Chapter 2

Distance Compatible Set-labeling of Graphs

We define distance compatible set-labeling (dcs) of a graph $G$ as an injective set-assignment $f : V(G) \to 2^X$, $X$ a nonempty ground set, such that the corresponding induced function $f^{\oplus} : V(G) \times V(G) \to 2^X - \emptyset$, defined by $f^{\oplus}(uv) = f(u) \oplus f(v)$ satisfies $|f^{\oplus}(uv)| = k_f(u,v)d(u,v)$ for all distinct $u, v \in V(G)$, where $d(u,v)$ is the distance between $u$ and $v$ and $k_f(u,v)$ is a constant, not necessarily an integer. We define dispersible dcs-graphs, edge-dispersible dcs-graphs, $(k,r)$-arithmetic dcs-graphs and $k$-uniform dcs-graphs as special cases of dcs-graphs. We define the dcs index $\delta_d$ of graph $G$ as the minimum cardinality of the ground set $X$ such that $G$ admits a dcs. In this chapter we initiate a study of above mentioned concepts.

2.1 Introduction

Acharya [1] introduced the notion of set-valuation as set analogue of number valuation. For a $(p,q)$ graph $G = (V,E)$ and a nonempty set $X$ of cardinality $n$, Acharya [2] defined set-indexer of $G$ as an
injective set-valued function \( f : V(G) \to 2^X \) such that the function \( f^{\oplus} : E(G) \to 2^X - \emptyset \) defined by \( f^{\oplus}(uv) = f(u) \oplus f(v) \) for every \( uv \in E(G) \) is also injective, where \( 2^X \) is the set of all subsets of \( X \) and “\( \oplus \)” is the symmetric difference of sets.

B. D. Acharya and Germina K. A. [6] introduced the particular instance of set-indexers when a metric (instead of topology), especially the size of the symmetric difference, is associated to edges. If this size is proportional to path-distance then, the graph is 1-1 embeddable. They [6] initiated a problem, whether we can study those graphs \( G = (V,E) \) that admit an injective ‘set-valuation’ \( f : V \to 2^X, X \) being a nonempty ‘ground set’, such that the cardinality of the symmetric difference \( f^{\oplus}(uv) = (f(u) - f(v)) \cup (f(v) - f(u)) \) is proportional to the usual path-distance \( d(u,v) \) between \( u \) and \( v \) in \( G \) for all pairs of distinct vertices \( u \) and \( v \) in \( G \). They called such a set-valuation \( f \) of \( G \), if it exists, a distance-compatible set-labeling (dcsl) of \( G \), and the ordered pair \( (G,f) \), a distance-compatible set-labeled (dcsl) graph. Thus we have the following definition.

**Definition 2.1.1.** Let \( G = (V,E) \) be any connected \((p,q)\) graph. A distance compatible set-labeling of a graph \( G \) is an injective set-assignment \( f : V(G) \to 2^X, X \) a nonempty ground set, such that the
corresponding induced function $f^\oplus : V(G) \times V(G) \rightarrow 2^X - \emptyset$, defined by $f^\oplus(uv) = f(u) \oplus f(v)$ satisfies $|f^\oplus(uv)| = k_{(u,v)} f d(u, v)$ for all distinct $u, v \in V(G)$, where $d(u, v)$ is the distance between $u$ and $v$ and $k_{(u,v)} f$ is a constant, not necessarily an integer. $G$ is distance compatible set-labeled (dcsl) graph if it admits a dcsl. We denote a dcsl-graph $G$ with a dcsl, $f$ by the ordered pair $(G, f)$. The corresponding ground set is called a dcsl-set.

Figure 2.1 gives a dcsl graph on five vertices.

One can identify various types of dcsl’s of a graph $G$. A dcsl $f$ of a $(p, q)$-graph $G$ is dispersive if the constants of proportionality $k_{uv} f$ with respect to $f$, $u \neq v$, $u, v \in V(G)$ are all distinct and $G$ is dispersible if it admits a dispersive dcsl. A dispersive dcsl $f$ of $G$ is $(k, r)$-arithmetic, if the constants of proportionality with respect to $f$ can be arranged
in the arithmetic progression, $k, k + r, k + 2r, \ldots, k + (q - 1)r$ and if $G$ admits such a dcsl then $G$ is a $(k, r)$-arithmetic dcsl-graph. A dcsl $f$ of $G$ is $k$-uniform if all the constants of proportionality with respect to $f$ are equal to $k$, and if $G$ admits such a dcsl then, $G$ is called a $k$-uniform dcsl-graph.

**Definition 2.1.2.** dcsl index of a dcsl-graph $(G, f)$ is the minimum cardinality of the dcsl-set and it is denoted by $\delta_d(G)$.

The main aim of this chapter is to present an account of what we know of such graphs and their corresponding distance compatible set-labeling.

As well known, apart from theoretical interest in the study of the distance matrix, such as the realization of a given matrix as the distance matrix of a graph (see Wai-Kai Chen [23]) it has found applications in many practically interesting areas such as Quantitative Structure-Activity Relation (QSAR) in discrete mathematical chemistry (see S.C. Basak *et. al.*, [8]) and, studies on the effect of indirect qualitative relationships between individuals in a social network (see Fiksel [13] and Kovchegov [18]).
Remark 2.1.1. Let $G = (V, E)$ be any $(p, q)$-graph, $M$ be an arbitrary nonempty proper subset of $V$ and $f$ be the corresponding dcsl. Then, the $M$-Weiner index $W_M(G)$ may be defined as the sum of the entries in the upper triangular half of the $M$-distance matrix $D_M(G)$; by a partial Weiner index $W'(G)$, we mean the $M$-Weiner index of $G$ for some nonempty proper subset $M$ of $V(G)$ and the well known Weiner index $W(G)$ (see Trinajstic, [22]) is then seen as the $M$-Weiner index with $M = V(G)$.

An interesting question for chemists would be the following.

Problem 2. Consider any structure-activity relationship $\mathcal{R}$ of a molecular graph that has been identified to be well correlated with the Weiner index. Is it possible to achieve such a correlation using $M$-Weiner index for a low cardinality dcsl-set $X$ as possible? [Choice of marker sets $X$ in the molecular graph might be very crucial and hence might involve deeper insights into the molecular characteristics.]

2.2 Classes of dcsl-graphs

Given a dcsl-graph $(G, f)$, its definition implies that

$$|f^\otimes(uv)| = k_{uv}^f d(u, v), \text{ for all } u, v \in V(G), \ u \neq v,$$  \hspace{1cm} (1)
where $k_{uv}^f$’s are the constants of proportionality. Let $\mathcal{K}_f(G) = \{k_{uv}^f : u \neq v, \ u, v \in V(G)\}$. Note that in the case of a dispersible dcsl-graph $(G, f)$ of order $n$, $\mathcal{K}_f(G)$ contains $\frac{n(n+1)}{2}$ distinct numbers and in the case of $k$-uniform dcsl-graphs it contains a single element $k$.

**Lemma 2.2.1.** If the complete graph $K_n$ is a dcsl-graph then for any dcsl $f$ of $K_n$,

$$|f \oplus (uv)| = k_{uv}^f, \text{ for all distinct } u, v \in V(K_n).$$  \hspace{1cm} (2)

**Proof.** This follows from the fact that a graph is complete if and only if every two distinct vertices in the graph are at unit distance. \hfill $\square$

**Lemma 2.2.2.** Every graph has a dcsl.

**Proof.** Let $G$ be any graph of order $n$. Let $X$ be any set such that the cardinality of the power set of $X$ is greater than the order of $G$. Define an injective set valuation $f : V(G) \rightarrow 2^X$ such that $f(v_i) = X_i, X_i \subseteq X$. Let $k_{(u,v)}^f = \frac{|f \oplus (uv)|}{d(u,v)}$ for every $u, v \in V(G)$. Then, $| f \oplus (uv) | = k_{(u,v)}^f d(u,v)$ for every $(u, v) \in V(G) \times V(G)$. Thus, $f$ is a dcsl and $G$ is a dcsl-graph. \hfill $\square$
2.3 Dispersible dcl-graphs

As defined already, a dcl $f$ of a graph $G$ is *dispersive* if the constants of proportionality $k_{uv}^f$, $u \neq v$, $u, v \in V(G)$ are all distinct and $G$ is *dispersible* if it admits a dispersive dcl.

