CHAPTER 2

A new global robust exponential stability criterion for $H_{\infty}$ control of uncertain stochastic neutral-type neural networks with both time-varying delays

2.1 Introduction

In recent years, researches on time delayed neural networks have found fruitful attention in several fields, such as biological systems, hydraulic systems, aircraft stabilization, electrical networks and nuclear reactor, [145, 147]. Obviously, time delay in the networks system frequently causes bad performance, oscillation and instability or chaos. Thus many essential stability results on the dynamical behaviors have been reported for time delayed neural networks, see [14, 19, 115] and references there in. Moreover, time delay can be categorized as two types: discrete and distributed time delays. Based on the huge length of the axon sizes we have taken both time delays into account when model our neural networks system. For
instance many research authors in [106, 122, 158] dealt with both time delays for neural networks. Neutral time delay is another type of delay which has drawn a lot of attention nowadays. Because, a neutral-type delay phenomenon contains delays both in its derivatives and state variables. Consequently, delayed neural networks with neutral terms has received much attention in recent years and a great number of results on this issue have been reported [45, 98].

During the physical implementation among the parameter fluctuation, some external disturbances naturally affected a real system. In general, two kinds of external disturbances to be considered here, i.e., uncertain parameters and stochastic noises. In real nervous system, the connection weights of the neuron depend on certain resistance and capacitance values that include uncertain parameters (modeling errors). Moreover, the uncertainties are inevitable while modeling a neural networks. So, it is noteworthy that the parameter fluctuation and errors during their implementation, which also outcomes in poor performance and instability of the neural networks. In [63], He et al. addressed the robust stability criteria of uncertain neutral-type systems with both time delays.

On the other hand, stochastic noises in neural networks built by introducing random variations into the network, either by giving them stochastic weights, or by giving the network's neurons stochastic transfer functions. This makes them useful tools for neural networks, since the random fluctuations help it to escape from instability. Thus, the consideration of stochastic perturbations in neural networks is necessary for stability property, see [90]. Thus, the stability performance have been analyzed for time-delayed neural networks with stochastic noises (see e.g. [218]) or uncertain parameters (see e.g. [149]).

Since Zames in [198] proposed the theory of $H_\infty$ control initially. $H_\infty$ designs are used in control theory to achieve stabilization with guaranteed performance, which has been successfully applied to an extensive variety of time-delay systems, the engineering fields and including uncertain systems. The time-delay dependent
problem of $H_\infty$ control for stochastic neural networks with uncertainties have received substantial observation among control area for the past years [17, 84]. The $H_\infty$ filtering problem of switched neural networks with random delays has been investigated in [16]. So, it is pointed out that there has been a small amount of results considering robust $H_\infty$ control for time-delayed stochastic neural networks with uncertainties, which remains challenging but open. However, to the best of our knowledge, up to now, the global robust exponential stability analysis problem for $H_\infty$ control of stochastic neutral-type neural networks with uncertainties and mixed time delays has not been properly addressed. Hence, this situation motivates our present research work.

Inspired by the above discussions, our main aim in this chapter is to investigate the global robust exponential stability problem for a class of uncertain stochastic neutral-type neural networks with discrete and distributed time-varying delays. By utilizing the Lyapunov stability theory and some inequality techniques, the stability problem of the concerned neural networks is converted into the feasibility problem of a set of linear matrix inequalities. By constructing the Lyapunov-Krasovskii functional, brand-new delay-dependent conditions for globally robustly exponential stability are formulated in terms of linear matrix inequalities, which can be verified easily by MATLAB LMI control toolbox. Also, three numerical examples with their simulations are provided to illustrate the effectiveness and feasibility with less conservatism of the proposed criteria.

The main contributions of this research work are highlighted as follows:

- Neutral-type, stochastic disturbances, uncertain parameters and discrete, distributed time-varying delays are taken into account in the stability analysis of addressed neural networks.

- By the implementation, Lyapunov-Krasovskii functionals, stability theory and some LMI techniques, a new-brand sufficient conditions for globally robustly exponential stability of $H_\infty$ control USNNNs are derived in terms of
LMIs.

- By handled both the time-varying delays and also neutral time delay terms in our uncertain stochastic neutral-type neural networks with $H_{\infty}$ control design, the allowable upper bounds of discrete, distributed & neutral time-varying delays are very large, when compared with the existing results, see Table 2.1 & Table 2.3 in Example 2.5.1 and 2.5.3, respectively. This explore the approach developed in this chapter is effective and less conservative than some previous works in the literature, which ensures the advantages of this research work.

### 2.2 Model description and preliminaries

In this chapter, we consider the stochastic neutral-type neural networks with structured uncertainties and mixed time-varying delays described by the following integro-differential equation:

\[
\begin{align*}
y(t) - (U_1 + \Delta U_1(t))y(t - d(t)) &= -(A + \Delta A(t))y(t) + (V_0 + \Delta V_0(t))f(y(t)) + (V_1 + \Delta V_1(t)) \\
&\quad \times g(y(t - k(t))) + (B + \Delta B(t))v(t) + Cu(t) + (V_2 + \Delta V_2(t)) \\
&\quad \times \int_{t - \sigma(t)}^{t} h(y(s))ds + [\Delta E(t)y(t) + \Delta E_k(t)y(t - k(t))]d\omega(t), \\
x(t) &= Fy(t) + Dv(t), \quad t > 0, \\
y(t) &= \varphi(t), \quad t \in [-K, 0], \\
\end{align*}
\]

where $y(t) \in \mathbb{R}^n$ is the state vector; $v(t) \in \mathbb{R}^m$ is the control input; $u(t) \in \mathbb{R}^p$ is the disturbance input belonging to $L_2[0, \infty)$; $x(t) \in \mathbb{R}^q$ is the system output; $f(y(t)) = (f_1(y(t)), f_2(y(t)), \ldots, f_n(y(t)))^T$, $g(y(t)) = (g_1(y(t)), g_2(y(t)), \ldots, g_n(y(t)))^T$ and $h(y(t)) = (h_1(y(t)), h_2(y(t)), \ldots, h_n(y(t)))^T$ are the neuron activation functions with $f(0) = g(0) = h(0) = 0$; the bound functions $k(t)$, $\sigma(t)$ and $d(t)$ respectively represents discrete, distributed and neutral delays of systems.
with \( 0 \leq k(t) \leq k_M, \dot{k}(t) \leq k_D < 1, 0 \leq \sigma(t) \leq \sigma_M, \dot{\sigma}(t) \leq \sigma_D < 1 \) and \( 0 \leq d(t) \leq d_M, \dot{d}(t) \leq d_D < 1 \), \( K = \max\{k_M, \sigma_M, d_M\} \); \( \omega(t) = (\omega_1(t), \omega_2(t), ..., \omega_n(t))^T \) is an \( n \)-dimensional Brownian motion defined on a complete probability space \((\mathcal{A}, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) with a Filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) satisfying the usual conditions (i.e. it is right continuous and \( \mathcal{F}_0 \) contains all \( \mathbb{P} \)-null sets) and \( \mathbb{E}\{d\omega(t)\} = 0, \mathbb{E}\{d\omega^2(t)\} = dt \). It is assumed that \( \varphi^T(t)\varphi(t) \leq \beta_0, t \in [-K, 0] \), where \( \beta_0 > 0 \) can be seen as an upper bound in the initial states. The diagonal matrix \( A = \text{diag}\{a_1, a_2, ..., a_n\} \) has positive entries \( a_i > 0 \). \( V_0 = (V_{0ij})_{n \times n}, V_1 = (V_{1ij})_{n \times n}, V_2 = (V_{2ij})_{n \times n}, U_1 = (U_{1ij})_{n \times n}, B = (B_{ij})_{n \times n}, C = (C_{ij})_{n \times n}, F = (F_{ij})_{n \times n} \) and \( D = (D_{ij})_{n \times n} \) are constant matrices with appropriate dimensions. \( \Delta A(t), \Delta V_0(t), \Delta V_1(t), \Delta V_2(t), \Delta U_1(t), \Delta B(t), \Delta E(t) \) and \( \Delta E_s(t) \) represents the structured uncertainties in stochastic neutral-type neural networks (2.2.1).

Throughout this chapter, to obtain the sufficient criteria, we need the following assumptions.

**Assumption 2.1.** The neuron activation functions \( f_i(\cdot), g_i(\cdot) \) and \( h_i(\cdot) \) \((i = 1, 2, ..., n)\) are bounded and there exists a positive diagonal matrix \( G = \text{diag}(\zeta_1, \zeta_2, ..., \zeta_n) \) such that

\[
0 \leq \frac{f_i(\varphi_1) - f_i(\varphi_2)}{\varphi_1 - \varphi_2} \leq \zeta_i, \quad \forall \varphi_1 \neq \varphi_2, \varphi_1 \& \varphi_2 \in \mathbb{R}, i = 1, 2, ..., n; \tag{2.2.2}
\]

where \( \zeta_i > 0 \) (\(i = 1, 2, ..., n\)) are some given constants. Similarly, the functions \( g_i(\cdot) \) and \( h_i(\cdot) \) satisfies the condition (2.2.2).

