_7 u(t) + \hat{u}^T(t)A_8 u(t) - u^T(t - \delta)A_1 \\
\times u(t - \sigma) - u^T(t)A_2 u(t - \sigma) - u^T(t - h_k(t))A_3 u(t - \sigma) \\
- u^T(t - \sigma)A_4 u(t - \sigma) - \hat{u}^T(t - \eta_k(t))A_5 u(t - \sigma) - g^T(u(t)) \\
\times A_6 u(t - \sigma) - g^T(u(t - h_k(t)))A_7 u(t - \sigma) - \hat{u}^T(t)A_8 \\
\times u(t - \sigma)] - 2[u^T(t - \delta)A_1 + u^T(t)A_2 + u^T(t - h_k(t))A_3 \\
+ u^T(t - \sigma)A_4 + \hat{u}^T(t - \eta_k(t))A_5 + g^T(u(t))A_6 \\
+ g^T(u(t - h_k(t)))A_7 + \hat{u}^T(t)A_8] \int_{t-\sigma}^{t} \hat{u}(\gamma)d\gamma + \eta_k^* \hat{u}^T(t) \\
\times L_2 \hat{u}(t) - \int_{t-h_k(t)}^{t} \hat{u}^T(s)L_2 \hat{u}(s)ds, \quad (4.3.7) \\
-2[u^T(t - \delta)K_1 + u^T(t)K_2 + u^T(t - h_k(t))K_3 + u^T(t - \sigma)K_4 + \hat{u}^T(t - \eta_k(t)) \\
\times K_5 + g^T(u(t))K_6 + g^T(u(t - h_k(t)))K_7 + \hat{u}^T(t)K_8 \\
\times \int_{t-h_k(t)}^{t} \hat{u}(\alpha)d\alpha \\
= -2[u^T(t - \delta)K_1 u(t) + u^T(t)K_2 u(t) + u^T(t - h_k(t))K_3 u(t) \\
+ u^T(t - \sigma)K_4 u(t) + \hat{u}^T(t - \eta_k(t))K_5 u(t) + g^T(u(t))K_6 u(t) \\
+ g^T(u(t - h_k(t)))K_7 u(t) + \hat{u}^T(t)K_8 u(t) - u^T(t - \delta)K_1 \\
\times u(t - h_k(t)) - u^T(t)K_2 u(t - h_k(t)) - u^T(t - h_k(t))K_3 \\
\end{align*}
\begin{align}
&\times u(t-h_k(t)) - u^T(t-\sigma)K_4u(t-h_k(t)) - u^T(t-\eta_k(t))K_5 \\
&\times u(t-h_k(t)) - g^T(u(t))K_6u(t-h_k(t)) - g^T(u(t-h_k(t))) \\
&\times K_7u(t-h_k(t)) - \hat{u}^T(t)K_8(u(t-h_k(t))) - h_5^\ast \hat{u}^T(t)L_1 \hat{u}(t) \\
&\quad + \int_{t-h_k^\ast}^t \hat{u}^T(s)L_1 \hat{u}(s)ds, \quad (4.3.8) \\
&-2[u^T(t-\delta)A_1 + u^T(t)A_2 + u^T(t-h_k(t))A_3 + u^T(t-\sigma)A_4 + \hat{u}^T(t-\eta_k(t))] \\
&\times A_5 + g^T(u(t))A_6 + g^T(u(t-h_k(t)))A_7 + \hat{u}^T(t)A_8 \\
&\times \int_{t-\sigma}^t \hat{u}(\gamma)d\gamma \\
&= -2\{[u^T(t-\delta)A_1u(t) + u^T(t)A_2u(t) + u^T(t-h_k(t))A_3u(t) \\
&\quad + u^T(t-\sigma)A_4u(t) + \hat{u}^T(t-\eta_k(t))A_5u(t) + g^T(u(t))A_6u(t) \\
&\quad + g^T(u(t-h_k(t)))A_7u(t) + \hat{u}^T(t)A_8u(t) - u^T(t-\delta)A_1 \\
&\quad \times u(t-\sigma) - u^T(t)A_2u(t-\sigma) - u^T(t-h_k(t))A_3u(t-\sigma) \\
&\quad - u^T(t-\sigma)A_4u(t-\sigma) - \hat{u}^T(t-\eta_k(t))A_5u(t-\sigma) \\
&\quad - g^T(u(t))A_6u(t-\sigma) - g^T(u(t-h_k(t)))A_7u(t-\sigma) - \hat{u}^T(t) \\
&\quad \times A_8u(t-\sigma)] - \eta_k^\ast \hat{u}^T(t)L_2 \hat{u}(t) + \int_{t-\eta_k^\ast}^t \hat{u}^T(s)L_2 \hat{u}(s)ds, \quad (4.3.9) \\
&2[u^T(t-\delta)M_1 + u^T(t)M_2 + \hat{u}^T(t)M_3 + \mathcal{H}^T(t)B_2^\ast]\left\{ -Au(t-\delta) + W_0 \\
&\times g(u(t)) + W_1^{(k)}g(u(t-h_k(t))) + W_3 \int_{t-\sigma}^t g(u(s))ds + \mathcal{H}(t) \\
&\quad - \hat{u}(t) + W_2u(t-\eta_k(t)) \right\} \\
&= 2\left\{ -u^T(t-\delta)AM_1u(t-\delta) - AM_2u^T(t)u(t-\delta) - AM_3 \\
&\times \hat{u}^T(t)u(t-\delta) + W_0M_1u^T(t-\delta)g(u(t)) + W_0M_2u^T(t)g(u(t)) \\
&\quad + W_0M_3u^T(t)g(u(t)) + W_1^{(k)}M_1u^T(t-\delta)g(u(t-h_k(t))) + M_2 \\
&\quad \times W_1^{(k)}u^T(t)g(u(t-h_k(t))) + W_1^{(k)}M_3u^T(t)g(u(t-h_k(t))) \\
&\quad + M_1W_3u^T(t-\delta) \int_{t-\sigma}^t g(u(s))ds + W_3M_2u^T(t) \int_{t-\sigma}^t g(u(s))ds \\
&\quad + W_3M_3u^T(t) \int_{t-\sigma}^t g(u(s))ds + u^T(t-\delta)M_1\mathcal{H}(t) + u^T(t)M_2 \right\}
\end{align}
\[
\begin{align*}
&\times \mathcal{H}(t) + u^T(t)M_3 \mathcal{H}(t) + \mathcal{H}^T(t)B_1^* \mathcal{H}(t) - u^T(t - \delta)M_1 \dot{u}(t) \\
&- u^T(t)M_2 \dot{u}(t) - \dot{u}^T(t)M_3 \dot{u}(t) + W_2 M_1 u^T(t - \delta)\dot{u}(t - \eta_k(t)) \\
&+ W_2 M_2 u^T(t)\dot{u}(t - \eta_k(t)) + W_2 M_3 \dot{u}^T(t)\dot{u}(t - \eta_k(t)) - \mathcal{H}^T(t) \\
&\times \Lambda B_1^* u(t - \delta) + \mathcal{H}^T(t)W_0 B_1^* g(u(t)) + \mathcal{H}^T(t)W_1^{(k)} B_1^* \\
&\times g(u(t - h_k(t))) + \mathcal{H}^T(t)W_2 B_1^* \left( \int_{t-\sigma}^{t} g(u(s))ds \right) - \mathcal{H}^T(t)B_1^* \\
&\times \dot{u}(t) + \mathcal{H}^T(t)B_1^* W_2 \dot{u}(t - \eta_k(t)) \right],
\end{align*}
\]