A dispersible labeling of path $P_4$ is illustrated in Figure 2.2.

![Figure 2.2: A dispersible dcl of $P_4$](image)

A natural question that arises is what are the classes of graphs which admits a dispersive dcl. Theorem 2.3.1 is an attempt to answer this question.

**Theorem 2.3.1.** $K_n$ is dispersible for all $n \geq 1$.

**Proof.** Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. Let $X = \{1, 2, \ldots, 2^{n-1}\}$. Define $f : V(K_n) \to 2^X$ by $f(v_i) = \{1, 2, 3, \ldots, 2^{i-1}\}$, $1 \leq i \leq n$. Clearly, $f(v_i) \subset f(v_j)$, for $i < j$. Now, we shall prove that the constants of
Figure 2.3: Dispersive dcsl of complete graphs $K_n$ for $n = 2, 3$ and 4

proportionality $k_{uv}^f$ are all distinct, for distinct $u, v \in V(K_n)$.

\[
k_{(v_1,v_i)}^f = |f^\oplus(v_1v_i)|, \quad 2 \leq i \leq n = \{1, 2^2 - 1, 2^3 - 1, \ldots, 2^{n-1} - 1\}
\]

\[
k_{(v_2,v_i)}^f = |f^\oplus(v_2v_i)|, \quad 3 \leq i \leq n = \{2^2 - 2, 2^3 - 2, \ldots, 2^{n-1} - 2\}
\]

\[
k_{(v_3,v_i)}^f = |f^\oplus(v_3v_i)|, \quad 4 \leq i \leq n = \{2^3 - 2^2, 2^4 - 2^2, \ldots, 2^{n-1} - 2^2\}
\]

\[
\vdots
\]

\[
\vdots
\]

\[
\vdots
\]

\[
k_{(v_{n-1},v_n)}^f = |f^\oplus(v_{n-1}v_n)| = 2^{n-1} - 2^{n-2}.
\]

Hence, $k_{(u,v)}^f$ are distinct for all distinct $u, v \in V(K_n)$, whence the dcsl $f$ of $K_n$ is dispersive, so that $K_n$ is dispersible dcsl-graph. \qed
Remark 2.3.1. Figure 2.4 shows that, $K_{1,2} \cong P_3$ and star $K_{1,3}$ is dispersible. We strongly feel that $K_{1,n}$ is dispersible for all finite values of $n$. Hence we pause the following Conjecture.

![Figure 2.4: Dispersive dcs of $K_{1,2}$ and $K_{1,3}$](image)

Conjecture 1. The star $K_{1,n}$ is dispersible dcs-graph for all finite values of $n$.

Problem 3. Characterization of dispersible dcs-graph is an open problem for further investigation.

Problem 4. For any dcs-graph $G$, the dispersivity $\nu(G)$ of $G$ is the least cardinality of a ground set $X$ such that $G$ admits a dispersive dcs. Finding $\nu(G)$ of a given graph $G$ is an interesting problem. In particular find $\nu(K_n)$. 
2.3.1 Edge-dispersible dcs{l}-graphs

As we have seen in the previous section all graphs need not be dispersible. However, to identify the characteristics of a graph model, sometimes it is enough to consider the constants of proportionality $k^f_{(u,v)}$ whenever $uv$ is an edge of $G$. Hence, we define a new concept namely, edge-dispersible dcs{l}-graphs as follows.

**Definition 2.3.1.** A dcs{l} $f$ of a $(p,q)$-graph $G$ is edge-dispersive if the constants of proportionality $k^f_{(u,v)} : uv \in E(G)$ are all distinct and $G$ is edge-dispersible graph if it admits an edge-dispersive dcs{l}.

**Remark 2.3.2.** All dispersible dcs{l}-graphs are edge-dispersible. However the converse need not be true. We may observe that dispersible dcs{l}-graphs are subclass of edge-dispersible dcs{l}-graphs. In the case of complete graphs dispersibility and edge dispersibility are the same, since every pair of vertices in a complete graph is adjacent. Following results depicts some classes of edge-dispersible dcs{l}-graphs.

**Theorem 2.3.2.** The star $K_{1,n}$ of order $n+1$ is edge-dispersible dcs{l}-graphs.

**Proof.** Consider $K_{1,n}$ with $n+1$ vertices. Let $V(K_{1,n}) = \{v_0, v_1, \ldots, v_n\}$. Let $X = \{1, 2, 3, \ldots, k\}$, where $k = \frac{n(n+1)}{2}$. 
Define \( f : V(K_{1,n}) \to 2^X \) defined by
\[
f(v_0) = \emptyset \\
f(v_1) = \{1\} \text{ and,} \\
f(v_i) = \{\max(f(v_{i-1})) + j, \ 1 \leq j \leq i\}, \ 2 \leq i \leq n. \text{ Then,} \\
| f^{(e)}(v_i) | = | f(v_0) \oplus f(v_i) | = | \{\max(f(v_{i-1})) + j, \ 1 \leq j \leq i\} | = i, \ 1 \leq i \leq n. \text{ That is, } k_{(v_0,v_i)}^f = i, \ 1 \leq i \leq n. \text{ Therefore } K_{1,n} \text{ is edge-dispersible.}
\]

**Theorem 2.3.3.** The paths \( P_n \) are edge-dispersible dcsl-graphs.

**Proof.** Let \( P_n \) be a path on \( n \) vertices. Let \( v_0, v_1, \ldots, v_{n-1} \) be the \( n \) vertices of \( P_n \). Let \( X = \{1,2,3,\ldots,l\}, l = \frac{n(n-1)}{2}. \)

Define \( f : V(P_n) \to 2^X \) defined by
\[
f(v_0) = \emptyset; \\
f(v_1) = \{1\}; \text{ and,} \\
f(v_i) = f(v_{i-1}) \cup \{\max(f(v_{i-1})) + j, \ 1 \leq j \leq i\}, \ 2 \leq i \leq n. \text{ Then,} \\
| f^{(e)}(v_i) | = | f(v_{i-1}) \oplus f(v_i) | = | \{\max(f(v_{i-1})) + j, \ 1 \leq j \leq i\} | = i = i(d(v_{i-1}v_i)). \text{ That is, } k_{(v_{i-1},v_i)}^f = i. \text{ Therefore, } P_n \text{ is edge-dispersible.} \]

**Theorem 2.3.4.** Every complete bipartite graph is edge-dispersible.

**Proof.** Let \( X \) and \( Y \) be the bipartition of the vertex set of \( K_{m,n} \), where \( V(X) = \{u_1,u_2,\ldots,u_m\} \)
\( V(Y) = \{v_1, v_2, \ldots, v_n\} \).

Let \( X^* = \{1, 2, \ldots, mn\} \).