**Assumption 2.2.** The perturbed uncertain matrices \( \Delta A(t), \Delta V_0(t), \Delta V_1(t), \Delta V_2(t), \Delta U_1(t), \Delta B(t), \Delta E(t) \) and \( \Delta E_s(t) \) are time-varying functions satisfying:

\[
\Delta V_0(t) = PF_0(t)N_{V_0}, \quad \Delta V_1(t) = PF_1(t)N_{V_1}, \quad \Delta V_2(t) = PF_2(t)N_{V_2}, \quad \Delta E(t), \Delta E_s(t)] = PF_3(t) [N_E, N_{E_s}] \quad \text{and} \quad \Delta A(t), \Delta B(t) = PF_4(t)[N_A, N_B], \quad \Delta U_1(t) = P\bar{F}_1(t)N_{U_1}, \quad \text{where} \quad P, N_A, N_B, N_{V_0}, N_{V_1}, N_{V_2}, N_E \quad \text{and} \quad N_{E_s} \quad \text{are given constant matrices.} \quad F_i(t) \quad (l = 0, 1, 2, 3, 4) \quad \text{and} \quad \bar{F}_1(t) \quad \text{are unknown real time-varying matrices which are in the following structure:} \quad F_i(t) = \text{blockdiag}\{\delta_{i1}(t)I_{r_1}, \delta_{i2}(t)I_{r_2}, ..., \delta_{ir}(t)I_{r_r}, F_i(t), ..., F_i(t)\}, \quad \delta_{il} \in \mathbb{R}, \quad |\delta_{il}| \leq 1, \quad 1 \leq i \leq k \quad \text{and} \quad F_i^T F_i \leq I, \quad 1 \leq j \leq r. \quad \text{We define the set} \quad \Delta_l =
{F_i^T F_i \leq I, F_i S_i = S_i F_i} for all S_i \in \Sigma_i$, where \( \Sigma_i = \{S_i = \text{blockdiag}[S_{i_1}, S_{i_2}, ..., S_{i_r}], S_{i_j} \text{ invertible } 1 \leq i \leq k \text{ and } s_{i_j} \in \mathbb{R}, s_{i_j} \neq 0 \text{ for } 1 \leq j \leq r \} \), \( S_{i_j} \) also have the same expression of \( F_i(t) \) \( (i = 0, 1, 2, 3, 4) \).

In this chapter, we implement the following control law to deal with the desired performance for the uncertain stochastic neutral-type neural networks (2.2.1):

\[
v(t) = Ly(t),
\]

where \( L \) denotes the gain matrices of the controller given by \( L = XZ^{-1} \).

**Remark 2.2.1.** Note that the Assumption 2.1 is used to linearize the nonlinear functions \( f(y(t)), g(y(t)) \) and \( h(y(t)) \). And also the activation functions used in Assumption 2.1, are less restrictive than the descriptions on Lipschitz-type activation functions as well as sigmoid activation functions. These functions will play a major role in our proposed methods to obtain the necessary conditions for the neural networks (2.2.1) which leads to be global robustly exponentially stable.

**Remark 2.2.2.** The firing rates and the weight coefficients of the neurons depend on certain resistance and capacitance values, which are subject to uncertain parameters in practical implementation of neural networks. So, it is necessary to take uncertainties into account in the \( H_{\infty} \) control neutral type neural networks. However, in [39], the stability problem was only studied for \( H_{\infty} \) control neural networks with discrete delays without uncertain parameters was investigated, no distributed delays term was proposed. In addition to that neutral terms are also not considered in [41, 155]. It means that the results, in [186] fail to deal with the cases for globally robustly exponential stability conditions and thus Theorem 2.4.1 presented here is more general and practical than the existing results in [39, 41, 155, 186].

**Definition 2.2.3.** The equilibrium point of time-delayed stochastic neural networks (2.2.1) is said to be asymptotically stable in the mean square sense if, when \( u(t) = 0 \), for all finite initial functions \( \varphi \in \mathbb{R}^n \) defined on \([-K, 0] \), the following equation is satisfied:

\[
\lim_{t \to \infty} \mathbb{E}\{ |y(t, \varphi)|^2 \} = 0.
\]
Definition 2.2.4. The time-delayed uncertain neural network system (2.2.1) is robustly mean square stabilized if the equilibrium point of the stochastic functional differential equation related to system (2.2.1) with $v(t) = Ly(t)$ is globally asymptotically stable for admissible uncertainties $\Delta A(t), \Delta B(t), \Delta E(t), \Delta E_5(t), \Delta V_0(t), \Delta V_1(t), \Delta V_2(t)$ and $\Delta U_1(t)$ in the mean square sense.

Definition 2.2.5. The zero solution of the system (2.2.1) when $u(t) = 0$ is said to be globally robustly exponential stable in the mean square sense if, there exist constants $\rho > 0$ and $\alpha > 0$ such that for $t \geq 0$

$$E\{\|y(t)\|^2\} \leq \rho e^{-2\alpha t}E\left\{\sup_{-k_m \leq s \leq 0} (\|\varphi(s)\|^2 + \|\phi(s)\|^2)\right\},$$

where $\varphi(t)$ is the initial function of $y_1$.

2.3 Robust stabilization in the mean square sense

Theorem 2.3.1. Under Assumptions 2.1 & 2.2, the time-delayed structured uncertain neutral-type neural networks (2.2.1) is robustly stabilized in the mean square sense with $u(t) = 0$ if $\exists$ symmetric positive definite matrices $Z, Q, M_1, J_0, J_1, J_2, J_3, J_4, M_2, M_3, M_4$ & $M_5$ and a matrix $X$ satisfying the following LMIs:

$$\begin{bmatrix} -Z & P J_3 \\ * & -J_3 \end{bmatrix} < 0,$$

(2.3.1)

$$\Omega = \begin{pmatrix} \Pi & \Xi \\ * & \Theta \end{pmatrix} < 0,$$

(2.3.2)

where

$$\Pi = \begin{bmatrix} \hat{\Omega}_1 & V_0 & V_3 & V_2 & 0 & 0 & 0 & 0 & \Phi_4 & \Phi_5 & \Phi_{10} \\ * & \hat{\Omega}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \hat{\Omega}_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \hat{\Omega}_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \hat{\Omega}_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \hat{\Omega}_6 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \hat{\Omega}_7 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \hat{\Omega}_8 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & \hat{\Omega}_9 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & \hat{\Omega}_{10} \end{bmatrix}.$$
\[
Z = \begin{bmatrix}
\Phi_{11} & \Phi_{12} & \Phi_{13} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \Phi_{14} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \Phi_{15} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \Phi_{16} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \Phi_{17} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]

\[
\Theta = \text{diag}\{\hat{\Omega}'_{11}, \hat{\Omega}'_{12}, \hat{\Omega}'_{13}, \hat{\Omega}'_{14}, \hat{\Omega}'_{15}, \hat{\Omega}'_{16}, \hat{\Omega}'_{17}\}, \hat{\Omega}'_{i1} = -ZA - AZ + BX
\]

\[+ X^T B^T + P J_4 P^T - ZM_5 Z; \quad \hat{\Omega}'_2 = -I + N_{10}^T J_0 N_{10}; \quad \hat{\Omega}'_3 = -(1 - k_D)
\]

\[\times M_2 + N_{v1}^T J_1 N_{v1}; \quad \hat{\Omega}'_4 = -M_3 + N_{v2}^T J_2 N_{v2}; \quad \hat{\Omega}'_5 = M_2; \quad \Phi_{16} = ZP;
\]

\[\hat{\Omega}'_6 = \sigma_M^2 M_3; \quad \hat{\Omega}'_7 = M_4 + d^2_M M_5; \quad \hat{\Omega}'_8 = -M_5; \quad \hat{\Omega}'_9 = -(1 - d_D) M_4;
\]

\[\hat{\Omega}'_{10} = \frac{1}{2} I_3; \quad \Phi_{17} = ZP; \quad \hat{\Omega}'_{11} = \frac{-1}{2} (1 - k_D) I_3; \quad \hat{\Omega}'_{12} = -I_4; \quad \hat{\Omega}'_{13} = -I;
\]

\[\hat{\Omega}'_{14} = -J_0; \quad \hat{\Omega}'_{15} = -J_1; \quad \hat{\Omega}'_{16} = -J_2; \quad \hat{\Omega}'_{17} = -T_1; \quad \Phi_8 = ZM_5 Z;
\]

\[\Phi_9 = ZU_1; \quad \Phi_{10} = ZN_{E_1}^T; \quad \Phi_{11} = ZN_{E_1}^T; \quad \Phi_{12} = (Z N_A Z - Z N_B X)^T;
\]

\[\Phi_{13} = ZGZ; \quad \Phi_{14} = ZP; \quad \Phi_{15} = ZP;
\]

and * denotes the symmetric terms of the matrix. In the above LMIs, \(J_0, J_1, J_2, J_3, I_4\) and \(T_1\) respectively take the forms of \((S_0 S_0^T)^{-1}\), \((S_1 S_1^T)^{-1}\), \((S_2 S_2^T)^{-1}\), \((S_3 S_3^T)^{-1}\), \((S_4 S_4^T)^{-1}\) and \((\bar{S}_1 \bar{S}_1^T)^{-1}\) for \(S_l, \bar{S}_l \in \Sigma_l (l = 0, 1, 2, 3, 4)\).