Thus,

\[
\dot{V}(u(t), t) \leq 2u^T(t)Q \left[ -Au(t - \delta) + W_0 g(u(t)) + W_1^{(k)} g(u(t - h_k(t))) \\
+ W_3 \int_{t-\sigma}^{t} g(u(s))ds + \mathcal{H}(t) + W_2 \dot{u}(t - \eta_k(t)) \right] + u^T(t)Y \\
u(t) - (1 - \tau_k)u^T(t - h_k(t)) Yu(t - h_k(t)) + g^T(u(t))S \\
g(u(t)) - (1 - \tau_k)g^T(u(t - h_k(t))) S g(u(t - h_k(t))) \\
+ \dot{u}^T(t) R \dot{u}(t) - (1 - \lambda_k)\dot{u}^T(t - \eta_k(t)) R \dot{u}(t - \eta_k(t)) \\
+ u^T(t)H_1 u(t) + g^T(u(t)) H_2 g(u(t)) - g^T(u(t - \sigma)) H_2 \\
g(u(t - \sigma)) + 2[\dot{u}^T(t - \delta)K_1 u(t) + u^T(t)K_2 u(t) \\
+ u^T(t - h_k(t))K_3 u(t) + u^T(t - \sigma)K_4 u(t) + \dot{u}^T(t - \eta_k(t)) \\
\times K_5 u(t) + g^T(u(t))K_6 u(t) + g^T(u(t - h_k(t))) K_7 u(t) \\
+ \dot{u}^T(t) K_8 u(t) - u^T(t - \delta) K_1 u(t - h_k(t)) - \dot{u}^T(t) K_2 \\
\times u(t - h_k(t)) - u^T(t - h_k(t)) K_3 u(t - h_k(t)) - u^T(t - \sigma) \\
\times K_4 u(t - h_k(t)) - \dot{u}^T(t - \eta_k(t)) K_5 u(t - h_k(t)) - g^T(u(t)) \\
\times K_6 u(t - h_k(t)) - g^T(u(t - h_k(t))) K_7 u(t - h_k(t)) - \dot{u}^T(t) \\
\times K_8 (u(t - h_k(t))) - h_k^* \dot{u}^T(t) L_1 \dot{u}(t) + \int_{t-h_k}^{t} \dot{u}^T(s) L_1 \dot{u}(s)ds \\
- 2[u^T(t - \delta)K_1 + u^T(t)K_2 + u^T(t - h_k(t))K_3 + u^T(t - \sigma) \\
\times K_4 + \dot{u}^T(t - \eta_k(t))K_5 + g^T(u(t))K_6 + g^T(u(t - h_k(t))) \\
\times K_7 + \dot{u}^T(t)K_8] \int_{t-h_k(t)}^{t} \dot{u}(s)ds + 2[u^T(t - \delta)A_1 u(t)
\right].
\]
+u^T(t)A_2u(t) + u^T(t - h_k(t))A_3u(t) + u^T(t - \sigma)A_4u(t)
+\dot{u}^T(t - \eta_k(t))A_5u(t) + g^T(u(t))A_6u(t) + g^T(u(t - h_k(t)))
\times A_7u(t) + \dot{u}^T(t)A_8u(t) - u^T(t - \delta)A_1u(t - \sigma) - u^T(t)
\times A_2u(t - \sigma) - u^T(t - h_k(t))A_3u(t - \sigma) - u^T(t - \sigma)A_4
\times u(t - \sigma) - u^T(t - \eta_k(t))A_5u(t - \sigma) - g^T(u(t))A_6u(t - \sigma)
\times L_2\dot{u}(t) + \int_{t-\eta_k(t)}^{t-\sigma} u^T(s)L_2s\dot{u}(s)ds - 2[u^T(t - \delta)A_1 + u^T(t)A_2
\times u^T(t - h_k(t))A_3 + u^T(t - \sigma)A_4 + u^T(t - \eta_k(t))A_5 + g^T(u(t))
\times A_6 + g^T(u(t - h_k(t)))A_7 + \dot{u}^T(t)A_8]\int_{t-\sigma}^{t} \dot{u}(t)dt + 2
\times \left\{-u^T(t - \delta)AM_1u(t - \delta) - AM_2u^T(t)u(t - \delta) - AM_3
\times \dot{u}^T(t)u(t - \delta) + W_0M_1u^T(t - \delta)g(u(t)) + W_0M_2u^T(t)
\times g(u(t)) + W_0M_3\dot{u}^T(t)g(u(t)) + W_1^{(k)}M_1u^T(t - \delta)
\times g(u(t - h_k(t))) + M_2W_1^{(k)}u^T(t)g(u(t - h_k(t))) + W_1^{(k)}M_3
\times \dot{u}^T(t)g(u(t - h_k(t))) + M_1W_3u^T(t - \delta)\int_{t-\sigma}^{t} g(u(s))ds
\times W_3M_2u^T(t)\int_{t-\sigma}^{t} g(u(s))ds + W_3M_3\dot{u}^T(t)\int_{t-\sigma}^{t} g(u(s))ds
\times u^T(t - \delta)M_1H(t) + u^T(t)M_2H(t) + \dot{u}^T(t)M_3H(t) + H^T(t)
\times B_1^*H(t) - u^T(t - \delta)M_1\dot{u}(t) - u^T(t)M_2\dot{u}(t) - \dot{u}^T(t)M_3\dot{u}(t)
\times W_2M_1u^T(t - \delta)\dot{u}(t - \eta_k(t)) + W_2M_2u^T(t)\dot{u}(t - \eta_k(t))
\times W_2M_3\dot{u}^T(t)\dot{u}(t - \eta_k(t)) - H^T(t)AB_1^*u(t - \delta) + H^T(t)W_0
\times B_1^*g(u(t)) + H^T(t)W_1^{(k)}B_1^*g(u(t - h_k(t))) + H^T(t)W_3B_1^*
\times \left(\int_{t-\sigma}^{t} g(u(s))ds\right) - H^T(t)B_1^*\dot{u}(t) + H^T(t)B_1^*W_2
\times \dot{u}(t - \eta_k(t))\right\} - 2g^T(u(t))Tg(u(t)) + 2u^T(t)TYg(u(t)). \tag{4.3.11}

According to Lemma 1.10.7, we obtain that

\[-2u^T(t)A_8\int_{t-\sigma}^{t} \dot{u}(t)dt \]
\begin{equation}
\leq -\hat{u}^T(t)A_8\Theta^{-1}_4\hat{u}(t) - \left(\int_{t-\sigma}^{t} \hat{u}(\gamma)d\gamma\right)^T A_8\Theta_1 \left(\int_{t-\sigma}^{t} \hat{u}(\gamma)d\gamma\right),
\end{equation}

\begin{equation}
2u^T(t)QW_3 \int_{t-\sigma}^{t} g(u(s))ds 
\leq u^T(t)QW_3\Theta^{-1}_2u(t) + \left(\int_{t-\sigma}^{t} g(u(s))ds\right)^T QW_3\Theta_2 \left(\int_{t-\sigma}^{t} g(u(s))ds\right),
\end{equation}

\begin{equation}
-2\hat{u}^T(t)K_8 \int_{t-h_k(t)}^{t} \hat{u}(\alpha)d\alpha 
\leq -\hat{u}^T(t)K_8\Theta^{-1}_3u(t) - \left(\int_{t-h_k(t)}^{t} \hat{u}(\alpha)d\alpha\right)^T K_8\Theta_3 \left(\int_{t-h_k(t)}^{t} \hat{u}(\alpha)d\alpha\right),
\end{equation}

\begin{equation}
\mathcal{H}^T(t)B^*_3W_3 \left(\int_{t-\sigma}^{t} g(u(s))ds\right) 
\leq \mathcal{H}^T(t)B^*_3W_3\Theta^{-1}_4\mathcal{H}(t) + \left(\int_{t-\sigma}^{t} g(u(s))ds\right)^T B^*_3W_3\Theta_4 \left(\int_{t-\sigma}^{t} g(u(s))ds\right).
\end{equation}

Substitute the equations (4.3.12)-(4.3.15) in (4.3.11), we have

\begin{equation}
\dot{V}(u(t), t) \leq e^T(t)\Xi e(t),
\end{equation}

where \(e(t) = \left[u^T(t) u^T(t - \delta) g^T(u(t)) g^T(u(t - h_k(t))) \left(\int_{t-\sigma}^{t} g(u(s))ds\right) \hat{u}^T(t - \eta_k(t)) u^T(t - h_k(t)) \hat{u}^T(t) g^T(u(t - \sigma)) u^T(t - \sigma) \left(\int_{t-h_k(t)}^{t} \hat{u}(\alpha)d\alpha\right) \left(\int_{t-h_k(t)}^{t} \hat{u}(\alpha)d\alpha\right)^T (\int_{t-\sigma}^{t} g(u(s))ds) \hat{u}(\gamma)d\gamma)^T \mathcal{H}^T(t)\right].