Define \( f : V(K_{m,n}) \to 2^{X^*} \), defined by

\[
\begin{align*}
    f(u_1) & = \emptyset; \\
    f(u_i) & = \{n + 1, n + 2, \ldots, in\}, \ 2 \leq i \leq m; \\
    f(v_j) & = \{1, 2, \ldots, j\}, \ 1 \leq j \leq n. \text{ Then,} \\
    | f^{\oplus}(u_1v_j) | & = | \{1, 2, \ldots, j\} | = j \text{ and, } | f^{\oplus}(u_iv_j) | = (i - 1)n + j, \ 2 \leq i \leq m, \ 1 \leq j \leq n. \text{ Now, } | f^{\oplus}(u_1v_j) | = | f^{\oplus}(u_iv_j) | \text{ implies } j = (i - 1)n + j, \text{ which implies } i = 1, \text{ not possible since, } 2 \leq i \leq m. \\
\end{align*}
\]

Thus \( K_{m,n} \) is edge-dispersible.

\[\square\]

### 2.4 \((k, r)\)-arithmetic dcsl-graphs

Again, recall that a dcsl \( f \) of a graph \( G = (V, E) \) is \((k, r)\)-arithmetic if the constants of proportionality with respect to \( f \) can be arranged in the arithmetic progression, \( k, k + r, k + 2r, \ldots, k + (q - 1)r \) and if \( G \) admits such a dcsl then, \( G \) is a \((k, r)\)-arithmetic dcsl-graph.

For example, \( K_4 \) is a \((1, 1)\)-arithmetic dcsl-graph, as evident from Figure 2.5.

**Theorem 2.4.1.** \ The path \( P_n \) is \((1, 1)\)-arithmetic dcsl if \( n \leq 3 \).
Proof. The subsets $\emptyset$, $\{1\}$ and $\{1,2,3\}$ of a set $X = \{1,2,3\}$, respectively assigned to the vertices $v_1, v_2$ and $v_3$ of $P_3$, we get a $(1,1)$-arithmetic dcsl-graph. Also $\emptyset$ and $\{1\}$ gives the $(1,1)$-arithmetic labeling of $P_2$ and $\emptyset$ gives a $(1,1)$-arithmetic labeling of $P_1$.

A $(1,1)$-arithmetic labeling of $P_3$ is shown in Figure 2.6.
Remark 2.4.1. We proved that paths $P_n$ are $(1,1)$-arithmetic dcsl-graphs if $n \leq 3$. We strongly believe that paths are not $(1,1)$-arithmetic dcsl-graphs for higher values of $n$. Thus we pose the Conjecture 2.

Conjecture 2. Path $P_n$ is $(1,1)$-arithmetic dcsl-graph if and only if $n \leq 3$.

In this section, we also establish that the complete graph $K_n$ admits a $(k, r)$-arithmetic dcsl if and only if $n \leq 4$ and $k = r$, a result which is strikingly similar to a result on ‘graceful’ complete graphs (see Golomb [15]).

Theorem 2.4.2. The complete graph $K_n$ is a $(k, r)$-arithmetic dcsl-graph if and only if $n \leq 4$ and $k = r$.

Proof. Necessity: Let $K_n$ admit a $(k, r)$-arithmetic dcsl $f : V(K_n) \rightarrow 2^X$, where $X = \{x_1, x_2, \ldots, \}$. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. We shall prove the theorem in two Parts.

PART 1: In this part, we prove that if $K_n$ is $(k, r)$-dcsl, then $k = r$.

We need to prove this in two cases, namely when the emptyset $\emptyset$ is or is not assigned to a vertex of $K_n$.

Case 1: $f(v_i) = \emptyset$ for some $v_i \in V(K_n)$. 

Without loss of generality, assume $f(v_1) = \emptyset$. Since $f$ is a $(k, r)$-arithmetic dcs, there exists an edge say $v_1v_j \in E(K_n)$ such that $|f^\oplus(v_1v_j)| = k$. Without loss of generality, assume that $|f^\oplus(v_1v_2)| = k$ so that $f(v_2) = \{x_1, x_2, \ldots, x_k\}$. Again, since $f$ is a $(k, r)$-arithmetic dcs, there exists an edge in $K_n$ such that the cardinality of the symmetric difference of the subsets of $X$ assigned to its end vertices is $k + r$. For this, there are two possibilities viz., $|f^\oplus(v_1v_t)| = k + r$ or $|f^\oplus(v_2v_t)| = k + r$. Without loss of generality, assume $v_t = v_3$. Then, we have either $|f^\oplus(v_2v_3)| = k + r$ or $|f^\oplus(v_1v_3)| = k + r$.

**First, we consider the possibility that $|f^\oplus(v_1v_3)| = k + r$.**

Then, we have the following possibilities.

$f(v_3) = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_r\}$

or $f(v_3) = \{y_1, y_2, \ldots, y_{k+r}\}$.

When $f(v_3) = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_r\}$, we get $|f^\oplus(v_2v_3)| = r$, which is not possible. Hence, $f(v_3) = \{y_1, y_2, \ldots, y_{k+r}\}$. This implies $|f^\oplus(v_2v_3)| = k + k + r = 2k + r$.

Therefore, the only possibility of getting $|f^\oplus(v_2v_3)| = k + 2r$ occurs when $k + 2r = 2k + r$, which implies $k = r$.

**Next, we consider the possibility that $|f^\oplus(v_2v_3)| = k + r$.**

In this case, there arise the following possibilities.
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\( f(v_3) = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_{k+r}\} \)

or \( f(v_3) = \{y_1, y_2, \ldots, y_r\} \)

When \( f(v_3) = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_{k+r}\} \), \( |f^\oplus(v_1v_3)| = 2k + r \); and when \( f(v_3) = \{y_1, y_2, \ldots, y_r\} \), \( |f^\oplus(v_1v_3)| = r \), which is not possible.

Hence, \( f(v_3) = \{y_1, y_2, \ldots, y_{k+r}\} \). Therefore, the only possibility of getting \( |f^\oplus(v_1v_3)| = k + 2r \) occurs when \( k + 2r = 2k + r \), which implies \( k = r \).

**Case 2:** \( f(v_i) \neq \emptyset \) for any \( v_i \in V(K_n) \).

Without loss of generality, assume \( f(v_1) = \{x_1, x_2, \ldots, x_t\} \). Since \( f \) is a \((k, r)\)-arithmetic dcs, there exists an edge say \( v_1v_j \in E(K_n) \) such that \( |f^\oplus(v_1v_j)| = k \). Without loss of generality, assume that \( |f^\oplus(v_1v_2)| = k \) so that \( f(v_2) = \{x_1, x_2, \ldots, x_{t+k}\} \) or \( f(v_2) = \{x_1, x_2, \ldots, x_t, y_1, y_2, \ldots, y_k\} \).

Again, since \( f \) is a \((k, r)\)-arithmetic dcs, there exists an edge in \( K_n \) such that the cardinality of the symmetric difference of the subsets of \( X \) assigned to its end vertices is \( k + r \). For this, there are two possibilities viz., \( |f^\oplus(v_1v_t)| = k + r \) or \( |f^\oplus(v_2v_t)| = k + r \). Without loss of generality, assume \( v_t = v_3 \). Thus, we have either \( |f^\oplus(v_2v_3)| = k + r \) or \( |f^\oplus(v_1v_3)| = k + r \).

Suppose \( f(v_2) = \{x_1, x_2, \ldots, x_{t+k}\} \). If \( |f^\oplus(v_2v_3)| = k + r \), then we have following possible assignments for the vertex \( v_3 \).
(i) \( f(v_3) = \{x_1, x_2, \ldots, x_{t+k}, y_1, y_2, \ldots, y_{k+r}\} \) or

(ii) \( f(v_3) = \{x_1, x_2, \ldots, x_t, y_1, y_2, \ldots, y_r\} \) or

(iii) \( f(v_3) = \{y_1, y_2, \ldots, y_{r-t}\} \)

or (iv) \( f(v_3) = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_{k+r-t}\} \)

(i) implies \( |f^\oplus(v_1v_3)| = 2k + r; \)

(ii) implies \( |f^\oplus(v_1v_3)| = t + r, \) which is not possible.

(iii) implies \( |f^\oplus(v_1v_3)| = t + r - t = r, \) which is not possible.

(iv) implies \( |f^\oplus(v_1v_3)| = 2k + r. \)

Therefore, the only possibility of getting \( |f^\oplus(v_1v_3)| = k + 2r \) occurs when \( k + 2r = 2k + r, \) which implies \( k = r. \)

The proof of the case when \( |f^\oplus(v_1v_3)| = k + r \) follows in a similar way.