Proof. First, by applying the controller \(v(t) = Ly(t)\) to the system (2.2.1) with \(u(t) = 0\), we have the following system:

\[
y(t) = \left[ -\mathcal{A}y(t) + \nabla_0 f(y(t)) + \nabla_1 g(y(t - k(t))) + \nabla_2 \right.
\]

\[\times \int_{t - \sigma(t)}^t h(y(s)) ds + [\Delta E_y(t) + \Delta E_z y(t - k(t))] d\omega(t)
\]

\[+ \overline{U}_1 y(t - d(t)) \right], \tag{2.3.3}
\]

where \(\mathcal{A} = (A + \Delta A(t)) - (B + \Delta B(t)) L, \nabla_0 = V_0 + \Delta V_0(t), \nabla_1 = V_1 + \Delta V_1(t),\)

and \(\nabla_2 = V_2 + \Delta V_2(t), \overline{U}_1 = U_1 + \Delta U_1(t)\). Now, we introduce the Lyapunov-
Krasovskii functional candidate $V(y_i, t)$ as follows:

$$
V(y_i, t) = y^T(t)Qy(t) + \int_{t-k(t)}^t y^T(s)M_1y(s)ds + \int_{t-k(t)}^t g^T(y(s))
\times M_2g(y(s))ds + \sigma_M \int_{t-\sigma_M}^t (s-(t-\sigma_M))h^T(y(s))
\times M_3h(y(s))ds + \int_{t-d(t)}^t \dot{y}^T(s)M_4\dot{y}(s)ds + d_M
\times \int_{t-d_M}^0 \int_{t+\theta}^t \dot{y}^T(s)M_5\dot{y}(s)ds\ d\theta, \quad (2.3.4)
$$

where $M_1$ is given by $M_1 = \frac{2}{1-k_D}N_1^{-1}E_3^{-1}N_{E_3}$, then the averaged derivative will be given by the following expressions:

$$
\mathcal{L}V(y_i) = y^T(t)\left[-Q\bar{A} - \bar{A}^TQ\right]y(t) + 2y^T(t)Q\bar{V}_0f(y(t)) + 2y^T(t)
\times Q\bar{V}_1g(y(t-k(t))) + 2y^T(t)Q\bar{V}_2 \int_{t-\sigma(t)}^t h(y(s))ds
+ 2y^T(t)Q\bar{U}_1\dot{y}(t-d(t)) + \left[\Delta E\dot{y}(t) + \Delta E_3y(t-k(t))\right]^T
\times Q[\Delta E\dot{y}(t) + \Delta E_3y(t-k(t))] + y^T(t)M_1y(t) - (1-\dot{k}(t))
\times y^T(t-k(t))M_1y(t-k(t)) + g^T(y(t))M_2g(y(t)) - (1-\dot{k}(t))
\times g^T(y(t-k(t)))M_2g(y(t-k(t))) + \sigma_M^2h^T(y(t))M_3h(y(t))
- \sigma_M \int_{t-\sigma_M}^t h^T(y(s))M_3h(y(s))ds + f^T(y(t))f(y(t)) - f^T(y(t))
\times f(y(t)) + \dot{y}^T(t)M_4\dot{y}(t) - (1-d_D)y^T(t-d(t))M_4\dot{y}(t-d(t))
+d_M^2\dot{y}^T(t)M_5\dot{y}(t) - d_M \int_{t-d_M}^t \dot{y}^T(s)M_5\dot{y}(s)ds. \quad (2.3.5)
$$

Let $Z = Q^{-1}$, then by Lemma 1.10.6, the inequality (2.3.1) will be equivalent to

$$
Q^{-1} - PF_3P^T > 0. \quad (2.3.6)
$$

Therefore, it follows from Lemma 1.10.3 & 1.10.7 that,

$$
\left[\Delta E\dot{y}(t) + \Delta E_3y(t-k(t))\right]^TQ[\Delta E\dot{y}(t) + \Delta E_3y(t-k(t))]
= \{PF_3(t)[N_E\dot{y}(t) + \Delta E_3y(t-k(t))]\}^TQ\{PF_3(t)[N_E\dot{y}(t) + \Delta E_3y(t-k(t))]\}
$$
\begin{align*}
&\leq [N_{E}y(t)]^T J_3^{-1}[N_{E}y(t)] + [N_{E}y(t - k(t))]^T J_3^{-1}[N_{E}y(t - k(t))] \\
&\quad \times y(t - k(t)) + [N_{E}y(t)]^T J_3^{-1}[N_{E}y(t)] + [N_{E}y(t - k(t))]^T \\
&\quad \times J_3^{-1}J_3^{-1}[N_{E}y(t - k(t))] \\
&\leq 2[N_{E}y(t)]^T J_3^{-1}[N_{E}y(t)] + 2[N_{E}y(t - k(t))]^T J_3^{-1}[N_{E}y(t - k(t))] \\
&\quad \times y(t - k(t)). \quad (2.3.7)
\end{align*}

From Lemma 1.10.1 and Assumption 2.1, we obtain
\begin{align*}
-\sigma_M \int_{t-\sigma_M}^{t} h^T(y(s)) M_3 h(y(s)) ds &
\leq -\sigma(t) \int_{t-\sigma(t)}^{t} h^T(y(s)) M_3 h(y(s)) ds \\
&\leq -\left[ \int_{t-\sigma(t)}^{t} h(y(s)) ds \right]^T M_3 \left[ \int_{t-\sigma(t)}^{t} h(y(s)) ds \right] \quad (2.3.8)
\end{align*}
\begin{align*}
f^T(y(t)) f(y(t)) &\leq y^T(t) gG y(t) \quad (2.3.9)
\end{align*}
\begin{align*}
-d_M \int_{t-d_M}^{t} \dot{y}^T(s) M_5 \dot{y}(s) ds &
\leq -d \int_{t-d(t)}^{t} \dot{y}^T(s) M_5 \dot{y}(s) ds \\
&\leq \left[ \begin{array}{c} \frac{y(t)}{y(t - d(t))} \\
M_R & -M_5 \end{array} \right] \left[ \begin{array}{c} -M_5 \\
M_5 & -M_5 \end{array} \right] \left[ \begin{array}{c} y(t) \\
y(t - d(t)) \end{array} \right]. \quad (2.3.10)
\end{align*}

Substituting the inequalities (2.3.7)-(2.3.10) into equation (2.3.5), we have
\begin{align*}
\mathcal{L}V(y(t)) &\leq y^T(t) \left[ -Q\overline{A} - \overline{A}^T Q \right] y(t) + 2y^T(t)Q \overline{V}_0 f(y(t)) + 2y^T(t)Q \\
&\times \overline{V}_1 g(y(t - k(t))) + 2y^T(t)Q \overline{V}_2 \int_{t-\sigma(t)}^{t} h(y(s)) ds + 2 \\
&\times y^T(t)Q \overline{U}_1 y(t - d(t)) + 2[N_{E}y(t)]^T J_3^{-1}[N_{E}y(t)] + 2 \\
&\times [N_{E}y(t - k(t))]^T J_3^{-1}[N_{E}y(t - k(t))] + y^T(t) M_1 y(t) \\
&\quad - (1 - k(t)) y^T(t - k(t)) M_1 y(t - k(t)) + g^T(y(t)) M_2 \\
&\quad \times g(y(t)) - (1 - k(t)) g^T(y(t - k(t))) g(y(t - k(t))) \\
&\quad + \sigma_M^2 h^T(y(t)) M_3 h(y(t)) - \left[ \int_{t-\sigma(t)}^{t} h(y(s)) ds \right]^T M_3 \\
&\quad \times \left[ \int_{t-\sigma(t)}^{t} h(y(s)) ds \right] + y^T(t) gG y(t) - f^T(y(t))
\end{align*}
\[
\begin{align*}
&\times f(y(t)) + y^T(t)M_4y(t) - (1 - d_D)y^T(t - d(t))M_4 \\
&\times y(t - d(t)) + d^2_My^T(t)M_5y(t) - y^T(t)M_5y(t) \\
&- y^T(t - d(t))M_5y(t - d(t)) + y^T(t)M_5y(t - d(t)) \\
&+ y^T(t - d(t))M_5y(t) \\
&= y^T(t)[-Q(A - BL) - (A - BL)^TQ + 2N_E^TJ_3^{-1}N_E \\
&+ \frac{2}{1 - k_D}N_E^TN_E^{-1} + GG - M_5]y(t) + f^T(y(t)) \\
&\times [-I]f(y(t)) + g^T(y(t - k(t)))[-(1 - k(t))M_2] \\
&\times g(y(t - k(t))) + \left( \int_{t-\sigma(t)}^t h(y(s))ds \right)^T[-M_3] \\
&\times \left( \int_{t-\sigma(t)}^t h(y(s))ds \right) + g^T(y(t))M_2g(y(t)) \\
&+ h^T(y(t))s_M^2M_3h(y(t)) + y^T(t)[M_4 + d^2_MM_5] \\
&\times y(t) + y^T(t - d(t))[-(1 - d_D)M_4]y(t - d(t)) \\
&+ y^T(t - d(t))[-M_5]y(t - d(t)) + 2y^T(t)Q\bar{V}_0f(y(t)) \\
&+ 2y^T(t)Q\bar{V}_1g(y(t - k(t))) + 2y^T(t)Q\bar{V}_2 \int_{t-\sigma(t)}^t h(y(s))ds \\
&\times ds + 2y^T(t)M_5y(t - d(t)) + 2y^T(t)Q\bar{U}_1y(t - d(t)).
\end{align*}
\]