Let \(\varphi(t) = g(u(t)) + g(u(t - h_k(t))) + \int_{t-\sigma}^{t} g(u(s))ds\) be the output of the neural networks. Hence, by (4.3.16), we can estimate that

\begin{equation}
\dot{V}(u(t), t) - 2\varphi^T(t)\mathcal{H}(t) - \mu\mathcal{H}^T(t)\mathcal{H}(t) \leq e^T(t)\Xi e(t).
\end{equation}

By applying the Schur complement lemma, it can conclude that \(\Xi < 0\) is equivalent to (4.3.1). If \(\Xi < 0\), then \(\dot{V}(u(t), t) - 2\varphi^T(t)\mathcal{H}(t) - \mu\mathcal{H}^T(t)\mathcal{H}(t) < 0\) and from which it follows that

\begin{equation}
2\int_{t_0}^{t_1} \varphi^T(t)\mathcal{H}(t)dt > V(u(t_1), t_1) - V(u(t_0), t_0) - \mu \int_{t_0}^{t_1} \mathcal{H}^T(t)\mathcal{H}(t)dt.
\end{equation}
Since \( V(u(t), t) > 0 \) for \( u(t) \neq 0 \) and \( V(u(t)) = 0 \) for \( u(t) = 0 \), which implies that \( t_0 \rightarrow t_1 \), the time delayed neural networks (4.2.2) is passive in Definition 4.2.3. Therefore, the proof of this theorem is completed. \( \square \)

**Remark 4.3.2.** Suppose that the neutral term in neural networks (4.2.2), is not exist, then we have the transformed neural networks as follows:

\[
\begin{align*}
\hat{u}(t) &= -Au(t-\delta) + W_0g(u(t)) + \sum_{k=1}^{c} W_1^{(k)} g(u(t-h_k(t))) + W_3 \\
&\quad \times \int_{t-\sigma}^{t} g(u(s))ds + \mathcal{H}(t), \\
\hat{v}(t) &= g(u(t)) + g(u(t-h_k(t))) + \int_{t-\sigma}^{t} g(u(s))ds + \mathcal{H}(t).
\end{align*}
\]

(4.3.19)

Then by Theorem 4.3.1 and Theorem 4.4.1, it is easy to have the following Corollary 4.3.3 and Corollary 4.4.2, respectively.

**Corollary 4.3.3.** Under Assumption 4.1, for given \( \tau_k < 1 \) and \( \lambda_k < 1 \), if there exists matrices \( Q > 0, S > 0, Y > 0, H_1, H_2 > 0 \) and diagonal matrix \( T > 0 \) and any matrices \( K_i(i = \{1, 2, ..., 8\} \{\{1, 2, 3\}) \), \( B_i^* \), and \( M_j(j = 1, 2, 3) \) with appropriate dimension such that the LMI holds as follows:

\[
\Pi = \begin{bmatrix}
E_{11} & E_{12} & E_{13} & E_{14} & E_{15} & E_{16} & E_{17} & E_{18} & E_{21} & E_{22} & E_{23} & E_{24} & E_{25} & E_{26} & E_{27} & E_{28} \\
E_{21} & E_{33} & E_{34} & E_{35} & E_{36} & E_{37} & E_{38} & E_{39} & E_{41} & E_{42} & E_{43} & E_{44} & E_{45} & E_{46} & E_{47} & E_{48} \\
E_{31} & E_{34} & E_{44} & E_{45} & E_{46} & E_{47} & E_{48} & E_{49} & E_{51} & E_{52} & E_{53} & E_{54} & E_{55} & E_{56} & E_{57} & E_{58} \\
& & & & & & & & & & & & & & & \\
E_{41} & E_{44} & E_{45} & E_{46} & E_{47} & E_{48} & E_{49} & E_{49} & E_{51} & E_{52} & E_{53} & E_{54} & E_{55} & E_{56} & E_{57} & E_{58} \\
& & & & & & & & & & & & & & & \\
E_{51} & E_{54} & E_{55} & E_{56} & E_{57} & E_{58} & E_{59} & E_{59} & E_{61} & E_{62} & E_{63} & E_{64} & E_{65} & E_{66} & E_{67} & E_{68} \\
& & & & & & & & & & & & & & & \\
E_{61} & E_{64} & E_{65} & E_{66} & E_{67} & E_{68} & E_{69} & E_{69} & E_{71} & E_{72} & E_{73} & E_{74} & E_{75} & E_{76} & E_{77} & E_{78} \\
& & & & & & & & & & & & & & & \\
E_{71} & E_{74} & E_{75} & E_{76} & E_{77} & E_{78} & E_{79} & E_{79} & E_{81} & E_{82} & E_{83} & E_{84} & E_{85} & E_{86} & E_{87} & E_{88} \\
& & & & & & & & & & & & & & & \\
\end{bmatrix} < 0,
\]

(4.3.20)

where

\[
\begin{align*}
E_{11} &= QW_3Q_2^{-1} + Y + H_1 + K_2 + A_2, \\
E_{12} &= -2QA + 2K_1 + 2A_1 - 2AM_2, \\
E_{13} &= 2W_0Q + 2K_6 + 2A_6 + 2W_0M_2 + 2TY, \\
E_{14} &= 2QW_1^{(k)} + 2K_7 + 2A_7 \\
&\quad + 2M_2W_1^{(k)}, \\
E_{1212} &= B_1^* + B_1^*W_3Q_4^{-1}, \\
E_{1010} &= -K_8Q_3, \\
E_{26} &= -2K_1, \\
E_{15} &= 2QW_3 + 2W_3M_2, \\
E_{16} &= 2K_3 - 2K_2 + 2A_3, \\
E_{712} &= M_3 - B_1^*, \\
E_{17} &= 2K_8 + 2A_8 - 2M_2, \\
E_{19} &= 2K_4 + 2A_4 - 2A_2, \\
E_{57} &= 2W_3M_3, \\
E_{24} &= 2W_1^{(k)}M_1, \\
E_{25} &= 2M_1W_3, \\
E_{412} &= W_1^{(k)}B_1^*, \\
E_{27} &= -2AM_3 - 2M_1, \\
E_{36} &= -2K_6, \\
E_{210} &= -2K_1, \\
E_{211} &= -2A_1, \\
E_{29} &= -2A_1, \\
E_{88} &= -H_2, \\
\end{align*}
\]

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\[ \Xi_{46}^* = -2K_7, \quad \Xi_{311}^* = -2A_6, \quad \Xi_{510}^* = -2K_6, \quad \Xi_{47}^* = 2W_1^{(k)} M_3, \]
\[ \Xi_{67}^* = -2K_8, \Xi_{69}^* = -2K_4 - 2A_3, \quad \Xi_{510}^* = -2K_3, \quad \Xi_{611}^* = -2A_3, \]
\[ \Xi_{22}^* = -2AM_1, \quad \Xi_{911}^* = -2A_4, \quad \Xi_{910}^* = -2K_4, \quad \Xi_{33}^* = S + H_2 - 2T, \]
\[ \Xi_{55}^* = QW_3 \Theta_2 + B_1^* W_3 \Theta_4, \quad \Xi_{66}^* = -(1 - \tau_k) Y - K_3, \quad \Xi_{711}^* = -2A_8, \]
\[ \Xi_{112}^* = M_2, \quad \Xi_{212}^* = M_1 - AB_1^*, \quad \Xi_{312}^* = W_0 B_1^*, \quad \Xi_{1111}^* = -A_8 \Theta_1, \]
\[ \Xi_{77}^* = -A_8 \Theta_1^{-1} - K_8 \Theta_3^{-1} - M_3, \quad \Xi_{1111}^* = -2A_2, \quad \Xi_{23}^* = 2W_0 M_1, \]
\[ \Xi_{39}^* = -2A_6, \quad \Xi_{411}^* = -2A_7, \quad \Xi_{710}^* = -2K_8, \quad \Xi_{44}^* = -(1 - \tau_k) S, \]
\[ \Xi_{37}^* = 2W_0 M_3, \quad \Xi_{410}^* = -2K_7, \quad \Xi_{79}^* = -2A_8, \quad \Xi_{110}^* = -2K_2, \]
\[ \Xi_{69}^* = -A_4, \]

then the neural networks (4.3.19) is passive.