Next, suppose \( f(v_2) = \{x_1, x_2, \ldots, x_t, y_1, y_2, \ldots, y_k\}. \) We should then have either \( |f^\oplus(v_2v_3)| = k + r \) or \( |f^\oplus(v_1v_3)| = k + r. \) Assume \( |f^\oplus(v_2v_3)| = k + r \) whence we have the following possibilities.

(i) \( f(v_3) = \{x_1, x_2, \ldots, x_t, y_1, y_2, \ldots, y_{k}, z_1, z_2, \ldots, z_{k+r}\} \) or

(ii) \( f(v_3) = \{x_1, x_2, \ldots, x_t, z_1, z_2, \ldots, z_r\} \) or

(iii) \( f(v_3) = \{y_1, y_2, \ldots, y_{k}, z_1, z_2, \ldots, z_{k+r-t}\} \) or

(iv) \( f(v_3) = \{z_1, z_2, \ldots, z_{r-t}\} \)

(i) implies \( |f^\oplus(v_1v_3)| = 2k + r; \)

(ii) implies \( |f^\oplus(v_1v_3)| = r, \) which is not possible.
(iii) implies $|f^\oplus(v_1v_3)| = 2k + r$, which is not possible.

(iv) implies $|f^\oplus(v_1v_3)| = r$.

Therefore, the only possibility of getting $|f^\oplus(v_1v_3)| = k + 2r$ occurs when $k + 2r = 2k + r$, which implies $k = r$.

Hence, if $K_n, n \geq 3$ is $(k, r)$-arithmetic dcsl, then $k = r$.

**PART 2:** In this part, we prove that if $K_n$ is $(r, r)$-arithmetic dcsl, then $n \leq 4$. If possible, suppose $n > 4$ and $K_n$ is $(r, r)$-arithmetic dcsl with an $(r, r)$-arithmetic dcsl $f$. Hence, the constants of proportionality with respect to $f$ can be arranged in the arithmetic progression, $r, 2r, 3r, 4r, 5r, 6r, 7r, 8r, 9r, 10r, \ldots$. Without loss of generality, assume $f(v_1) = X_1 = \{x_1, x_2, \ldots, x_k\}$. Then, to have the constant of proportionality $r$ on one of the edges of $K_5$ we have the following selection of the set $X_2$, say at the vertex $v_2$;

1. $f(v_2) = X_2 = \{x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_{2k+r}\}$ or
2. $f(v_2) = X_2 = \{x_1, x_2, \ldots, x_k\} \cup \{y_1, y_2, \ldots, y_{k+r}\}$.

Now, in order to get the constant of proportionally $2r$ on one of the other edges of $K_n$, we have the following assignment of subsets of the ground set $X$ to the vertices say at $v_3$.

1. $f(v_3) = X_3 = \{x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_{k+2r}\}$;
2. $f(v_3) = X_3 = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_{2r}\}$;
Consider (1.a1): Then,
\[ |f(v_1) \oplus f(v_2)| = r; \]
\[ |f(v_1) \oplus f(v_3)| = 2r \text{ and } |f(v_2) \oplus f(v_3)| = r, \]
a contradiction to the assumption that \( f \) is \((r, r)\)-dclsl. Hence, the labeling (1.a1) is not admissible.

Consider (1.a2) Then,
\[ |f(v_1) \oplus f(v_2)| = r; \]
\[ |f(v_1) \oplus f(v_3)| = 2r \text{ and } |f(v_2) \oplus f(v_3)| = 3r, \]
which are admissible.

Consider (1.a3) Then,
\[ |f(v_1) \oplus f(v_2)| = r; \]
\[ |f(v_1) \oplus f(v_3)| = 2r \text{ and } |f(v_2) \oplus f(v_3)| = r, \]
a contradiction to the assumption that \( f \) is \((r, r)\)-dclsl. Therefore, the labeling (1.a3) is not admissible.

Hence, until at this stage, we have the admissible assignment of subsets of \( X \) as follows:
\[ f(v_1) = X_1 = \{x_1, x_2, \ldots, x_k\} \]
\[ f(v_2) = X_2 = \{x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_{2k+r}\} \]
\[ f(v_3) = X_3 = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_{2r}\} \]

By symmetry, the possibility of the assignment (2), \( f(v_2) = X_2 = \{x_1, x_2, \ldots, x_k\} \cup \{y_1, y_2, \ldots, y_{k+r}\} \), reduces to the same choice namely,
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\[ f(v_3) = X_3 = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_{2r}\}. \]

Hence, without loss of generality, assume
\[ f(v_3) = X_3 = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_{2r}\}. \]

Now, to have the constant of proportionality 4r on pairs of vertices involving \( v_1, v_2, v_3 \) we have the following three possible selections.

II.a. \( f(v_4) = X_4 = \{x_1, x_2, \ldots, x_{k+4}, y_1, y_2, \ldots, y_{2r}\} \) or

II.b. \( f(v_4) = X_4 = \{x_1, x_2, \ldots, x_{k+r}, y_1, y_2, \ldots, y_{3r}\} \) or

II.c. \( f(v_4) = X_4 = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_{2r}, t_1, t_2, \ldots, t_{2r}\} \).

Consider (II.a): In this case, we get
\[
|f(v_1) \oplus f(v_2)| = r; \\
|f(v_1) \oplus f(v_3)| = 2r; \\
|f(v_2) \oplus f(v_3)| = 3r; \\
|f(v_1) \oplus f(v_4)| = 4r; \\
|f(v_2) \oplus f(v_4)| = 3r, \] a contradiction to our assumption that \( f \) is \((r, r)\)-dcsl. Hence, the labeling (1.a1) is not admissible.

Consider (II.b): In this case,
\[
|f(v_1) \oplus f(v_2)| = r; \\
|f(v_1) \oplus f(v_3)| = 2r; \\
|f(v_2) \oplus f(v_3)| = 3r; \\
|f(v_1) \oplus f(v_4)| = 4r; \\
|f(v_2) \oplus f(v_4)| = 3r, \] a contradiction to the assumption that \( f \) is \((r, r)\)-dcsl. Hence, the labeling (1.a1) is not admissible.

Consider (II.c): In this case,
\[
|f(v_1) \oplus f(v_2)| = r; \\
|f(v_1) \oplus f(v_3)| = 2r; \\
|f(v_2) \oplus f(v_3)| = 3r; \\
|f(v_1) \oplus f(v_4)| = 4r; \\
|f(v_2) \oplus f(v_4)| = 3r; \]
\[ |f(v_2) \oplus f(v_4)| = 5r \text{ and } |f(v_3) \oplus f(v_4)| = 6r, \text{ whence (1.a1) is admissible.} \]

Hence, until at this stage of completing the assignment of the vertices \( v_1, v_2, v_3, v_4 \), we have the admissible assignments as follows:

\[
\begin{align*}
  f(v_1) &= X_1 = \{x_1, x_2, \ldots, x_k\} \\
  f(v_2) &= X_2 = \{x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_{2k+r}\} \\
  f(v_3) &= X_3 = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_{2r}\}. \\
  f(v_4) &= X_4 = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_{2r}, t_1, t_2, \ldots, t_{2r}\}.
\end{align*}
\]

Now, to have the constant of proportionality \( 7r \) on pairs with the vertices \( v_1, v_2, v_3, v_4 \) we have the following four selections.