\[
\mathcal{L}V(y_t) \leq \Xi^T(t)\Omega\Xi(t), \quad (2.3.11)
\]

where

\[
\Xi(t) = \begin{bmatrix}
y^T(t), f^T(y(t)), g^T(y(t - k(t))), \left( \int_{t-\sigma(t)}^t h(y(s))ds \right)^T, g^T(y(t))
y^T(t), \dot{y}^T(t), y^T(t - d(t)), \dot{y}^T(t - d(t))
\end{bmatrix}^T,
\]

\[
\Omega_1 = \begin{bmatrix}
\Omega_{11} & QV_0 & QV_1 & 0 & 0 & 0 & 0 & M_4 & 0 \\
* & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}.
\]
with \( \Omega_{11} = -Q(A - BL) - (A - BL)^TQ + 2N_E^T J_3^{-1} N_E + \frac{2}{1 - \kappa_D} N_{E_3}^T J_3^{-1} N_{E_3} + GG - M_5 - QPF_4(t) \times (N_A - N_B L) - QPF_4(t) (N_A - N_B L)^T; \)

\[
Q_1 = \begin{bmatrix}
QP \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix},
S_0 = [0 \ N_V_0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0],
\]

\[
S_1 = [0 \ 0 \ N_V_1 \ 0 \ 0 \ 0 \ 0 \ 0], S_2 = [0 \ 0 \ 0 \ N_V_2 \ 0 \ 0 \ 0 \ 0],
\]

and \( \bar{S}_1 = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ N_U_1]. \)

By Lemma 1.10.6 and 1.10.7, we can get that \( \Omega \leq \Omega', \) where

\[
\Omega' = \begin{bmatrix}
\Pi' & \Xi' \\
* & \Theta'
\end{bmatrix},
\]

(2.3.13)

where

\[
\Pi' = \begin{bmatrix}
\hat{\Omega}_1 & \Psi_2 & \Psi_3 & \Psi_4 & 0 & 0 & 0 & M_5 & \Psi_9 & \Psi_{10} \\
* & \hat{\Omega}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & \hat{\Omega}_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & \hat{\Omega}_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & \hat{\Omega}_5 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & \hat{\Omega}_6 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & \hat{\Omega}_7 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & \hat{\Omega}_8 & 0 & 0 \\
* & * & * & * & * & * & * & * & \hat{\Omega}_9 & 0 \\
* & * & * & * & * & * & * & * & * & \hat{\Omega}_{10}
\end{bmatrix},
\]

\[
\Xi' = \begin{bmatrix}
\Psi_{11} & \Psi_{12} & \Psi_{13} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \Psi_{14} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \Psi_{15} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \Psi_{16} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \Psi_{17} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \Psi_{17}
\end{bmatrix},
\]

41
\[ \Theta' = \text{diag}\left\{ \bar{\Omega}'_{11}, \bar{\Omega}'_{12}, \bar{\Omega}'_{13}, \bar{\Omega}'_{14}, \bar{\Omega}'_{15}, \bar{\Omega}'_{16}, \bar{\Omega}'_{17} \right\}; \bar{\Omega}'_1 = -Q(A - BL) - (A - BL)^T Q + (QP)J_4(QP)^T - M_5; \Psi_2 = QV_0; \Psi_3 = QV_1; \Psi_4 = QV_2; \Psi_9 = QU_U; \Psi_{10} = N_E^T; \Psi_{11} = N_E^T; \Psi_{13} = G; \Psi_{14} = QP; \Psi_{15} = QP; \Psi_{16} = QP; \Psi_{17} = QP \text{ and } \Psi_{12} = (N_A - N_B L)^T. \]

Now pre and post-multiply the matrix \( \bar{\Omega}' \) by \( \text{diag}(Z, I, I, I, I, I, I, I, I, I, I, I) \) and let \( X = LZ \), we can see that \( \bar{\Omega}' < 0 \) is equivalent to LMI (2.3.2).

Note that LMI (2.3.2) implies that there exists a scalar \( \eta > 0 \) such that

\[ \Phi + \text{dia}(\eta I, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) < 0. \]

From this and the inequality (2.3.12), we obtain \( LV(y) \leq -\eta |y(t)|^2 \).

According to Itô’s formula, the neural networks (2.2.1) is robustly stabilized in the mean square sense. The proof of this theorem is completed. \( \square \)

**Remark 2.3.2.** In Theorem 2.3.1, the time delay-dependent robust stabilized criterion for the system (2.2.1) with \( u(t) = 0 \) be established. For consequence, a different Lyapunov-Krasovskii functional handled in the following Theorem 2.4.1 to guarantee the stochastic neutral-type neural networks (2.2.1) with \( u(t) = 0 \) is global robust exponentially stable in the mean square sense.

**Remark 2.3.3.** In [28], Cheng et al., discussed about the stability results of the state vector for neutral NNs and, in [115], Liu et al. investigated the \( H_\infty \) behavior for time-delayed neural networks. The authors in [43], conversed the robust exponential stability criteria for neural networks with multiple delays. Also, uncertain parameters in \( H_\infty \) problem have been analyzed by Li et al. [110].

In all the above said references the robustly stabilized problem for \( H_\infty \) control of neural networks is considered only with mixed time-delays, but the uncertain parameters, stochastic noise, neutral terms has not been taken into account and also no one investigates the global robust exponentially stable together at a time. So, consider the above facts are very challenged in this chapter.
2.4 Globally robustly exponential stability in the mean square sense

Theorem 2.4.1. Under Assumptions 2.1 & 2.2, the equilibrium point of the time-delayed neural networks (2.2.1) is said to be global robust exponentially stable with $u(t) = 0$ in the mean square sense, if there exist symmetric positive definite matrices $Z, J_3, Q, M_1, M_2, M_3, M_4, M_5$ and a matrix $X$ satisfying the following LMIs:

$$\begin{bmatrix} -Z & P_{13} \\ * & -J_3 \end{bmatrix} < 0, \tag{2.4.1}$$

$$\Omega^* = \begin{bmatrix} \Pi^* & \Xi^* \\ * & \Theta^* \end{bmatrix} < 0, \tag{2.4.2}$$

holds, where

$$\Pi^* = \begin{bmatrix} \bar{Y}_1 & \Lambda_2 & \Lambda_3 & \Lambda_4 & 0 & 0 & 0 & M_5 & \Lambda_9 & \Lambda_{10} \\ * & \bar{Y}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \bar{Y}_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \bar{Y}_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \bar{Y}_5 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \bar{Y}_6 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \bar{Y}_7 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \bar{Y}_8 & 0 & 0 \\ * & * & * & * & * & * & * & * & \bar{Y}_9 & 0 \end{bmatrix}$$

$$\Xi^* = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Lambda_{14} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Lambda_{15} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Lambda_{16} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_{17} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_{17} \end{bmatrix},$$

$$\Theta^* = \text{diag} \{ \bar{Y}_{11}, \bar{Y}_{12}, \bar{Y}_{13}, \bar{Y}_{14}, \bar{Y}_{15}, \bar{Y}_{16}, \bar{Y}_{17} \}; \bar{Y}_1 = -Q(A - BL) - (A - BL)^TQ + 2N_E^TJ_3^{-1}N_E + \frac{2}{1-k_D}N_E^TJ_3^{-1}N_E + GG - M_5 - QP(N_A - N_BL) - QP(N_A - N_BL)^T;$$

$$\bar{Y}_2 = -I; \bar{Y}_3 = -(1 - k_D)e^{-2\alpha x_M}M_2; \bar{Y}_4 = -M_3; \bar{Y}_5 = M_2; \bar{Y}_6 = \sigma_M^2 M_3; \bar{Y}_7 =$$
$M_4 + d_{3,1}M_5; \hat{y}_8 = -M_5; \hat{y}_9 = -(1 - d_D)e^{-2\alpha_dM}M_4; \hat{y}_{10} = -e_1I; \hat{y}_{11} = -e_2I; \hat{y}_{12} = -e_3I; \hat{y}_{13} = -e_4I; \hat{y}_{14} = -e_1I; \hat{y}_{15} = -e_2I; \hat{y}_{16} = -e_3I; \hat{y}_{17} = -e_4I; \Lambda_2 = Q\nu_0; \Lambda_3 = Q\nu_1; \Lambda_4 = Q\nu_2; \Lambda_9 = QU_1; \Lambda_{10} = \epsilon_1Q; \Lambda_{11} = \epsilon_2Q; \Lambda_{12} = \epsilon_3Q; \Lambda_{13} = \epsilon_4Q; \Lambda_{14} = N_{\nu_0}^T; \Lambda_{15} = N_{\nu_1}^T; \Lambda_{16} = N_{\nu_2}^T; \Lambda_{17} = N_{U_1}^T$ and $*$ denotes the symmetric terms of the matrices.