**Remark 4.3.4.** The authors in [60], investigates the impact of multiple time-delays in neural networks to find the stability conditions via LMIs, and M. V. Thuan, H. Trinh, L. V. Hien, discussed about the passivity study on time-delayed neural networks by using some inequality techniques in [166]. The neutral terms in neural networks are fully addressed by R. Rakkiyappan & P. Balasubramaniam in [142] to verify the stability criteria in the sense of asymptotic. The authors in [104], studied and obtained some new results on problem of stability for the time delayed neural networks with leakage terms. Also, in [182], Wu et al., conversed the exponential passivity performance of neural networks through different Lyapunov-Krasovskii functionals.

To the best of our knowledge, from the above said references, we have clearly understood the passivity problem of neural networks is considered only with discrete time-varying delays and external inputs, but the distributed delays, multiple discrete time-delays, neutral time-delays and leakage time delays has not been taken into account and also no one investigates the passivity as well as exponential passivity criteria for NNs at a time. Although the considered facts are complicated, we took this as a challenging task and admits us to do the proposed research work.
4.4 Exponential passivity results

Theorem 4.4.1. Suppose that Assumption 4.1 holds. For given scalars \( \tau_k < 1 \) and \( \lambda_k < 1 \), then the time-delayed neural networks (4.2.2) is passive in the sense of exponential, if there exists positive definite matrices \( C_r (r = 1, 2, 3, \ldots, 10) \), \( F^* \), \( D_L \), positive diagonal matrices \( \mathcal{O} \), \( J_1 \) and \( E^T_1 = [E^T_{11}, E^T_{12}, E^T_{13}, E^T_{14}, E^T_{15}, E^T_{16}, E^T_{17}, E^T_{18}, E^T_{19}, E^T_{110}, E^T_{111}, E^T_{112}, E^T_{113}, E^T_2 = [E^T_{21}, E^T_{22}, E^T_{23}, E^T_{24}, E^T_{25}, E^T_{26}, E^T_{27}, E^T_{28}, E^T_{29}, E^T_{210}, E^T_{211}, E^T_{212}, E^T_{213}, E^T_3 = [E^T_{31}, E^T_{32}, E^T_{33}, E^T_{34}, E^T_{35}, E^T_{36}, E^T_{37}, E^T_{38}, E^T_{39}, E^T_{310}, E^T_{311}, E^T_{312}, E^T_{313}, E^T_4 = [E^T_{41}, E^T_{42}, E^T_{43}, E^T_{44}, E^T_{45}, E^T_{46}, E^T_{47}, E^T_{48}, E^T_{49}, E^T_{410}, E^T_{411}, E^T_{412}, E^T_{413} \) such that the LMIs are satisfied as follows:

\[
\Lambda^* = \begin{bmatrix}
\sum \sqrt{h_k^2} E_1 & \sqrt{h_k^2} E_2 & \sqrt{\sigma} E_3 & \sqrt{\sigma} E_4 W_3 \\
* & -C_9 & 0 & 0 \\
* & * & -D_1 & 0 \\
* & * & * & -D_1 \\
\end{bmatrix} < 0, \tag{4.4.1}
\]

where

\[
\Sigma = \begin{bmatrix}
\Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} & \Gamma_{15} & \Gamma_{16} & \Gamma_{17} & \Gamma_{18} & \Gamma_{19} & \Gamma_{110} & \Gamma_{111} & \Gamma_{112} & \Gamma_{113} \\
* & \Gamma_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & \Gamma_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & \Gamma_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & \Gamma_{15} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & \Gamma_{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & \Gamma_{17} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & \Gamma_{18} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & \Gamma_{19} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & \Gamma_{110} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} < 0,
\]

\[
\begin{align*}
\Gamma_{11} &= C_3 + C_4 + C_5 + h_k^* C_8 + \eta_k^* f^* + 2 E_{11}, & \Gamma_{12} &= 2 E_{12} - E_{42} A^T - \Lambda E_{42}^T, \\
\Gamma_{22} &= -\Lambda E_{42}^T - A^T E_{42}, & \Gamma_{15} &= 2 J_1 Y + 2 E_{15} + W_0 E_{45}^T + E_{45} W_0^T, \\
\Gamma_{13} &= 2 C_1 + 2 E_{13} - E_{43}^T - E_{43}, & \Gamma_{14} &= 2 E_{14}, & \Gamma_{111} &= 2 E_{11}, & \Gamma_{1313} &= -2 E_{313}, \\
\Gamma_{16} &= 2 E_{16} + W_1^{(k)} E_{46}^T + E_{46} W_1^{(k)T}, & \Gamma_{17} &= 2 E_{17}, & \Gamma_{18} &= 2 E_{18}, & \Gamma_{19} &= 2 E_{19}, \\
\Gamma_{113} &= 2(\Gamma_{13} - E_{33}), & \Gamma_{35} &= 2 C_3 - E_{45}^T - E_{45} + W_0 E_{45}^T + W_0^T E_{45}, \\
\Gamma_{110} &= 2 E_{10} + E_{410}^T + E_{410} - 2 E_{310}, & \Gamma_{213} &= -2 E_{313}, & \Gamma_{77} &= -C_4 - 2 E_{27}, \\
\Gamma_{45} &= -2 E_{15} + 2 E_{25} + E_{45} W_0^T + W_0 E_{45}^T, & \Gamma_{513} &= E_{43} W_0^T + E_{43}^T W_0, \\
\Gamma_{1010} &= -E_{410}^T + E_{410}, & \Gamma_{33} &= C_3 + C_7 + 2 h_k^* C_9 + \eta_k^* C_10 - E_{43}^T - E_{43}, \\
\Gamma_{313} &= -2 E_{313} - E_{413}^T - E_{413}, & \Gamma_{44} &= -(1 - \tau_k) C_3 - 2 E_{14} + 2 E_{24},
\end{align*}
\]
\[ \Gamma_{46} = -2 \varepsilon_{16} + 2 \varepsilon_{26} + 2 J_1 Y + W_{11}^{(k)} \varepsilon_{46}^T + \varepsilon_{46} W_{11}^{(k)}^T, \quad \Gamma_{47} = -2 \varepsilon_{17} + 2 \varepsilon_{27}, \]
\[ \Gamma_{56} = \varepsilon_{46}^T W_0 + W_0^T \varepsilon_{46} + W_1^{(k)} \varepsilon_{46}^T + \varepsilon_{46} W_1^{(k)}^T, \quad \Gamma_{710} = -2 \varepsilon_{210}, \]
\[ \Gamma_{413} = -2 \varepsilon_{113} + 2 \varepsilon_{213} - 2 \varepsilon_{313}, \quad \Gamma_{66} = -(1 - \tau_k) C_2 - 2 J_1 - W_{11}^{(k)} \]
\[ \times \varepsilon_{46}^T - \varepsilon_{46} W_{11}^{(k)}^T, \quad \Gamma_{613} = -2 \varepsilon_{313} - W_{11}^{(k)} \varepsilon_{413}^T - \varepsilon_{413} W_{11}^{(k)}^T, \quad \Gamma_{79} = -2 \varepsilon_{29}, \]
\[ \Gamma_{48} = W_{22}^T \varepsilon_{48} + \varepsilon_{48} W_2 - 2 \varepsilon_{18} + 2 \varepsilon_{28}, \quad \Gamma_{78} = -2 \varepsilon_{28} + W_{22}^T \varepsilon_{48} + \varepsilon_{48} W_2^T, \]
\[ \Gamma_{713} = -2 \varepsilon_{213} - 2 \varepsilon_{313}, \quad \Gamma_{88} = -(1 - \lambda_k) C_5 + W_2 \varepsilon_{48}^T + W_{22}^T \varepsilon_{48}, \]
\[ \Gamma_{99} = -2 \varepsilon_{39}, \quad \Gamma_{1013} = -2 \varepsilon_{313} + \varepsilon_{413} + \varepsilon_{413}^T, \quad \Gamma_{1111} = -(1 - \lambda_k) C_6, \]
\[ \Gamma_{18} = 2 \varepsilon_{18} + W_2^T \varepsilon_{48} + \varepsilon_{48} W_2, \quad \Gamma_{a10} = -2 \varepsilon_{310} + \varepsilon_{410} + \varepsilon_{410}^T, \quad a = 1, 2, 3, \ldots, 9; \]
\[ \Gamma_{813} = -2 \varepsilon_{313} + W_2 \varepsilon_{413}^T + W_{22}^T \varepsilon_{413}, \quad \Gamma_{1113} = -2 \varepsilon_{313}, \quad \Gamma_{1213} = -2 \varepsilon_{313}, \]
\[ \Gamma_{1212} = -C_7, \quad \Gamma_{55} = C_2 + 3 \sigma D_1 + 2 \varepsilon_{35} - 2 J_1 + \varepsilon_{45} W_0^T + \varepsilon_{45}^T W_0, \]
\[ \Gamma_{711} = -2 \varepsilon_{211}, \quad \Gamma_{712} = -2 \varepsilon_{212}. \]