**II.a.** With the vertex \( v_1 \), \( f(v_5) = X_5 = \{x_1, x_2, \ldots, x_{k+7r}\} \) or

**II.b.** With the vertex \( v_2 \):

**II.b1.** \( f(v_5) = X_5 = \{x_1, x_2, \ldots, x_k, x_{k+8r}, \ldots, y_1, y_2, \ldots, y_{3r}\} \)

or

**II.b2.** \( f(v_5) = X_5 = \{x_1, x_2, \ldots, x_k, x_{k+r}, \ldots, y_1, y_2, \ldots, y_{6r}\} \)

**II.c.** With \( v_3 \): **II.c1.** \( f(v_5) = X_5 = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_{9r}\} \)

or

**II.c2.** \( f(v_5) = X_5 = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_{2r}, h_1, h_2, \ldots, h_{7r}\} \)

**II.d.** With \( v_4 \):

**II.d1.** \( f(v_5) = X_5 = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_{2r}, t_1, t_2, \ldots, t_{9r}\} \).
or

II.d2. \( f(v_5) = X_5 = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_{2r}, t_1, t_2, \ldots, t_{2r}, z_1, z_2, \ldots, z_{7r}\} \)

Consider II.a. Then, we get

\[
|f(v_1) \oplus f(v_2)| = r; \\
|f(v_1) \oplus f(v_3)| = 2r; \quad |f(v_2) \oplus f(v_3)| = 3r; \quad |f(v_1) \oplus f(v_4)| = 4r; \\
|f(v_2) \oplus f(v_4)| = 5r; \quad |f(v_3) \oplus f(v_4)| = 6r; \quad |f(v_2) \oplus f(v_4)| = 6r = |f(v_3) \oplus f(v_4)|, \]a contradiction to the assumption that \( f \) is \((r, r)\)-dcsl.
Hence, the labeling (II.a) is not admissible.

Consider II.b1. Then, we get

\[
|f(v_1) \oplus f(v_2)| = r; \\
|f(v_1) \oplus f(v_3)| = 2r; \quad |f(v_2) \oplus f(v_3)| = 3r; \quad |f(v_1) \oplus f(v_4)| = 4r; \\
|f(v_2) \oplus f(v_4)| = 5r; \quad |f(v_3) \oplus f(v_4)| = 6r; \quad |f(v_1) \oplus f(v_5)| = 8r; \\
|f(v_2) \oplus f(v_5)| = 7r; \quad |f(v_3) \oplus f(v_5)| = 10r; \quad |f(v_4) \oplus f(v_5)| = 12r, \]a contradiction to the assumption that \( f \) is \((r, r)\)-dcsl. Hence, the labeling (II.b1) is not admissible.

Consider II.b2. Then, we get

\[
|f(v_1) \oplus f(v_2)| = r; \\
|f(v_1) \oplus f(v_3)| = 2r; \quad |f(v_2) \oplus f(v_3)| = 3r; \quad |f(v_1) \oplus f(v_4)| = 4r; \\
|f(v_2) \oplus f(v_4)| = 5r; \quad |f(v_3) \oplus f(v_4)| = 6r; \quad |f(v_2) \oplus f(v_5)| = 6r = \]
\[|f(v_3) \oplus f(v_4)|,\] a contradiction to the assumption that \(f\) is \((r, r)\)-dcsl.

Hence, the labeling (II.b2) is not admissible.

Consider II.c1. Then, we get

\[|f(v_1) \oplus f(v_2)| = r;\]
\[|f(v_1) \oplus f(v_3)| = 2r; \quad |f(v_2) \oplus f(v_3)| = 3r; \quad |f(v_1) \oplus f(v_4)| = 4r;\]
\[|f(v_2) \oplus f(v_4)| = 5r; \quad |f(v_3) \oplus f(v_4)| = 6r; \quad |f(v_1) \oplus f(v_5)| = 11r,\] which is greater than \(10r\), the maximum constant of proportionality and hence a contradiction to the assumption that \(f\) is \((r, r)\)-dcsl. Hence, the labeling (II.c1) is not admissible.

Consider II.c2. Then, we get

\[|f(v_1) \oplus f(v_2)| = r;\]
\[|f(v_1) \oplus f(v_3)| = 2r; \quad |f(v_2) \oplus f(v_3)| = 3r; \quad |f(v_1) \oplus f(v_4)| = 4r;\]
\[|f(v_2) \oplus f(v_4)| = 5r; \quad |f(v_3) \oplus f(v_4)| = 6r; \quad |f(v_1) \oplus f(v_5)| = 9r;\]
\[|f(v_2) \oplus f(v_5)| = 10r; \quad |f(v_3) \oplus f(v_5)| = 7r; \quad |f(v_4) \oplus f(v_5)| = 9r = |f(v_1) \oplus f(v_5)|,\] a contradiction to the assumption that \(f\) is \((r, r)\)-dcsl.

Hence, the labeling (II.c2) is not admissible.

Hence, all the possible choices of assignment of the subsets of \(X\) to the vertices fail to define \(f\) as an \((r, r)\)-dcsl. Hence, for \(n \geq 5\), there exists no \((r, r)\)-dcsl for \(K_n\).
Sufficiency: Let \( n \leq 4 \) and \( k = r \). We then display an \((r, r)\)-arithmetic dcsl for \( K_n \) for each \( n \).

For \( n = 1, 2 \), assign \( f(v_1) = \emptyset \), and \( f(v_1) = \emptyset \), \( f(v_2) = \{x_1, x_2, \ldots, x_r\} \) respectively. In each case, it is easy to see that \( f \) so defined is indeed an \((r, r)\)-arithmetic dcsl of \( K_n \).

Next, let \( V(K_3) = \{v_1, v_2, v_3\} \) and let \( f : V(K_3) \to 2^X \) be defined by \( f(v_1) = X_1 = \{x_1, x_2, \ldots, x_r\} \); \( f(v_2) = X_2 = \{x_1, x_2, \ldots, x_r\} \cup \{y_1, y_2, \ldots, y_{2r}\} \), \( x_i \neq y_i \); \( f(v_3) = X_3 = \{x_1, x_2, \ldots, x_r, x_{r+1}, \ldots, x_{2r}\} \).

Then, since \( K_3 \) is complete, \( d(v_i, v_j) = 1 \) for all distinct \( i, j \in \{1, 2, 3\} \).

Also, \(|f(v_1) \oplus f(v_2)| = r; |f(v_1) \oplus f(v_3)| = 2r \) and \(|f(v_2) \oplus f(v_3)| = 3r \).

Thus, \( K_3 \) is an \((r, r)\)-arithmetic dcsl-graph.

Let \( V(K_4) = \{v_1, v_2, v_3, v_4\} \). Define \( f : V(K_4) \to 2^X \) so that \( f(v_1) = X_1 = \{x_1, x_2, \ldots, x_r\} \); \( f(v_2) = X_2 = \{x_1, x_2, \ldots, x_r\} \cup \{y_1, y_2, \ldots, y_{2r}\} \), \( x_i \neq y_i \); \( f(v_3) = X_3 = \{x_1, x_2, \ldots, x_r, x_{r+1}, \ldots, x_{2r}\} \) and \( f(v_4) = X_4 = \{x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_{2r}, t_1, t_2, \ldots, t_{2r}\} \). Since \( K_4 \) is complete, \( d(v_i, v_j) = 1 \) for all distinct \( i, j \in \{1, 2, 3, 4\} \). Then,

\[
|f(v_1) \oplus f(v_2)| = r; |f(v_1) \oplus f(v_3)| = 2r; |f(v_1) \oplus f(v_4)| = 4r; \\
|f(v_2) \oplus f(v_3)| = 3r; |f(v_2) \oplus f(v_4)| = 5r \text{ and } |f(v_3) \oplus f(v_4)| = 6r. 
\]

Hence, \( K_f(K_4) = \{r, 2r, 3r, 4r, 5r, 6r\} \) and the proof is complete.
2.5 1-uniform distance compatible set-labeled graphs

As defined already, a dcsl $f$ of a graph $G = (V, E)$ is $k$-uniform if all the constants of proportionality with respect to $f$ are equal to $k$, and if $G$ admits such a dcsl then $G$ is a $k$-uniform dcsl-graph. In this section, we study particularly on classes of graphs which admit a 1-uniform dcsl.

We start with the following proposition.