**Proof.** Define the following the Lyapunov-Krasovskii functional:

$$
V(y) = e^{2\alpha}\dot{y}^T(t)Qy(t) + \int_{t-k(t)}^{t} e^{2\alpha s}\dot{y}^T(s)M_1y(s)ds + \int_{t-k(t)}^{t} e^{2\alpha s}g^T(y(s))
\times M_2g(y(s))ds + \sigma_M \int_{t-\sigma_M}^{t} e^{2\alpha \theta}(s - (t - \sigma_M))h^T(y(s))M_3
\times h(y(s))ds + \int_{t-d(t)}^{t} e^{2\alpha s}\dot{y}^T(s)M_4y(s)ds + \dot{d}_M \int_{t+\theta}^{t} e^{2\alpha \theta}
\times \dot{y}^T(s)M_5\dot{y}(s)ds \, d\theta. \tag{2.4.3}
$$

where $M_1$ is given by $M_1 = \frac{2}{1-k_D}N_{E_1}^TI_3^{-1}N_{E_1}$, then by the following expressions, the averaged derivative will be given as

$$
\mathcal{L}V(y) = 2\alpha e^{2\alpha t}\dot{y}^T(t)Qy(t) + e^{2\alpha t}\dot{y}^T(t)[-Q\hat{A} - \hat{A}^TQ]y(t) + 2e^{2\alpha t}\dot{y}^T(t)
\times Q\nu_0f(y(t)) + 2e^{2\alpha t}\dot{y}^T(t)Q\nu_1g(y(t) - k(t))) + 2e^{2\alpha t}\dot{y}^T(t)
\times Q\nu_2 \int_{t-\sigma(t)}^{t} h(y(s))ds + 2e^{2\alpha t}\dot{y}^T(t)QU_1\dot{y}(t - d(t)) + e^{2\alpha t}
\times [\Delta E\dot{y}(t) + \Delta E\dot{y}(t - k(t))]^TQ[\Delta E\ddot{y}(t) + \Delta E\ddot{y}(t - k(t))]
+ e^{2\alpha t}\dot{y}^T(t)M_1y(t) - (1 - k(t))e^{2\alpha t-k(t)}\dot{y}^T(t - k(t))M_1
\times (y(t - k(t)) + e^{2\alpha t}g^T(y(t))M_2g(y(t))) - (1 - k(t))e^{2\alpha (t-k(t))}
\times g^T(y(t) - k(t)))M_2g(y(t) - k(t))) + \sigma_M^2 e^{2\alpha t}h^T(y(t))M_3
\times h(y(t)) - \sigma_M \int_{t-\sigma_M}^{t} e^{2\alpha s}h^T(y(s))M_3h(y(s))ds + e^{2\alpha t}f^T(y(t))
\times f(y(t)) - e^{2\alpha t}f^T(y(t))f(y(t)) + e^{2\alpha t}\dot{y}^T(t)M_4\dot{y}(t)
-(1 - d_D)e^{2\alpha (t-d(t))}\dot{y}^T(t - d(t))M_4\dot{y}(t - d(t)) + \dot{d}_M e^{2\alpha t}\dot{y}^T(t)
\times M_5\dot{y}(t) - \dot{d}_M \int_{t-d(t)}^{t} e^{2\alpha s}\dot{y}^T(s)M_5\dot{y}(s)ds \tag{2.4.4}
$$

Put $Z = Q^{-1}$, then by Lemma 1.10.6, the inequality (2.4.1) will be equivalent to

$$
Q^{-1} - P\lambda P^T > 0. \tag{2.4.5}
$$

44
By proceeding similar way of Theorem 2.3.1, we get,

\[
\mathcal{L}V \leq e^{2\alpha t} \{ y^T(t)[-Q(A - BL) - (A - BL)^TQ + 2N^T_E J^T_3 N_E \\
+ \frac{2}{1 - k_D} N^T_E J^T_3 N_E s + GG - M_5]y(t) + f^T(y(t))[-I] \\
\times f(y(t)) + g^T(y(t - k(t)))[-(1 - \kappa(t))M_2] \\
\times g(y(t - k(t))) + \left( \int_{t-\sigma(t)}^{t} \hspace{0.3cm} h(y(s)) ds \right)^T[-M_3] \\
\times \left( \int_{t-\sigma(t)}^{t} \hspace{0.3cm} h(y(s)) ds \right) + \tilde{g}^T(y(t)) M_2 g(y(t)) + h^T(y(t)) \hspace{0.5cm} \times \bar{g}^2 M_3 h(y(t)) + \bar{y}^T[M_4 + d_{3a}^2 M_5] \bar{y}(t) + \bar{y}^T(t - d(t)) \\
[-(1 - d_D)M_4]y(t - d(t)) + 1[y^T(t - d(t))][-M_5] \\
\times y(t - d(t)) + 2y^T(t)QV_0 f(y(t)) + 2y^T(t)QV_1 \\
\times g(y(t - k(t))) + 2y^T(t)QV_2 \int_{t-\sigma(t)}^{t} \hspace{0.3cm} h(y(s)) ds + 2y^T(t) \\
\times M_5 y(t - d(t)) + 2y^T(t)Q\bar{U}_1 \bar{y}(t - d(t)) \}
\]

\[
\mathcal{L}V(y_t) \leq e^{2\alpha t} \Xi^T(t) \Omega^* \Xi(t), \tag{2.4.6}
\]

where

\[
\Omega^* = \Omega^*_1 + Q_1 F_0(t)S_0 + S_0^T F_0^T(t)Q_1^T + Q_1 F_1(t)S_1 + S_1^T F_1^T(t)Q_1^T + Q_1 F_2(t)S_2 \\
+ S_2^T F_2^T(t)Q_1^T + Q_1 F_1(t)S_1^T + S_1^T F_1^T(t)Q_1.
\]

\[
\Xi(t) = \left[ y^T(t), f^T(y(t)), g^T(y(t - k(t))), \left( \int_{t-\sigma(t)}^{t} \hspace{0.3cm} h(y(s)) ds \right)^T, g^T(y(t)), \\
\hspace{1cm} h^T(y(t)), y^T(t), y^T(t - d(t)), \bar{y}^T(t - d(t)) \right]^T,
\]

\[
\Omega^*_1 = \begin{bmatrix}
\Omega^*_{11} & QV_0 & QV_1 & QV_2 & 0 & 0 & 0 & M_5 & QU_1 \\
* & * & * & * & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

with \( \Omega^*_{11} = -Q(A - BL) - (A - BL)^TQ + 2N^T_E J^T_3 N_E + \frac{2}{1 - k_D} N^T_E J^T_3 N_E s + GG - M_5 - QP (N_A - N_B L) - QP (N_A - N_B L)^T; \Omega_{99} = -(1 - d_D) e^{-2\alpha d(t)} M_4; \)

\[
S_1 = [0 \hspace{1cm} 0 \hspace{1cm} 0 \hspace{1cm} 0 \hspace{1cm} 0 \hspace{1cm} 0 \hspace{1cm} 0 \hspace{1cm} 0 \hspace{1cm} 0 \hspace{1cm} 0], S_2 = [0 \hspace{1cm} 0 \hspace{1cm} 0 \hspace{1cm} 0 \hspace{1cm} 0 \hspace{1cm} 0 \hspace{1cm} 0 \hspace{1cm} 0 \hspace{1cm} 0 \hspace{1cm} 0],
\]

45
\[
\bar{S}_1 = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ N_{U_1}],
\]

\[
Q_1 = \begin{bmatrix}
QP \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

and \( S_0 = [0 \ N_{V_0} \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]. \)

Now by using Lemma 1.10.4, we get

\[
\Omega = \Omega_1 + F_0(t) \begin{bmatrix}
QP \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
0 \\
N_{V_0} \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}^T + \begin{bmatrix}
0 \\
N_{T_0} \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
P \ M^T \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
QP \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

\[
\times F_1(t) + \begin{bmatrix}
0 \\
N_{V_1} \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}^T + \begin{bmatrix}
F_1^T (t) \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
F_2(t) \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
F_2^T (t) \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
0 \\
0 \\
N_{V_2} \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
Q \ P^T \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
F_1(t) \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
N_{U_1}
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
Q \ P^T \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
F_1^T (t) \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

46
Thus, we have

$$\mathcal{L}V(y_t) \leq -\lambda_1 \{ ||y_t||^2 + ||\dot{y}_t||^2 \},$$

(2.4.7)

where \( \lambda_1 = \lambda_{\text{min}}(\Omega) \)

By taking expectation on both sides of (2.4.7) yields

$$\mathbb{E}\{\mathcal{L}V(y_t)\} \leq -\lambda_1 \mathbb{E}\{||y_t||^2\}.$$  

(2.4.8)

Now, let us define \( Y(t) = e^{2at}V(t) \). Then its infinitesimal operator \( \mathcal{L} \) is given by

$$\mathcal{L}Y(t) = 2ae^{2at}V(t) + e^{2at}\mathcal{L}V(t).$$

(2.4.9)