**Proof.** Consider the following Lyapunov-Krasovskii functional for neural networks (4.2.2) as

\[ V(u(t), t) = \sum_{l=1}^{5} V_l(u(t), t), \quad (4.4.2) \]

where

\[ V_1(u(t), t) = u^T(t) C_1 u(t) + 2 \sum_{k=1}^{m} a_k \int_0^t g_k(\xi) d\xi, \]
\[ V_2(u(t), t) = \int_{t-h_{l}(t)}^{t} (g^T(u(\xi)) C_2 g(u(\xi))) + \int_{t-h_{l}(t)}^{t} u^T(\xi) \times C_4 u(\xi) d\xi, \]
\[ V_3(u(t), t) = \int_{t-h_{l}(t)}^{t} (u^T(\xi) C_5 u(\xi) + u^T(\xi) C_6 u(\xi)) d\xi + \int_{t-h_{l}(t)}^{t} u^T(\xi) C_7 u(\xi) d\xi, \]
\[ V_4(u(t), t) = \int_{t-h_{l}(t)}^{t} (u^T(\xi) C_8 u(\xi) + 2 u^T(\xi) C_9 u(\xi)) d\xi + \int_{t-h_{l}(t)}^{t} u^T(\xi) C_7 u(\xi) d\xi, \]
\[ V_5(u(t), t) = \int_{t-h_{l}(t)}^{t} (u^T(\xi) C_{10} u(\xi) + u^T(\xi) F^* u(\xi)) d\xi, \]
\[ V_6(u(t), t) = 3 \int_{t-h_{l}(t)}^{t} g^T(u(\xi)) D_1 g(u(\xi)) d\xi. \]
Calculating the derivative of $V_l(u(t), t)$, along the trajectory $u(t)$ of (4.2.2) are

$$\dot{V}(u(t), t) = \sum_{l=1}^{6} V_l(u(t), t), \quad (4.4.3)$$

where

$$\dot{V}_1(u(t), t) = 2u^T(t)C_1 \dot{u}(t) + 2g^T(u(t))O \ddot{u}(t),$$

$$\dot{V}_2(u(t), t) = g^T(u(t))C_2 g(u(t)) - (1 - \dot{h}_k(t))g^T(u(t - h_k(t)))C_2$$
$$\times g(u(t - h_k(t))) + u^T(t)C_3 u(t) - (1 - \dot{h}_k(t))$$
$$\times u^T(t - h_k(t))C_3 u(t - h_k(t)) + u^T(t)C_4 u(t)$$
$$- u^T(t - h_k^*)C_4 u(t - h_k^*),$$

$$\dot{V}_3(u(t), t) = u^T(t)C_5 \dot{u}(t) - (1 - \dot{\eta}_k(t))u^T(t - \eta_k(t))C_5 \dot{u}(t - \eta_k(t))$$
$$+ u^T(t)C_6 u(t) - (1 - \dot{\eta}_k(t))u^T(t - \eta_k(t))C_6$$
$$\times u(t - \eta_k(t)) + \dot{u}^T(t)C_7 u(t - \eta_k^*),$$

$$\dot{V}_4(u(t), t) = h_k^* u^T(t)C_8 u(t) - \int_{t-h_k^*}^{t} u^T(\xi)C_9 u(\xi) d\xi + 2h_k^* u^T(t)C_9$$
$$\times \dot{u}(t) - 2 \int_{t-h_k^*}^{t} u^T(\xi)C_9 \dot{u}(\xi) d\xi,$$

$$\dot{V}_5(u(t), t) = \eta_k^* \dot{u}^T(t)C_{10} \dot{u}(t) - \int_{t-\eta_k^*}^{t} u^T(\xi)C_{10} \dot{u}(\xi) d\xi + \eta_k^* u^T(t)F^*$$
$$- \int_{t-\eta_k^*}^{t} u^T(\xi)F^* u(\xi) d\xi,$$

$$\dot{V}_6(u(t), t) = 3\sigma g^T(u(t))D_1 g(u(t)) - 3 \int_{t-\sigma}^{t} g^T(u(\xi))D_1 g(u(\xi)) d\xi$$
$$\leq 3\sigma g^T(u(t))D_1 g(u(t)) - \left( \int_{t-\sigma}^{t} g(u(\xi)) d\xi \right)D_1$$
$$\times \left( \int_{t-\sigma}^{t} g(u(\xi)) d\xi \right).$$

By utilizing the Newton-Leibnitz formula and (4.2.2), for any matrices $E_3$, $E_4$, $E_3$ and $E_4$ with appropriate dimensions, one can get that

$$e_1 = 2\sigma^T(t)E_1 \left[ u(t) - u(t - h_k(t)) - \int_{t-h_k(t)}^{t} \dot{u}(\xi) d\xi \right] = 0, \quad (4.4.4)$$

$$e_2 = 2\sigma^T(t)E_2 \left[ u(t - h_k(t)) - u(t - h_k^*) - \int_{t-h_k}^{t} u(\xi) d\xi \right] = 0, \quad (4.4.5)$$
\[ \varphi_3 = 2 \zeta^T(t) \mathcal{E}_3 \left[ g(u(t)) - g(u(t - \sigma)) - \int_{t-\sigma}^t g(u(\xi))d\xi \right] = 0, \quad (4.4.6) \]

\[ \varphi_4 = 2 \zeta^T(t) \mathcal{E}_4 \left[ -Au(t - \delta) + W_0 g(u(t)) + W_1^{(k)} g(u(t - h_k(t))) - W_3 \int_{t-\sigma}^t g(u(s))ds + W_2 \dot{u}(t - \eta_k(t)) + \mathcal{H}(t) - \ddot{u}(t) \right] = 0, \quad (4.4.7) \]

where

\[ \zeta^T(t) = \begin{bmatrix} u(t) \\ u(t-\delta) \\ \dot{u}(t) \\ u(t-h_k(t)) \\ \dot{g}(u(t)) \\ g(u(t-h_k(t))) \\ u(t-h_k^*(t)) \\ \dot{u}(t-\eta_k(t)) \\ u(t-\sigma) \\ \dot{\mathcal{H}}(t) \\ u(t-\eta_k^*(t)) \\ \dot{g}(u(t-\sigma)) \end{bmatrix}. \]