**Proposition 2.5.1.** If $(G, f)$ is any 1-uniform dcsl-graph then, no two adjacent vertices in $G$ receive subsets of the same cardinality.

**Proof.** Suppose $G$ has two adjacent vertices $u$ and $v$ with $|f(u)| = |f(v)|$.

**Case 1:** $f(u) \cap f(v) = \emptyset$

\[|f(u) \oplus f(v)| = |f(u) \cup f(v)| = |f(u) \cup f(v)| \geq 2,\text{ whereas } d(u, v) = 1, \text{ a contradiction.}

**Case 2:** $f(u) \cap f(v) \neq \emptyset$.

\[|f(u)| = |f(v)| \text{ and } f \text{ is injective implies } |f^{\oplus}(uv)| = |f(u) \oplus f(v)| = |f(u) - f(v)| + |f(v) - f(u)| \geq 2.\text{ Which is again a contradiction, since } d(u, v) = 1. \text{ Hence no two adjacent vertices in a 1-uniform dcsl-graph } G \text{ receive subsets of the same cardinality.} \]

**Theorem 2.5.2.** The complete graph $K_n$ admits a 1-uniform dcsl if and only if $n \in \{1, 2\}$. 
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Figure 2.7: A 1-uniform dcs of $K_1$ and $K_2$

Proof. A 1-uniform dcs each of $K_1$ and $K_2$ are given in Figure 2.7. Conversely, let $n \geq 3$ and suppose $K_n$ admits a 1-uniform dcs say, $f$ with the dcs-set $X$. Let the vertices $v_i$, be labeled by subsets $X_i$ of $X$. We first claim that none of these subsets $X_i$ can be empty. For, suppose $X_i = \emptyset$ for some $i$, say for $i = 1$. Then, by definition, $|X_1 \oplus X_2| = |X_2| = 1$. Similarly, $|X_1 \oplus X_3| = |X_3| = 1$. By injectivity of $f$, we then get $|X_2 \oplus X_3| = 2$, whereas $d(v_2, v_3) = 1$. This implies that, the constant of proportionality $k^f_{(v_2, v_3)} = 2 > 1$, which is a contradiction. Thus, $|X_i| \geq 1$, for each $i$, $1 \leq i \leq n$. Again, by hypothesis, $|X_1 \oplus X_2| = 1$, $|X_1 \oplus X_3| = 1$ and $|X_2 \oplus X_3| = 1$. Since, $X_i \oplus X_j = (X_i - X_j) \cup (X_j - X_i)$, $|X_1 \oplus X_2| = 1 \Rightarrow$ either $|X_1 - X_2| = 1$ or $|X_2 - X_1| = 1$. Without loss of generality, let $|X_1 - X_2| = 1$. Then $X_2 - X_1 = \emptyset \Rightarrow X_1 = X_2$ or $X_2 \subset X_1$. The
first possibility contradicts the injectivity of $f$. Therefore, $X_2 \subset X_1$.

Thus, we can find an element $k_1$ in $X$ such that

$$X_1 = X_2 \cup \{k_1\} \ldots \ldots \text{(i)}.$$  

Similarly, $|X_2 \oplus X_3| = 1 \Rightarrow X_2 \subset X_3$ or $X_3 \subset X_2$. If $X_2 \subset X_3$, then, there exists $k_2 \in X$ ($k_1 \neq k_2$) such that,

$$X_3 = X_2 \cup \{k_2\} \ldots \ldots \text{(ii)}.$$  

If $X_3 \subset X_2$, then there exists $k_3 \in X$ such that

$$X_2 = X_3 \cup \{k_3\} \ldots \ldots \text{(iii)}.$$  

From (i) and (ii) we get, $|X_1 \oplus X_3| = |\{k_1, k_2\}| = 2$, a contradiction, since $|X_1 \oplus X_3| = 1$. Also from (i) and (iii), we get $X_1 = X_3 \cup \{k_1, k_3\}$. So that, $|X_1 \oplus X_3| = |\{k_1, k_3\}| = 2$, again a contradiction. Thus, there cannot exists a 1-uniform dcsl, $f$ and subsets $X_1$, $X_2$ and $X_3$ such that $|X_1 \oplus X_2| = 1$, $|X_1 \oplus X_3| = 1$ and $|X_2 \oplus X_3| = 1$. This completes the proof. \hfill \square

Thus, the following interesting result is suggested by the proof of the Theorem \ref{thm:triangle-free}.

**Proposition 2.5.3.** If $G$ is a graph that admits a 1-uniform dcsl then $G$ is triangle-free.

**Proof.** Suppose $G$ has a triangle, with vertices $v_1, v_2$ and $v_3$, and $G$
has a 1-uniform dcsl \( f \). Two cases arise, viz.,

(i) one of the vertices receives the empty set as its \( f \)-value and
(ii) none of the vertices of the triangle receives \( \emptyset \) as its \( f \)-value.

A contradiction can be derived in each case as in the proof of Theorem 2.5.2, establishing the result by contraposition.

\[ \square \]

**Corollary 2.5.4.** For any graph \( G \) of order at least six, at most one of \( G \) and its complement \( \overline{G} \) can admit a 1-uniform dcsl.

**Proof.** This follows from Theorem 2.5.2 and the well known Theorem of Ramsey that ‘for every graph \( G \) of order at least six, either \( G \) or \( \overline{G} \) contains a triangle’ (see, [11]).

\[ \square \]

**Remark 2.5.1.** If \( G \) is a maximal outerplanar graph, then its central subgraph \( \langle C(G) \rangle \) is isomorphic to one of the seven graphs given in Figure 2.8 (see, [11]). By Proposition 2.5.3, the only maximal outerplanar graphs that admit a 1-uniform dcsl are \( K_1 \) and \( K_2 \).

**Proposition 2.5.5.** Any 1-uniform dcsl-graph \( G \) of order at least 3 has at most one vertex of full degree.

**Proof.** Let \( G \) be a 1-uniform dcsl-graph of order \( n \), where \( n \geq 3 \). Suppose, if possible, \( G \) has two vertices say, \( u \) and \( v \), of full degree.
That is, \( d(u) = n - 1 = d(v) \). Then, both \( u \) and \( v \) are adjacent to a vertex \( w \) of \( G \) and these three vertices \( u, v \) and \( w \) form a triangle in \( G \), which is not possible due to Proposition 2.5.3. Hence, \( G \) can have at most one vertex of full degree.

\[
\text{Proposition 2.5.6. } \text{In a 1-uniform dcsl-graph } G \text{ of order at least 3, one vertex is of full degree then all other vertices are pendant.}
\]

\[
\text{Proof. Suppose } G \text{ is a 1-uniform dcsl-graph with } n \text{ vertices and } v_1 \text{ be the vertex such that } d(v_1) = n - 1. \text{ That is, } v_1 \text{ is adjacent to all other vertices of } G. \text{ Now, if any two other vertices say } u \text{ and } w \text{ of } G \text{ are adjacent, then } u, w, v_1 \text{ form a triangle. Hence by Proposition 2.5.3, the result follows by contraposition.}
\]
2.5.1 Classes of 1-uniform dcsI-graphs

Here we discuss the existence of 1-uniform dcsI for different classes of graphs. We begin with path $P_n$.

**Theorem 2.5.7.** All paths are 1-uniform dcsI-graphs.

*Proof.* Consider $P_n$, the path with $n$ vertices. Let $v_1, v_2, \ldots, v_n$ be the vertices of $P_n$. Define $f : V(P_n) \rightarrow \{1, 2, 3, \ldots, n\}$ defined by $f(v_i) = \{1, 2, 3, \ldots, i\}$, $1 \leq i \leq n$; so that, $f^{\oplus}(v_i v_k) = f(v_i) \oplus f(v_k) = \{1, 2, 3, \ldots, i\} \oplus \{1, 2, 3, \ldots, k\}$, $i \leq k$

$= \{i + 1, i + 2, i + 3, \ldots, k\}$.