Integrating on both sides of (2.4.9) and taking expectation gives

$$\mathbb{E}\{Y(t)\} - \mathbb{E}\{Y(0)\} = \int_0^t \mathbb{E}\{2ae^{2as}V(s) + e^{2as}\mathcal{L}V(s)\} \, ds$$
Choose $2\alpha \leq \min \left\{ \frac{\lambda_{\text{max}}}{\lambda_{\text{max}}(Q)} \right\}$ which implies that $\mathbb{E}\{Y(t)\} - \mathbb{E}\{Y(0)\} \leq 0$,

**i.e.,** $e^{2\alpha t}\mathbb{E}\{V(t)\} \leq \mathbb{E}\{V(0)\}$

\[
\leq \rho_1 \sup_{-k_M, -d_M \leq s \leq 0} \mathbb{E}\left\{ \|\phi(s)\|^2 + \|\dot{\phi}(s)\|^2 \right\}, \tag{2.4.11}
\]

where

\[
\rho_1 = \max \left\{ \lambda_{\text{max}}(Q), k_0\lambda_{\text{max}}(M_1) + k_0\lambda_{\text{max}}(M_2) + \sigma_M^2 \times \lambda_{\text{max}}(M_3) + d_0\lambda_{\text{max}}(M_4) + d_M^2\lambda_{\text{max}}(M_5) \right\}.
\]

Meanwhile, it follows from (2.4.3) that

\[
\mathbb{E}\{V(t)\} \geq \mathbb{E}\left\{ \lambda_{\text{min}}(Q)\|y_t\|^2 \right\}. \tag{2.4.12}
\]

Combining with (2.4.11) and (2.4.12) leads to

\[
\mathbb{E}\{\|y_t\|^2\} \leq \rho e^{-2\alpha t}\mathbb{E}\left\{ \sup_{-k_M, -d_M \leq s \leq 0} (\|\phi(s)\|^2 + \|\dot{\phi}(s)\|^2) \right\}, \tag{2.4.13}
\]

where $\rho = \frac{\rho_1}{\lambda_{\text{min}}(Q)}$. This completes the proof. \qed

**Remark 2.4.2.** As much as know, all the existing results concerning the dynamical behaviors of neural networks [165, 173, 177, 204] have not considered the globally robustly exponential stability performance in the mean square and time-varying delayed situation, which are investigated via LMI approach in this chapter. Therefore, our conclusions are brand-new and compare to previous results.

**Remark 2.4.3.** This chapter fully considers the relationship between time-varying delays and their upper bounds. Also, some inequality techniques are used to obtain the maximum upper bounds of discrete, distributed and neutral time-varying delays, see Example 2.5.1 & 2.5.3. Hence this techniques and methods may lead to less conservative criterions.
Remark 2.4.4. Table 2.1 and Table 2.3 shows that the maximum upper bounds of \( k_M, \sigma_M \& d_M \) which guarantees the global robust exponentially stable of the addressed neural networks (2.2.1). These tables demonstrate the effectiveness of our proposed method.

Remark 2.4.5. Suppose, the uncertain parameters are not present in neural networks (2.2.1), then the modified nominal neutral type neural networks are as follows:

\[
\dot{y}(t) - U_1 y(t - d(t)) = -Ay(t) + V_0 f(y(t)) + V_1 g(y(t - k(t))) + Bv(t) + C u(t) + V_2 \int_{t - \sigma(t)}^{t} h(y(s)) ds,
\]

\[
x(t) = F y(t) + D v(t), \quad t > 0,
\]

\[
y(t) = \varphi(t), \quad t \in [-K, 0]. \tag{2.4.14}
\]

Then by Theorem 2.4.1, it is easy to have the following Corollary 2.4.6.

Corollary 2.4.6. Under Assumptions 2.1 \& 2.2, the equilibrium point of the time-delayed neutral NNs (2.2.1) is said to be global robust exponentially stable with \( u(t) = 0 \) in the mean square sense, if there exist symmetric positive definite matrices \( Z, Q, M_1, M_2, M_3, M_4, M_5 \) and a matrix \( X \) satisfying the following LMIs:

\[
-Z < 0, \tag{2.4.15}
\]

\[
\Omega^* = \begin{bmatrix} \Pi^*_1 & \Xi^*_1 \\ \ast & \Theta^*_1 \end{bmatrix} < 0, \tag{2.4.16}
\]

hold, where

\[
\Pi^*_1 = 
\begin{bmatrix}
\hat{Y}_1 & \Lambda_2 & \Lambda_3 & \Lambda_4 & 0 & 0 & 0 & M_5 & \Lambda_9 & \Lambda_{10} \\
\ast & \hat{Y}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\ast & \ast & \hat{Y}_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\ast & \ast & \ast & \hat{Y}_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \hat{Y}_5 & 0 & 0 & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \hat{Y}_6 & 0 & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \hat{Y}_7 & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \hat{Y}_8 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \hat{Y}_9 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \hat{Y}_{10}
\end{bmatrix},
\]

49
\[ \Xi_1^* = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \]

\[ \Theta_1^* = \text{diag}\{\hat{Y}_{11}, \hat{Y}_{12}, \hat{Y}_{13}\}; \quad \hat{Y}_1 = -Q(A - BL) - (A - BL)^TQ + G^TG - M_5; \quad \hat{Y}_2 = -I; \]
\[ \hat{Y}_3 = -(1 - k_1)e^{-2nk_1}G_1; \quad \hat{Y}_4 = -M_3; \quad \hat{Y}_5 = M_2; \quad \hat{Y}_6 = \sigma_1^2M_3; \quad \hat{Y}_7 = M_4 + d_1^2M_5; \quad \hat{Y}_8 = -M_5; \quad \hat{Y}_9 = -(1 - d_1)e^{-2nd_1}M_4; \quad \hat{Y}_{10} = -\epsilon_1I; \quad \hat{Y}_{11} = -\epsilon_2I; \quad \hat{Y}_{12} = -\epsilon_3I; \quad \hat{Y}_{13} = -\epsilon_4I; \quad \Lambda_2 = QV_0; \quad \Lambda_3 = QV_1; \quad \Lambda_4 = QV_2; \quad \Lambda_9 = QU_1; \quad \Lambda_{10} = \epsilon_1QP; \quad \Lambda_{11} = \epsilon_2QP; \quad \Lambda_{12} = \epsilon_3QP; \quad \Lambda_{13} = \epsilon_4QP \text{ and } * \text{ denotes the symmetric terms of the matrices.} \]

**Remark 2.4.7.** In case, the neutral terms and stochastic noises will not be considered in (2.2.1), then the mixed time delayed neural networks (2.2.1) reduces to

\[
\begin{align*}
\dot{y}(t) &= -(A + \Delta A(t))y(t) + (V_0 + \Delta V_0(t))f(y(t)) + (V_1 + \Delta V_1(t)) \\
&\quad \times g(y(t - k(t))) + (B + \Delta B(t))\nu(t) + Cu(t) + (V_2 + \Delta V_2(t)) \\
&\quad \times \int_{t-\sigma(t)}^{t} h(y(s))ds, \\
x(t) &= Fy(t) + D\nu(t), \quad t > 0, \\
y(t) &= \varphi(t), \quad t \in [-K, 0].
\end{align*}
\]

(4.2.17)

Then by Theorem 2.4.1, it is easy to have the following Corollary 2.4.8.

**Corollary 2.4.8.** Under Assumptions 2.1 & 2.2, the equilibrium point of the time-delayed neural networks (2.2.1) is said to be globally robustly exponential stable with \( u(t) = 0 \) in the mean square sense, if there exist symmetric positive definite matrices \( Z, J_3, Q, M_1, M_2, M_3, M_4, M_5 \) and a matrix \( X \) satisfying the following LMIs:

\[
\begin{bmatrix} -Z & P \end{bmatrix} J_3 < 0,
\]

(4.2.18)

\[ \Omega^* = \begin{bmatrix} \Pi_2^* & \Xi_2^* \\ * & \Theta_2^* \end{bmatrix} < 0, \]