From the elementary inequality, for any matrix \( C_9, D_1 > 0 \), we obtain

\[ -2 \zeta^T(t) \mathcal{E}_1 \int_{t-h_k(t)}^t \dot{u}(\xi)d\xi \leq h_k^* \zeta^T(t) \mathcal{E}_1 C_9^{-1} \zeta(t) + \int_{t-h_k(t)}^t \dot{u}^T(\xi) \times C_9 \dot{u}(\xi) d\xi, \quad (4.4.8) \]

\[ -2 \zeta^T(t) \mathcal{E}_2 \int_{t-h_k^*}^{t-h_k(t)} \dot{u}(\xi)d\xi \leq h_k^* \zeta^T(t) \mathcal{E}_2 C_9^{-1} \zeta(t) + \int_{t-h_k^*}^{t-h_k(t)} \dot{u}(\xi) \times C_9 \dot{u}(\xi) d\xi, \quad (4.4.9) \]

\[ -2 \zeta^T(t) \mathcal{E}_3 \int_{t-\sigma}^t g(u(\xi))d\xi \leq \sigma \zeta^T(t) \mathcal{E}_3 D_1^{-1} \zeta(t) + \int_{t-\sigma}^t \dot{g}(u(\xi)) \times D_1 g(u(\xi)) d\xi, \quad (4.4.10) \]

\[ 2 \zeta^T(t) \mathcal{E}_4 W_3 \int_{t-\sigma}^t g(u(s))ds \leq \sigma \zeta^T(t) (\mathcal{E}_4 W_3 D_1^{-1} (\mathcal{E}_4 W_3)^T \zeta(t)) + \int_{t-\sigma}^t \dot{g}(u(s)) D_1 g(u(s))ds \quad (4.4.11) \]

\[ \varphi_4 = \zeta^T(t) \{ \mathcal{E}_4 \mathcal{M}^* + \mathcal{M}^{*T} \mathcal{E}_4 \} \zeta(t), \quad (4.4.12) \]

where \( \mathcal{M}^* = [0, -A, -I, 0, W_0, W_1^{(k)}, 0, W_2, 0, I, 0, 0, 0, 0] \).
Then, from Assumption 4.1, it can be deduced that for any positive diagonal matrix $\mathcal{J}_1$,
\begin{align}
\varphi_5 &= 2g^T(u(t))\mathcal{J}_1[Yu(t) - g(u(t))] \geq 0, \\
\varphi_6 &= 2g^T(u(t-h_k(t)))\mathcal{J}_1[Yu(t-h_k(t)) - g(u(t-h_k(t)))] \geq 0,
\end{align}
where $Y = \text{diag}(\beta_1, \beta_2, \ldots, \beta_n)$, $\beta_j > 0$, $j = 1, 2, 3, \ldots, n$ are provided in Assumption 4.1.

From (4.4.3)-(4.4.14), it follows that
\begin{align}
V(u(t), t) &= 2\varphi^T(t)\mathcal{H}(t) \\
&\leq 2u^T(t)C_1\dot{u}(t) + 2g^T(u(t))O\dot{u}(t) + g^T(u(t))C_2g(u(t)) \\
&\quad - (1 - h_k(t))g^T(u(t-h_k(t)))C_2g(u(t-h_k(t))) \\
&\quad + u^T(t)C_3u(t) - (1 - h_k(t))u^T(t-h_k(t))C_3u(t-h_k(t)) \\
&\quad + u^T(t)C_4u(t) - u^T(t-h_k^*(t))C_4u(t-h_k^*(t)) + \dot{u}^T(t)C_5\dot{u}(t) \\
&\quad - (1 - \eta_k(t))u^T(t-\eta_k(t))C_5u(t-\eta_k(t)) + u^T(t)C_6u(t) \\
&\quad - (1 - \eta_k(t))u^T(t-\eta_k(t))C_6u(t-\eta_k(t)) + \dot{u}^T(t)C_7\dot{u}(t) \\
&\quad + u^T(t-\eta_k^*(t))C_7u(t-\eta_k^*(t)) + \dot{u}^T(t-\eta_k^*(t))C_8u(t-\eta_k^*(t)) - \int_{t-h_k^*}^{t} \dot{u}^T(\zeta) \\
&\quad \times C_8u(\zeta)d\zeta + 2h_k^*\dot{u}^T(t)C_9u(t) - 2\int_{t-h_k^*}^{t} \dot{u}^T(\zeta)C_9u(\zeta)d\zeta \\
&\quad + \eta_k^*\dot{u}^T(t)C_{10}\dot{u}(t) - \int_{t-h_k^*}^{t} \dot{u}^T(\zeta)C_{10}\dot{u}(\zeta)d\zeta + \eta_k^*\dot{u}^T(t)\mathcal{F}^* \\
&\quad \times u(t) - \int_{t-\eta_k^*}^{t} \dot{u}^T(\zeta)\mathcal{F}^*u(\zeta)d\zeta + 3g^T(u(t))D_1g(u(t)) \\
&\quad + 3\left( \int_{t-\zeta}^{t} g(u(\zeta))d\zeta \right)^T D_1 \left( \int_{t-\zeta}^{t} g(u(\zeta))d\zeta \right) - 2\varphi^T(t) \\
&\quad \times \mathcal{H}(t) + \sum_{q=1}^{6} \varphi_q
\end{align}
From the inequality (4.4.15), we have
\begin{align}
V(u(t), t) - 2\varphi^T(t)\mathcal{H}(t) &\leq \xi^T(t)\Lambda^*\xi(t) - \int_{t-h_k^*}^{t} \dot{u}^T(\zeta)C_8u(\zeta)d\zeta - \int_{t-h_k^*}^{t} \dot{u}^T(\zeta)
\end{align}
where \( \Lambda^* = \Sigma + h_k^* E_1 C_9^{-1} E_1^T + h_k^* E_2 C_9^{-1} E_2^T + \sigma E_3 D_1^{-1} E_3^T + \sigma (E_4 W_3) D_1^{-1} (E_4 W_3)^T \).

By Lemma 1.10.7, the LMI (4.4.1) is equivalent to \( \Lambda^* < 0 \). Here, we note that \( |\xi(t)| \leq |u(t)| \). So,

\[
\dot{V}(u(t), t) - 2v^T(t)H(t) \leq \lambda_{\max}(\Lambda^*) |u(t)|^2 - \lambda_{\min}(\mathcal{C}_9) \int_{t-h_k^*}^t |u(\xi)|^2 d\xi \\
- \lambda_{\max}(\mathcal{C}_9) \int_{t-h_k^*}^t |u(\xi)|^2 d\xi - \lambda_{\min}(\mathcal{C}_{10}) \\
\times \int_{t-h_k^*}^t |u(\xi)|^2 d\xi - \lambda_{\min}(\mathcal{F}^*) \int_{t-h_k^*}^t |u(\xi)|^2 d\xi \\
- \lambda_{\min}(\mathcal{D}_1 Y^2) \int_{t-h_k^*}^t |u(\xi)|^2 d\xi. \tag{4.4.17}
\]

On the other hand, it can be easily checked that

\[
\dot{V}(u(t), t) \leq (\|C_1\| + \|\mathcal{O}\| \|Y\|) |u(t)|^2 + (\|C_2\| \|Y\|^2 + \|C_3\| + \|C_4\|
+h_k^* \|\mathcal{C}_9\|) \int_{t-h_k^*}^t |u(\xi)|^2 d\xi + (\|C_5\| + \|C_7\| + \eta_k^* \|\mathcal{C}_{10}\|)
\times \int_{t-h_k^*}^t |u(\xi)|^2 d\xi + (\|\mathcal{C}_6\|) \int_{t-h_k^*}^t |u(\xi)|^2 d\xi + 3\sigma \\
\times (\|\mathcal{D}_1\| \|Y\|^2) \int_{t-h_k^*}^t |u(\xi)|^2 d\xi + 2h_k^* \|\mathcal{C}_9\| \int_{t-h_k^*}^t |u(\xi)|^2 d\xi. \tag{4.4.18}
\]

Put \( \gamma \) be sufficiently small such that

\[
\gamma (\|C_1\| + \|\mathcal{O}\| \|Y\|) + \lambda_{\max}(\Lambda^*) < 0, \tag{4.4.19}
\]

\[
\gamma (\|C_2\| \|Y\|^2 + \|C_3\| + \|C_4\| + h_k^* \|\mathcal{C}_9\|) - \lambda_{\min}(\mathcal{C}_9) < 0, \tag{4.4.20}
\]

\[
\gamma (\|C_5\| + \|C_7\| + \eta_k^* \|\mathcal{C}_{10}\|) - \lambda_{\min}(\mathcal{C}_{10}) < 0, \tag{4.4.21}
\]

\[
\gamma (2h_k^* \|\mathcal{C}_9\|) - \lambda_{\min}(\mathcal{F}^*) < 0, \tag{4.4.22}
\]

\[
\gamma (3\sigma (\|\mathcal{D}_1\| \|Y\|^2)) - \lambda_{\min}(\mathcal{D}_1 Y^2) < 0. \tag{4.4.24}
\]