Thus,

$| f^{\oplus}(v_i v_k) | = k - i = d(v_i, v_k), 1 \leq i < k \leq n. \quad \Box$

**Remark 2.5.2.** However, 1-uniform dcsI of $P_n$ given in the proof of Theorem 2.5.7 is not unique. Figure 2.9 displays a 1-uniform dcsI of $P_5$ with $| X | = 4$. Theorem 2.5.7 gives a 1-uniform dcsI of $P_n$ with the cardinality of the underlying set $X$ being equal to $n$. Hence to find the minimum cardinality of the underlying set $X$ with respect to which $P_n$ admits a 1-uniform dcsI is an interesting problem to be investigated.
Theorem 2.5.8. All finite stars are 1-uniform dcsl-graphs.

Proof. Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of $K_{1,n}$ with $v_1$ as its center. Define $f : V(K_{1,n}) \rightarrow 2^X$ defined by

\[ f(v_1) = \{1\}, \quad f(v_i) = \{1, i\}, \quad 2 \leq i \leq n + 1. \]

Then, $f^\oplus(v_1v_j) = f(v_1) \oplus f(v_j) = \{1\} \oplus \{1, j\} = \{j\}.$

Thus $|f^\oplus(v_1v_j)| = 1 = d(v_1, v_j).$ Also, $f^\oplus(v_i v_j) = f(v_i) \oplus f(v_j) = \{1, i\} \oplus \{1, j\} = \{i, j\},$

and hence, $|f^\oplus(v_i v_j)| = 2 = d(v_i v_j).$ Thus, $f$ is a 1-uniform dcsl. \qed

A dcsl of $K_{1,9}$ is shown in Figure 2.10.

Combining Proposition 2.5.6 and Theorem 2.5.8 we have the following theorem.

Theorem 2.5.9. A graph $G$ with a vertex of full degree is 1-uniform dcsl if and only if $G \cong K_{1,n}.$

Theorem 2.5.10. Every even cycle is 1-uniform dcsl-graph.
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Figure 2.10: A dcl of $K_{1,9}$

**Proof.** Consider $C_n$, $n \geq 4$ even. Let $V(C_n) = \{v_1, v_2, v_3, \ldots, v_n\}$ and let $X = \{1, 2, 3, \ldots, k\}$ where $k = \frac{n}{2} + 1$.

Define $f : V(C_n) \rightarrow 2^X$ defined by,

$f(v_1) = X,$

$f(v_{i+1}) = f(v_i) - \{i + 2\}, \quad 1 \leq i \leq \frac{n}{2},$

$f(v_k) = \{1, 2, 3\},$

$f(v_{k+j}) = f(v_{k+j-1}) \cup \{j + 3\}, \quad 1 \leq j \leq \frac{n-4}{2}$. It can be easily verified that the set-valuation $f$ is a 1-uniform dcl.

The 1-uniform dcl given in Theorem 2.5.10 is not unique. Figure 2.11 gives a 1-uniform dcl of $C_6$ with a dcl set $X$ of cardinality four.

**Theorem 2.5.11.** Cycles $C_n$ with $n \geq 3$ and $n$ odd are not 1-uniform dcl-graphs.
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**Figure 2.11:** 1-uniform dcsl of $C_6$

---

**Proof.** Let $C_n$ be a cycle with $n \geq 3$ and $n$ odd. Let $(v_1, v_2, \ldots, v_n)$ be the vertices of $C_n$ read in that order around the cycle. If possible, suppose that $f$ is a 1-uniform dcsl of $C_n$ and, without loss of generality, $X_1, X_2, \ldots, X_n$ be the distinct subsets of $X$ assigned by $f$ to the vertices $v_1, v_2, \ldots, v_n$ of $C_n$ respectively. That is, $f(v_i) = X_i$, $1 \leq i \leq n$.

**Claim** $X_i \neq \emptyset$ for any $i$, $1 \leq i \leq n$.

Suppose $X_i = \emptyset$ for some $i$, say for $i = 1$. Then,

$$1 = |X_2| = |X_n|, 2 = |X_3| = |X_{n-1}|, \ldots, \left\lfloor \frac{n}{2} \right\rfloor |X_{\left\lfloor \frac{n}{2} \right\rfloor}| = |X_{\left\lfloor \frac{n}{2} \right\rfloor + 1}| = \left\lceil \frac{n}{2} \right\rceil.$$ Therefore $|f^\oplus(v_{\left\lfloor \frac{n}{2} \right\rfloor}, v_{\left\lfloor \frac{n}{2} \right\rfloor + 1})| \geq 2$. But, $d(v_{\left\lfloor \frac{n}{2} \right\rfloor}, v_{\left\lfloor \frac{n}{2} \right\rfloor + 1}) = 1$, which is a contradiction. Therefore, it follows that $|X_i| \geq 1$, for every $i$, $1 \leq i \leq n$. 

Let $|X_1| = |f(v_1)| = m$. Now, $1 = |X_1 \oplus X_2| = |X_1 - X_2| + |X_2 - X_1|$; 
⇒ either $|X_1 - X_2| = 1$ and $|X_2 - X_1| = 0$; 
or $|X_2 - X_1| = 1$ and $|X_1 - X_2| = 0$; 
Suppose, $|X_1 - X_2| = 1$ and $|X_2 - X_1| = 0$. Then, $X_2 \subseteq X_1$. On the 
other hand, if $|X_1 - X_2| = 0$ and $|X_2 - X_1| = 1$, then $X_1 \subseteq X_2$.
Similarly, for each $i$, $2 \leq i \leq n$, we can show that, either $X_i \subseteq X_{i+1}$ or 
$X_{i+1} \subseteq X_n$, where the indices are reduced modulo $n$.
Considering the different possible cardinalities of the sets $X_i$, $2 \leq i \leq n$
we have the following.

\[
|X_2| = |X_n| = \begin{cases} 
m + 1 \\
\text{or} \\
m - 1
\end{cases}
\]
\[
|X_3| = |X_{n-1}| = \begin{cases} 
m + 2 \\
m \\
m - 2
\end{cases}
\]
\[
|X_4| = |X_{n-2}| = \begin{cases} 
m + 3 \\
m + 1 \\
m - 1 \\
m - 3
\end{cases}
\]
\[ |X_{\lfloor \frac{n}{2} \rfloor}| = |X_{\lfloor \frac{n}{2} \rfloor+1}| = \begin{cases} 
  m + \lfloor \frac{n}{2} \rfloor - 1 \\
  m + \lfloor \frac{n}{2} \rfloor - 3 \\
  m + \lfloor \frac{n}{2} \rfloor - 5 \\
  \quad \ldots \\
  m + 2 \\
  m \\
  m - 2 \\
  \quad \ldots \\
  m - \lfloor \frac{n}{2} \rfloor - 5 \\
  m - \lfloor \frac{n}{2} \rfloor - 3 \\
  m - \lfloor \frac{n}{2} \rfloor - 1 
\end{cases} \]

if \( \lfloor \frac{n}{2} \rfloor \) is odd
and

\[
\begin{align*}
|X_{\lfloor \frac{n}{2} \rfloor}| &= |X_{\lfloor \frac{n}{2} \rfloor + 1}| = \\
&= \begin{cases} 
  m + \lfloor \frac{n}{2} \rfloor - 1 \\
  m + \lfloor \frac{n}{2} \rfloor - 3 \\
  m + \lfloor \frac{n}{2} \rfloor - 5 \\
  \vdots \\
  m - \lfloor \frac{n}{2} \rfloor - 5 \\
  m - \lfloor \frac{n}{2} \rfloor - 3 \\
  m - \lfloor \frac{n}{2} \rfloor - 1 
\end{cases}
\end{align*}
\]

if \( \lfloor \frac{n}{2} \rfloor \) is even.