(4.2.19)
\section*{Chapter 2: Global Robust Exponential Stability of USNNs}

\[ \Pi_2^* = \begin{bmatrix}
\dot{Y}_1^* & \Lambda_2 & \Lambda_3 & \Lambda_4 & 0 & 0 & 0 & M_5 & \Lambda_9 & \Lambda_{10} \\
\ast & \dot{Y}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\ast & \ast & \dot{Y}_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\ast & \ast & \ast & \dot{Y}_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
\ast & \ast & \ast & \ast & \dot{Y}_5 & 0 & 0 & 0 & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \dot{Y}_6 & 0 & 0 & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \dot{Y}_7 & 0 & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \dot{Y}_8 & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \dot{Y}_9 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \dot{Y}_{10}
\end{bmatrix}, \]

\[ \Xi_2^* = \begin{bmatrix}
\Lambda_{11} & 0 & 0 & 0 \\
0 & \Lambda_{12} & 0 & 0 \\
0 & 0 & \Lambda_{13} & 0 \\
0 & 0 & 0 & \Lambda_{14} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \]

\[ \Theta_2^* = \text{diag}\{\dot{Y}_{11}, \dot{Y}_{12}, \dot{Y}_{13}, \dot{Y}_{14}\}; \dot{Y}_1^* = -Q(A - BL) - (A - BL)^TQ + GG - M_5 - QP(N_A - N_{B_1}L) - QP(N_A - N_{B_2}L)^T; \dot{Y}_2 = -I; \dot{Y}_3 = -(1 - k_D)e^{-2a_k}\sigma_{M_2}; \dot{Y}_4 = -M_3; \dot{Y}_5 = M_2; \dot{Y}_6 = \sigma_{M_2}^2 M_3; \dot{Y}_7 = M_4; \dot{Y}_8 = -M_5; \dot{Y}_9 = -e_1 I; \dot{Y}_{10} = -e_2 I; \dot{Y}_{11} = -e_3 I; \dot{Y}_{12} = -e_1 I; \dot{Y}_{13} = -e_2 I; \dot{Y}_{14} = -e_3 I; \Lambda_2 = QV_0; \Lambda_3 = QV_1; \Lambda_4 = QV_2; \Lambda_9 = e_1 QP; \Lambda_{10} = e_2 QP; \Lambda_{11} = e_3 QP; \Lambda_{12} = N_{V_0}; \Lambda_{13} = N_{V_1}^T; \Lambda_{14} = N_{V_2}^T \text{ and } * \text{ denotes the symmetric terms of the matrices.} \]

\begin{remark}
In case, the neutral terms and stochastic noises are not exists in neural networks (2.2.1), then we have the following simplified neural networks:

\[ \dot{y}(t) = -Ay(t) + V_0 f(y(t)) + V_1 g(y(t - k(t))) + B\sigma(t) + Cu(t) + V_2 \int_{t-\sigma(t)}^{t} h(y(s))ds, \]

\[ x(t) = Fy(t) + D\sigma(t), \quad t > 0, \]

\[ y(t) = \varphi(t), \quad t \in [-K, 0]. \]

Then from Corollary 2.4.6, it is easy to have one corollary for the above mentioned remark with \( \Lambda_9 = 0 \) and the other entries are same as Corollary 2.4.6.

\textit{Remark 2.4.9.}
Remark 2.4.10. Based on the Example 2.5.1–2.5.3, it is easy to see that the conclude results are better than the ones in [1, 122, 129]. Hence, the proposed method is a improved over some existing literatures.

2.5 Numerical examples

In this section, we provide three numerical examples with their simulations to illustrate the superiority and benefits of the proposed criteria.

Example 2.5.1. Consider a two dimensional uncertain neutral-type neural networks (2.2.1) with the following associated parameters:

\[
A = \begin{bmatrix} 1.2 & 0 \\ 0 & 1.5 \end{bmatrix}, \quad V_0 = \begin{bmatrix} 0.3 & 0.2 \\ -0.3 & 0.4 \end{bmatrix}, \quad V_1 = \begin{bmatrix} 0.3 & 0.2 \\ -0.1 & 0.4 \end{bmatrix},
\]

\[
B = \begin{bmatrix} 0.1 & -0.3 \\ 0.1 & -0.2 \end{bmatrix}, \quad C = \begin{bmatrix} 0.05 & -0.01 \\ 0.02 & 0.02 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0.3 & -0.2 \\ 0.1 & -0.5 \end{bmatrix},
\]

\[
U_1 = \begin{bmatrix} 0.3 & 0.2 \\ -0.3 & 0.4 \end{bmatrix}, \quad F = \begin{bmatrix} 0.06 & 0.02 \\ 0.3 & 0.02 \end{bmatrix}, \quad D = \begin{bmatrix} 0.03 & 0.01 \\ 0.05 & 0.02 \end{bmatrix},
\]

\[
X = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad Z = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad N_{V_0} = \begin{bmatrix} 0.02 & 0.03 \\ -0.01 & 0.03 \end{bmatrix},
\]

\[
N_{V_1} = \begin{bmatrix} 0.01 & 0.05 \\ 0.01 & 0.01 \end{bmatrix}, \quad N_{V_2} = \begin{bmatrix} 0.04 & 0.01 \\ 0.02 & 0.01 \end{bmatrix}, \quad M = \begin{bmatrix} 0.01 & 0.02 \\ 0.03 & 0.01 \end{bmatrix},
\]

\[
N_E = \begin{bmatrix} 0.2 & -0.1 \\ 0.3 & -0.2 \end{bmatrix}, \quad N_{E_4} = \begin{bmatrix} 0.1 & -0.5 \\ 0.2 & -0.3 \end{bmatrix}, \quad N_A = \begin{bmatrix} 0.05 & -0.01 \\ -0.03 & 0.02 \end{bmatrix},
\]

\[
N_B = \begin{bmatrix} 0.02 & 0.03 \\ -0.02 & 0.01 \end{bmatrix}, \quad G = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.02 \end{bmatrix}, \quad P = \begin{bmatrix} 0.02 & 0.03 \\ 0.02 & 0.04 \end{bmatrix},
\]

\[
I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad k_D = 0.8, \quad \sigma_M = 38.7, \quad d_D = 0.5, \quad d_M = 24.8.
\]

The following activation functions are play a key role in neural networks (2.2.1):

\[
f(y(t)) = g(y(t)) = h(y(t)) = 0.5 \times \tanh(y(t)).
\]
Table 2.1: MAUB of discrete ($k_M$) & distributed ($\sigma_M$) time delays.

<table>
<thead>
<tr>
<th>Methods</th>
<th>$k_M &gt; 0$</th>
<th>$\sigma_M &gt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>In Ref [154]</td>
<td>0.6</td>
<td>0.5</td>
</tr>
<tr>
<td>In Ref [83]</td>
<td>1</td>
<td>0.5</td>
</tr>
<tr>
<td>In Ref [1]</td>
<td>1.7536</td>
<td>-</td>
</tr>
<tr>
<td>In Ref [93]</td>
<td>2.7374</td>
<td>-</td>
</tr>
<tr>
<td>In Ref [156]</td>
<td>3.2376</td>
<td>0.2</td>
</tr>
<tr>
<td>Theorem 2.3.1</td>
<td>-</td>
<td>38.237</td>
</tr>
<tr>
<td>Theorem 2.4.1</td>
<td>18.35</td>
<td>24.7</td>
</tr>
</tbody>
</table>

Taking the disturbance input $u(t) = 0.3 \times \sin(0.2 \times \exp(0.6 \times t))$ and by solving the LMIs in Theorem 2.3.1 using the MATLAB LMI control toolbox, one can obtain the feasible solutions as follows:

$$J_0 = \begin{bmatrix} 1.6794 & 0.0747 \\ 0.0747 & 4.9416 \end{bmatrix}, \quad J_1 = \begin{bmatrix} 53.6657 & 2.3626 \\ 2.3626 & 31.8322 \end{bmatrix},$$

$$J_3 = \begin{bmatrix} 67.9485 & 1.2770 \\ 1.2770 & 66.7433 \end{bmatrix}, \quad J_4 = \begin{bmatrix} 42.8300 & -0.0057 \\ -0.0057 & 42.8246 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 57.4405 & 0.6314 \\ 0.6314 & 57.9711 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0.1691 & -0.0000 \\ -0.0000 & 0.1781 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} 0.0887 & 0.0118 \\ 0.0118 & 0.0753 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 122.1460 & 0.0601 \\ 0.0601 & 126.9022 \end{bmatrix},$$

$$T_1 = \begin{bmatrix} 42.8094 & -0.0019 \\ -0.0019 & 42.8073 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 23.9807 & -4.5098 \\ -4.5098 & 43.6891 \end{bmatrix}.$$  

Thus, the state responses of the stochastic neutral type NNs (2.2.1) are shown in Figure 2.1. By solving the LMIs (2.3.1) & (2.3.2) in Theorem 2.3.1 we can find the above feasible solutions. The obtained distributed time-delay upper bounds of $\sigma$ for neural networks (2.2.1) are very large, which are given in Table 2.1. Therefore by Theorem 2.3.1, we conclude that the neural networks (2.2.1) is robustly stabilized in the mean square for the maximum allowable upper bounds $\sigma = 38.237$. This shows that the contributions of this research work is more effective and less conservative than some existing literatures.
Example 2.5.2. Consider the following two dimensional uncertain stochastic neutral-type neural networks (2.2.1) with the associated parameters as follows:

\[
A = \begin{bmatrix} 6 & 0 \\ 0 & 5 \end{bmatrix}, \quad V_0 = \begin{bmatrix} 0.03 & 0.02 \\ -0.03 & 0.04 \end{bmatrix}, \quad V_1 = \begin{bmatrix} 0.04 & 0.01 \\ -0.02 & 0.03 \end{bmatrix},
\]

\[
B = \begin{bmatrix} 0.01 & -0.03 \\ 0.03 & -0.02 \end{bmatrix}, \quad C = \begin{bmatrix} 0.05 & -0.01 \\ 0.02 & 0.02 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0.01 & -0.02 \\ 0.03 & -0.04 \end{bmatrix},
\]

\[
U_1 = \begin{bmatrix} 0.04 & 0.01 \\ -0.02 & 0.03 \end{bmatrix}, \quad F = \begin{bmatrix} 0.02 & 0.02 \\ 0.3 & 0.02 \end{bmatrix}, \quad D = \begin{bmatrix} 0.03 & 0.01 \\ 0.04 & 0.02 \end{bmatrix},
\]