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So,
\[ \mathcal{V}(u(t), t) + \gamma \mathcal{V}(u(t), t) - 2v^T(t)\mathcal{H}(t) \leq 0. \] (4.4.25)

According to (4.4.25) and Definition 4.2.4, the delayed neural networks (4.2.2) is exponentially passive. \qed

**Corollary 4.4.2.** Suppose that Assumption 4.1 holds. For a given scalars \( \tau_k < 1 \) and \( \lambda_k < 1 \), then the time-delayed neural networks (4.3.19) is passive in the sense of exponential, if there exists positive definite matrices \( \mathcal{C}_r (r = 1, 2, 3, ..., 10), \mathcal{F}^*, \mathcal{D}_1 \), positive diagonal matrices \( \mathcal{O}_1, \mathcal{J}_1 \) and \( \mathcal{E}_1^T = [\mathcal{E}_{11}^T, \mathcal{E}_{12}^T, \mathcal{E}_{13}^T, \mathcal{E}_{14}^T, \mathcal{E}_{15}^T, \mathcal{E}_{16}^T, \mathcal{E}_{17}^T, \mathcal{E}_{18}^T, \mathcal{E}_{19}^T, \mathcal{E}_{10}^T, \mathcal{E}_{111}^T, \mathcal{E}_{112}^T, \mathcal{E}_{113}^T, \mathcal{E}_2^T = [\mathcal{E}_{21}^T, \mathcal{E}_{22}^T, \mathcal{E}_{23}^T, \mathcal{E}_{24}^T, \mathcal{E}_{25}^T, \mathcal{E}_{26}^T, \mathcal{E}_{27}^T, \mathcal{E}_{28}^T, \mathcal{E}_{29}^T, \mathcal{E}_{210}^T, \mathcal{E}_{211}^T, \mathcal{E}_{212}^T, \mathcal{E}_{213}^T], \mathcal{E}_3^T = [\mathcal{E}_{31}^T, \mathcal{E}_{32}^T, \mathcal{E}_{33}^T, \mathcal{E}_{34}^T, \mathcal{E}_{35}^T, \mathcal{E}_{36}^T, \mathcal{E}_{37}^T, \mathcal{E}_{38}^T, \mathcal{E}_{39}^T, \mathcal{E}_{310}^T, \mathcal{E}_{311}^T, \mathcal{E}_{312}^T, \mathcal{E}_{313}^T], \mathcal{E}_4^T = [\mathcal{E}_{41}^T, \mathcal{E}_{42}^T, \mathcal{E}_{43}^T, \mathcal{E}_{44}^T, \mathcal{E}_{45}^T, \mathcal{E}_{46}^T, \mathcal{E}_{47}^T, \mathcal{E}_{48}^T, \mathcal{E}_{49}^T, \mathcal{E}_{410}^T, \mathcal{E}_{411}^T, \mathcal{E}_{412}^T, \mathcal{E}_{413}^T] \) such that the following LMIs holds
\[
\Lambda^* = \begin{bmatrix}
\sum^* & \sqrt{\sigma} \mathcal{E}_3 & \sqrt{\sigma} \mathcal{E}_4 W_3 \\
* & -\mathcal{D}_1 & 0 \\
* & * & -\mathcal{D}_1
\end{bmatrix} < 0, \quad (4.4.26)
\]
\[
\sum = \begin{bmatrix}
\Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} & \Gamma_{15} & \Gamma_{16} & \Gamma_{17} & \Gamma_{18} & \Gamma_{19} & \Gamma_{110} & \Gamma_{111} & \Gamma_{112} \\
* & \Gamma_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Gamma_{210} & 0 & \Gamma_{213} \\
* & * & \Gamma_{33} & 0 & 0 & 0 & 0 & 0 & 0 & \Gamma_{310} & 0 & \Gamma_{313} \\
* & * & * & \Gamma_{44} & \Gamma_{45} & \Gamma_{46} & \Gamma_{47} & 0 & 0 & \Gamma_{410} & 0 & \Gamma_{413} \\
* & * & * & * & \Gamma_{55} & \Gamma_{56} & 0 & 0 & 0 & \Gamma_{510} & 0 & \Gamma_{513} \\
* & * & * & * & * & \Gamma_{66} & 0 & 0 & 0 & 0 & \Gamma_{613} \end{bmatrix} < 0,
\]
where \( \Gamma_{33}^* = -\mathcal{E}_{43}^T \). Also, the columns and rows of 8th and 12th terms in Theorem 4.4.1 are zero, the remaining values of this corollary are same as in Theorem 4.4.1.

**Remark 4.4.3.** There are no passivity and exponential passivity results on NNNs with multiple time delays (4.2.2), established via LMIs in available literature. Hence, the above gap is filled by Theorem 4.3.1 and 4.4.1 in this chapter. For consequence, a suitable Lyapunov-Krasovskii functional handled to guarantee the neutral-type neural networks (4.2.2) is passive and exponentially passive, which means that our results generalize the previous ones.
Remark 4.4.4. Based on the Example 4.5.1-4.5.2, it is easy to see that the conclude results are less conserved than the existing ones in the references cited [160, 188, 199]. Therefore, the proposed method is an improvement over the available sources.

4.5 Numerical examples.

In this section, we provide two numerical examples with their simulations to demonstrate the superiority and benefits of the proposed criteria.

Example 4.5.1. Consider a two dimensional multiple delayed neutral-type neural networks (4.2.2) with the following associated parameters:

\[
A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad W_0 = \begin{bmatrix} -0.5 & 0.4 \\ 0.3 & 0.1 \end{bmatrix}, \quad W_1^{(1)} = \begin{bmatrix} 0.1 & 0.2 \\ -0.08 & 0.4 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0.09 & -0.2 \\ 0.6 & -0.1 \end{bmatrix}, \quad W_3 = \begin{bmatrix} -0.6 & 0.3 \\ 0.5 & 0.2 \end{bmatrix}, \quad \Theta_1 = \begin{bmatrix} 0.2 \\ 0.06 \end{bmatrix}, \quad \Theta_3 = \begin{bmatrix} 0.4 \\ 0.03 \end{bmatrix}, \quad \Theta_4 = \begin{bmatrix} 0.1 \\ 0.03 \end{bmatrix}, \quad Y = \begin{bmatrix} 0.5219 & 0 \\ 0 & 1.8993 \end{bmatrix}, \quad W_1^{(2)} = \begin{bmatrix} 0.3 \\ 0.01 \end{bmatrix}, \quad \Theta_2 = \begin{bmatrix} 0.1 \\ 0.4 \end{bmatrix}, \quad 0.5. \]

\[
\delta = 2, \quad \sigma = 0.5, \quad h_k^* = 4.5, \quad \eta_k^* = 2.7, \quad \tau_k = 0.3, \quad \lambda_k = 0.5. \]

The following activation functions are playing in neural networks (4.2.2):

\[
g(u(t)) = \frac{1}{3} \times ((u(t) + 1) - |(u(t) - 1)|) .
\]

Taking the external input \( \mathcal{H}(t) = |(0.7 \times t)| \), and by solving the LMI in Theorem 4.3.1 using the MATLAB LMI control toolbox, one can obtain the feasible solutions as follows:

\[
Q = 10^{-3} \times \begin{bmatrix} 0.0659 & 0.1101 \\ 0.1101 & 0.3124 \end{bmatrix}, \quad S = \begin{bmatrix} 0.0024 & 0.0004 \\ 0.0004 & 0.0013 \end{bmatrix}, \quad R = 10^3 \times \begin{bmatrix} 1.7897 & -0.0000 \\ -0.0000 & 1.7897 \end{bmatrix}, \quad Y = \begin{bmatrix} 0.0023 & 0.0038 \\ 0.0038 & 0.0072 \end{bmatrix} .
\]
Chapter 4: Passivity and exponential passivity for NNNs