\[
X = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad Z = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.3 \end{bmatrix}, \quad N_{V_0} = \begin{bmatrix} 0.05 & 0.01 \\ -0.02 & 0.03 \end{bmatrix},
\]

\[
N_{V_1} = \begin{bmatrix} 0.02 & 0.01 \\ 0.03 & 0.02 \end{bmatrix}, \quad N_{V_2} = \begin{bmatrix} 0.01 & 0.03 \\ 0.04 & 0.02 \end{bmatrix}, \quad M = \begin{bmatrix} 0.02 & 0.02 \\ 0.03 & 0.01 \end{bmatrix},
\]

\[
N_E = \begin{bmatrix} 0.04 & -0.02 \\ 0.02 & -0.01 \end{bmatrix}, \quad N_{E_4} = \begin{bmatrix} 0.01 & -0.03 \\ 0.05 & -0.03 \end{bmatrix}, \quad N_A = \begin{bmatrix} 0.02 & -0.04 \\ -0.02 & 0.03 \end{bmatrix},
\]

\[
N_B = \begin{bmatrix} 0.03 & 0.01 \\ -0.01 & 0.02 \end{bmatrix}, \quad G = \begin{bmatrix} 0.03 & 0 \\ 0 & 0.02 \end{bmatrix}, \quad L = \begin{bmatrix} 0.06 & 0.02 \\ -0.01 & 0.03 \end{bmatrix},
\]

\[
P = \begin{bmatrix} 0.03 & 0.03 \\ -0.02 & 0.03 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad k_D = 0.4, \quad \sigma_M = 24.7,
\]
Figure 2.2: State trajectories of concerned uncertain neutral NNs (2.2.1)

\[ d_D = 0.6, \quad d_M = 26, \quad k_M = 18.35, \quad \alpha = 0.46. \]

The same activation functions in Theorem 2.3.1, are used in this example. Taking the disturbance input parameter \( u(t) = 0.3 \times \sin(0.2 \times \exp(0.6 \times t)) \). Then the following feasible solutions for the LMIs in Theorem 2.4.1 are obtained:

\[
J_3 = 10^4 \times \begin{bmatrix} 1.0074 & -0.5649 \\ -0.5649 & 0.5529 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 7.4329 & -0.3468 \\ -0.3468 & 7.4172 \end{bmatrix},
\]

\[
M_3 = 10^5 \times \begin{bmatrix} 2.0326 & 0.0000 \\ 0.0000 & 2.0326 \end{bmatrix}, \quad M_4 = 10^8 \times \begin{bmatrix} 7.2289 & -0.1317 \\ -0.1317 & 4.1366 \end{bmatrix},
\]

\[
M_5 = 10^6 \times \begin{bmatrix} 1.0135 & -0.0201 \\ -0.0201 & 0.5600 \end{bmatrix}, \quad Q = \begin{bmatrix} 39.1032 & 4.5172 \\ 4.5172 & 34.9593 \end{bmatrix},
\]

\[
\varepsilon_1 = 758.9261, \quad \varepsilon_2 = 758.9886, \quad \varepsilon_3 = 758.9399, \quad \varepsilon_4 = 423.7296.
\]

Therefore, by solving the LMIs (2.4.1) & (2.4.2) in Theorem 2.4.1 we can find the feasible solutions. Also, the state trajectories of the concerned neural networks (2.2.1) are shown in Figure 2.2. The allowable maximum upper bounds of discrete and distributed time delays for neural networks (2.2.1) are very large, which are given and compared with some existing literature in Table 2.1. Therefore by Theorem 2.4.1, we obtain that (2.2.1) is
Table 2.2: The values of $\varepsilon_i (i = 1, 2, 3, 4)$ for given convergence rate ($\alpha$) in Theorem 2.4.1

<table>
<thead>
<tr>
<th>$\alpha &gt; 0$</th>
<th>$\varepsilon_1$</th>
<th>$\varepsilon_2$</th>
<th>$\varepsilon_3$</th>
<th>$\varepsilon_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>3.9288</td>
<td>3.8900</td>
<td>3.9305</td>
<td>3.9295</td>
</tr>
<tr>
<td>0.14</td>
<td>14.2941</td>
<td>13.4206</td>
<td>14.2966</td>
<td>14.2963</td>
</tr>
<tr>
<td>0.21</td>
<td>157.6576</td>
<td>147.0769</td>
<td>157.6651</td>
<td>157.6936</td>
</tr>
<tr>
<td>0.28</td>
<td>513.5290</td>
<td>507.9365</td>
<td>511.1780</td>
<td>514.4296</td>
</tr>
</tbody>
</table>

Table 2.3: MAUB of neutral delays ($d_M$).

<table>
<thead>
<tr>
<th>Methods</th>
<th>$d_M &gt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>In Ref [210]</td>
<td>0.2951</td>
</tr>
<tr>
<td>In Ref [43]</td>
<td>3</td>
</tr>
<tr>
<td>In Ref [3]</td>
<td>4.6473</td>
</tr>
<tr>
<td>In Ref [120]</td>
<td>5</td>
</tr>
<tr>
<td>Corollary 2.4.6</td>
<td>28</td>
</tr>
</tbody>
</table>

global robust exponentially stable in the mean square sense for the maximum upper bounds $k_M = 18.35$ & $\sigma_M = 24.7$. Hence, this experiment demonstrate the effectiveness and less conservatism of the proposed research work.

Example 2.5.3. Consider the two-neuron neutral-type neural networks (2.4.14) with the following parameters:

\[
A = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}, \quad V_0 = \begin{bmatrix} 0.01 & 0.07 \\ -0.04 & 0.02 \end{bmatrix}, \quad V_1 = \begin{bmatrix} 0.05 & 0.01 \\ -0.02 & 0.04 \end{bmatrix},
\]

\[
B = \begin{bmatrix} 0.02 & -0.02 \\ 0.01 & -0.03 \end{bmatrix}, \quad C = \begin{bmatrix} 0.03 & -0.08 \\ 0.02 & 0.05 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0.03 & -0.04 \\ 0.02 & -0.02 \end{bmatrix},
\]

\[
U_1 = \begin{bmatrix} 0.04 & 0.01 \\ -0.05 & 0.03 \end{bmatrix}, \quad F = \begin{bmatrix} 0.03 & 0.05 \\ 0.01 & 0.04 \end{bmatrix}, \quad D = \begin{bmatrix} 0.02 & 0.02 \\ 0.06 & 0.03 \end{bmatrix},
\]

\[
X = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix}, \quad Z = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.4 \end{bmatrix}, \quad G = \begin{bmatrix} 0.03 & 0 \\ 0 & 0.05 \end{bmatrix},
\]

\[
L = \begin{bmatrix} 0.02 & 0.04 \\ -0.03 & 0.05 \end{bmatrix}, \quad P = \begin{bmatrix} 0.04 & 0.07 \\ 0.01 & 0.02 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]
Figure 2.3: State trajectories of neutral-type neural networks (2.4.14)

Now, let \( k_D = 0.5, \sigma_M = 25.5, d_D = 0.6, d_M = 28, k_M = 18.29, \alpha = 0.32. \)

Here also the same activation functions in Theorem 2.3.1, are used for neural networks (2.4.14).

Taking the disturbance input \( u(t) = 0.3 \times \sin(0.2 \times \exp(0.6 \times t)) \). Then solving the LMI in Corollary 2.4.6 by aid of Matlab LMI control toolbox, one can obtain the following feasible solutions:

\[
M_2 = \begin{bmatrix}
-3.0217 & -0.1811 \\
-0.1811 & 3.0243
\end{bmatrix},
M_3 = \begin{bmatrix}
378.5930 & 17.0042 \\
17.0042 & 391.0258
\end{bmatrix},
\]

\[
M_4 = 10^5 \times \begin{bmatrix}
3.8890 & 0.7009 \\
0.7009 & 4.5935
\end{bmatrix},
M_5 = \begin{bmatrix}
408.2757 & 83.1658 \\
83.1658 & 503.1650
\end{bmatrix},
\]

\[
Q = \begin{bmatrix}
15.9011 & 7.8500 \\
7.8500 & 6.3157
\end{bmatrix},
\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = 79.4453.
\]

By utilizing the Matlab LMI control toolbox, we have listed the comparison of upper bounds of neural delay in Table 2.3. Moreover, the state response of (2.4.14) is explored in Figure 2.3. Therefore, according to Corollary 2.4.8, we conclude that (2.4.14) is globally robustly exponential stable in the mean square sense.
2.6 Conclusions

In this chapter, the globally robustly exponential stability problem has been proposed for $H_{\infty}$ control of stochastic neutral-type neural networks with parameter uncertainties and both time-varying delays. By a key role of the Lyapunov technique, M-matrix theory, stability analysis procedure and Inequality techniques, some brand-new LMIs conditions are derived to explore the uncertain neutral-type neural networks is globally robustly exponential stable in the mean square sense. Additionally, our proposed method provide the allowable upper bounds of the discrete, distributed and neutral time-varying delays are very large when compared with some previous literature, which may possess highly important significance in the design of time-delayed addressed neural networks with uncertain parameters. Finally, three numerical examples with simulations are given to demonstrate the superiority of the proposed theoretical outcomes.