![Figure 4.1: State trajectories of addressed neutral-type neural networks (4.2.2) in Example 4.5.1](image)

\[ L_1 = 10^3 \times \begin{bmatrix} 2.6727 & -0.6990 \\ -0.6990 & 3.0479 \end{bmatrix}, \quad L_2 = 10^3 \times \begin{bmatrix} 1.8852 & -0.2038 \\ -0.2038 & 1.9944 \end{bmatrix}, \]

\[ H_1 = \begin{bmatrix} 0.0017 & 0.0028 \\ 0.0028 & 0.0051 \end{bmatrix}, \quad H_2 = 10^{-3} \times \begin{bmatrix} 0.4577 & 0.3649 \\ 0.3649 & 0.7104 \end{bmatrix}, \]

\[ T = \begin{bmatrix} 0.0046 & 0 \\ 0 & 0.0016 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0.0134 & 0.0216 \\ 0.0183 & 0.0385 \end{bmatrix}. \]

Due to page limitation, some of the feasible solutions are ignored here. Hence the state trajectories of the neutral-type neural networks (4.2.2) are depicted in Figure 4.1. By solving the LMI (4.3.1), in Theorem 4.3.1 we can find the feasible solutions. The obtained upper bounds of multiple discrete time-delay & neutral delay \( h_k^* \) and \( \eta_k^* \) for neural networks.

<table>
<thead>
<tr>
<th>Methods</th>
<th>( \tau_k = 0 )</th>
<th>( \tau_k = 0.3 )</th>
<th>( \tau_k = 0.5 )</th>
<th>( \tau_k = 0.9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>In Ref [46]</td>
<td>0.5763</td>
<td>0.5679</td>
<td>0.5273</td>
<td></td>
</tr>
<tr>
<td>In Ref [153]</td>
<td>5.346</td>
<td>-</td>
<td>4.592</td>
<td>-</td>
</tr>
<tr>
<td>In Ref [27]</td>
<td>8.7788</td>
<td>-</td>
<td>0.8912</td>
<td>0.3463</td>
</tr>
<tr>
<td><strong>Theorem 4.3.1</strong></td>
<td>43.58</td>
<td>46.731</td>
<td>37.23</td>
<td>40.8221</td>
</tr>
</tbody>
</table>
Table 4.2: MAUB ($\eta_k^*$) of neutral delay for different $\lambda_k$

<table>
<thead>
<tr>
<th>$\lambda_k$</th>
<th>0.2</th>
<th>0.5</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>In Ref [133]</td>
<td>-</td>
<td>4.5306</td>
<td>4.1632</td>
</tr>
<tr>
<td>In Ref [75]</td>
<td>-</td>
<td>4.7388</td>
<td>1.5708</td>
</tr>
<tr>
<td><strong>Theorem 4.3.1</strong></td>
<td>36.91</td>
<td>32.066</td>
<td>27.36</td>
</tr>
</tbody>
</table>

(4.2.2) are maximum, which are given in Table 4.1 and Table 4.2, respectively. Therefore by Theorem 4.3.1, we conclude that the neural networks (4.2.2) is passive. This demonstrate that the contributions made from this work is more effective and less conservative than some existing literatures.

**Example 4.5.2.** Consider a two dimensional multiple discrete delayed neutral-type neural networks (4.2.2) with the following associated parameters:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \quad W_0 = \begin{bmatrix} 0.05 & 0.1 \\ 0.2 & 0.07 \end{bmatrix}, \quad W_1^{(1)} = \begin{bmatrix} 0.3 \\ 0.1 \\ 0.4 \end{bmatrix}, \quad W_1^{(2)} = \begin{bmatrix} 0.5 & 0.04 \\ 0.03 & 0.1 \end{bmatrix},$$

$$W_2 = \begin{bmatrix} 0.4 & -0.1 \\ 1 & -0.3 \end{bmatrix}, \quad W_3 = \begin{bmatrix} -0.2 & 0.3 \\ -0.2 & 0.8 \end{bmatrix}, \quad Y = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.03 \end{bmatrix},$$

$\delta = 1$, $h_k^* = 2.5$, $\eta_k^* = 1.9$, $\tau_k = 0.3$, $\lambda_k = 0.5$, $\sigma = 0.3$. The following activation

![Figure 4.2: State performance of concerned neural networks (4.2.2) in Example 4.5.2](image)
functions are playing in neural networks (4.2.2):

\[ g(u(t)) = 0.05 \sin(u(t)). \]

Taking the external input \( \mathcal{H}(t) = \sin(-0.4) \times (\lfloor 0.5 \times t \rfloor) \), and the LMIs in Theorem 4.5.2 solved by using the LMI control toolbox in Matlab software, one can obtain the following feasible solutions as:

**Table 4.3: MAUB \( h_x^* \) in Example 4.5.2 for different \( \tau_c \)**

<table>
<thead>
<tr>
<th>Methods</th>
<th>0.3</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>In Ref [195]</td>
<td>-</td>
<td>2.9436</td>
<td>2.5630</td>
<td>2.2150</td>
</tr>
<tr>
<td>In Ref [148]</td>
<td>1.4021</td>
<td>1.3916</td>
<td>-</td>
<td>1.3734</td>
</tr>
<tr>
<td><strong>Theorem 4.4.1</strong></td>
<td>24.33</td>
<td>19.031</td>
<td>12.59</td>
<td>11.84</td>
</tr>
</tbody>
</table>

\[ C_1 = \begin{bmatrix} 6.8429 & -0.0361 \\ -0.0361 & 2.5242 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 6.8429 & -0.0361 \\ -0.0361 & 2.5242 \end{bmatrix}, \]

\[ C_3 = \begin{bmatrix} 0.1389 & -0.0127 \\ -0.0127 & 0.0131 \end{bmatrix}, \quad C_4 = \begin{bmatrix} 0.2488 & -0.0170 \\ -0.0170 & 0.0871 \end{bmatrix}, \]

\[ C_5 = \begin{bmatrix} 2.3924 & -0.6297 \\ -0.6297 & 0.9981 \end{bmatrix}, \quad C_6 = \begin{bmatrix} 0.3082 & -0.0231 \\ -0.0231 & 0.0927 \end{bmatrix}, \]

\[ C_7 = \begin{bmatrix} 1.9778 & -0.4183 \\ -0.4183 & 0.9551 \end{bmatrix}, \quad C_8 = \begin{bmatrix} 0.0648 & -0.0054 \\ -0.0054 & 0.0156 \end{bmatrix}, \]

\[ C_9 = \begin{bmatrix} 3.1116 & -0.3155 \\ -0.3155 & 0.7907 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 5.3859 & -3.2280 \\ -3.2280 & 13.7230 \end{bmatrix}, \]

\[ C_{10} = \begin{bmatrix} 1.0468 & -0.2840 \\ -0.2840 & 0.3629 \end{bmatrix}, \quad \mathcal{F}^* = \begin{bmatrix} 0.0846 & -0.0070 \\ -0.0070 & 0.0209 \end{bmatrix}. \]

Due to page limitation, some of the feasible solutions are ignored here. By using Matlab LMI control toolbox, we have compared the upper bounds of multiple discrete delay and listed in Table 4.3 for the passivity investigation. Moreover, the state responses of (4.2.2) are explored in Figure 4.2. Thus, according to Theorem 4.4.1, we conclude that the neural network system (4.2.2) is exponentially passive.
4.6 Conclusions

This chapter mainly focuses on exploring a LMI approach to study the passivity and exponential passivity for a class of neutral-type neural networks with leakage, multiple discrete delay and distributed time delay. By employing the Lyapunov theory, inequality techniques and matrix theory, some new-brand sufficient conditions were derived for the addressed neural networks to guarantee the passivity and exponential passivity. Additionally, the upper bounds of the multiple discrete time-varying delay for our concerned design is very large when compared with the previous results, which may possess highly important significance in the proposed time-delayed neural networks. The obtained theoretical findings are less conserved and it can be verified by two numerical examples with their simulations, which shows the validity of the research work.

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