There are \( \lfloor \frac{n}{2} \rfloor \) different possibilities for the cardinality of \( |X_{\lfloor \frac{n}{2} \rfloor}| \) and \( |X_{\lfloor \frac{n}{2} \rfloor + 1}| \). Considering the \( \lfloor \frac{n}{2} \rfloor^2 \) combinations of these possibilities, we can see that \( |X_{\lfloor \frac{n}{2} \rfloor}| \oplus |X_{\lfloor \frac{n}{2} \rfloor + 1}| \geq 2 \). But, \( d(v_{\lfloor \frac{n}{2} \rfloor}, v_{\lfloor \frac{n}{2} \rfloor + 1}) = 1 \), again a contradiction. Hence the theorem seems to be proved, by method of
contradiction.

In view of Theorem 2.5.10 and Theorem 2.5.11 we have the characterization of 1-uniform dcsl-cycles.

**Theorem 2.5.12.** Cycle $C_n$, $(n \geq 3)$ is 1-uniform dcsl if and only if $n$ is even.

The following result asserts more than what Theorem 2.5.11 reflects.

**Theorem 2.5.13.** If a graph $G$ has an odd cycle as an induced subgraph then $G$ does not admit a 1-uniform dcsl.

**Proof.** First we will prove that any graph contains $C_3$ or $C_5$ as an induced subgraph, does not admit a 1-uniform dcsl. Suppose $G$ is a graph on $m$ vertices which contains an odd cycle $C_3$ as an induced subgraph and $G$ admits a 1-uniform dcsl $f$, with the ground set $X$. Let $X_1, X_2$ and $X_3$ are the subsets of $X$ which are assigned to the vertices of $C_3$ under the 1-uniform dcsl $f$. Then, the given 1-uniform dcsl $f$ induces a 1-uniform dcsl on $C_3$. But this is not possible by Proposition 2.5.3. Again, if it contains $C_5$ as an induced subgraph and if there exist a chord of length two between any pair of vertices in $C_5$, then we get a $C_3$ as an induced subgraph and by previous argument $G$ cannot have a 1-uniform dcsl. Thus, any graph contains $C_5$ as an induced subgraph
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is not 1-uniform dcsl-graph. Now consider $C_7$. Radius of $C_7$ is three. Existence of a 2-chord gives rise to a $C_5$ and hence do not admit a 1-uniform dcsl. Chord of length three makes no difference between the lengths of different pairs of vertices in $C_7$ and it is not 1-uniform dcsl-graph by Theorem 2.5.11. Thus, any graph contains $C_7$ as an induced subgraph is not a 1-uniform dcsl-graph.

In general, if $C_{2n+1}$ is an odd cycle with radius $\lfloor \frac{2n+1}{2} \rfloor$, then existence of any of the $p$-chords ($2 \leq p \leq \lfloor \frac{2n+1}{2} \rfloor - 1$) makes an induced odd cycle of length less than $2n + 1$ in the graph $G$. By induction, it follows that if a graph $G$ has an odd cycle as an induced subgraph, then $G$ does not admit a 1-uniform dcsl. 

We already proved that even cycles are dcsl-graphs. It is interesting to check whether a cycle with chords is dcsl-graph. Also Theorem 2.5.13 states that no odd cycle with chords can admit a dcsl. Thus, we have the following characterization of dcsl cycle with chords.

**Theorem 2.5.14.** The $n$ cycle $C_n$ with chords is dcsl-graph if and only if $n$ is even and the maximum number of chords is $\frac{n}{2} - 2$

**Proof.** Let $G$ be a cycle of length $n$ where $n$ is odd with $\frac{n}{2} - 2$ chords. Since odd cycles are forbidden for dcsl-graphs, $G$ is not a dcsl-graph.
Conversely, let $C_n$ be an $n$ cycle. Let $v_1, v_2, \ldots, v_n$ are the vertices of $C_n$. Join the non-adjacent vertices $v_i$ and $v_j$ in $C_n$ when $d(v_i, v_j)$ is odd. At each step we get a dcsl-graph. There are maximum $\frac{n}{2} - 2$ such chords. Finally we get the graph which is nothing but $P_2 \times P_{\frac{n}{2}}$ and invoking Theorem 2.5.26 $P_2 \times P_{\frac{n}{2}}$ is a dcsl-graph. □

**Remark 2.5.3.** Note that the dcsl-graph obtained in the above theorem is a maximal outerplanar graph. Thus maximum number of chords that can be added to an even cycle $C_n$ such that the resulting graph is maximal outerplanar dcsl-graph is $\frac{n}{2} - 2$.

**Corollary 2.5.15.** For $n > 4$, no graph $C_n$ has a 1-uniform dcsl $f$ such that $f(u) = \emptyset$ for some $u \in C_n$.

**Proof.** Let $C_n$ be any cycle on $n$ vertices $v_1, v_2, \ldots, v_n$, $n > 4$. Suppose $C_n$ has a 1-uniform dcsl $f$ such that $f(v_1) = \emptyset$

**Case 1: $n$ is even.**

If $n$ is even, then $f(v_1) \subset f(v_2) \subset f(v_3) \subset \cdots \subset f(v_{\frac{n}{2}})$ with $|f(v_i)| = i - 1, 1 \leq i \leq \frac{n}{2}$ and $f(v_1) \subset f(v_{\frac{n}{2}}) \subset f(v_{n-1}) \subset \cdots \subset f(v_{n+1})$ with $|f(v_{n-j})| = |f(v_{n-j+1})| + 1, 1 \leq j \leq \frac{n}{2} + 1$. Thus, $|f(v_{\frac{n}{2}})| = |f(v_{\frac{n}{2}+1})|$ and, $|f^\oplus(v_{\frac{n}{2}}, v_{\frac{n}{2}+1})| \geq 2$, which is a contradiction, since $v_{\frac{n}{2}}$ is adjacent to $v_{\frac{n}{2}+1}$. 

**Case 2:** $n$ is odd.

If $n$ is odd, then by Theorem 2.5.11 $f(u) \neq \emptyset$, for any $u \in C_n$. Hence, for $n > 4$, no graph $C_n$ has a 1-uniform dcsl $f$ such that $f(u) = \emptyset$, for some $u \in C_n$.

**Theorem 2.5.16.** Complete bipartite graph $K_{m,n}$ is 1-uniform dcsl if and only if it is isomorphic to $K_{1,n}$ or $K_{2,2}$.

**Proof.** Due to Theorem 2.5.8, $K_{1,n}$ is 1-uniform dcsl. Since $K_{2,2} \cong C_4$, by Theorem 2.5.10, $K_{2,2}$ is also 1-uniform dcsl.

Now, suppose if possible, there exists a 1-uniform dcsl, $f$ of $K_{m,n}$, where $m, n \geq 3$. Let $X$ be the dcsl-set satisfying, $X \geq d(K_{m,n})$ and $|2^X| \geq mn$. Let $V_1 = \{u_1, u_2, u_3, \ldots, u_m\}$ and $V_2 = \{v_1, v_2, v_3, \ldots, v_n\}$ be the bipartition of vertex set of $K_{m,n}$. Now, since $f$ is a 1-uniform dcsl, $|f(u_i) \oplus f(v_j)| = 1$, $|f(u_i) \oplus f(u_j)| = 2$ and $|f(v_i) \oplus f(v_j)| = 2$, $1 \leq i \leq m$, $1 \leq j \leq n$.

**Case 1:** $|f(u_1)| = |f(u_2)| = |f(u_3)| = m$. Then, $|f(v_1)| = m+1$ or $m-1$.

**Subcase (i):** If $|f(v_1)| = m+1$, then, $f(v_1)$ must be the union of $f(u_1), f(u_2)$ and $f(u_3)$, Since $f$ is a 1-uniform dcsl. Now, since $f(v_1)$ must be the union of $f(u_1), f(u_2)$ and $f(u_3)$, by injectivity of $f$, we
get, either $|f(v_2)| \geq m + 2$, a contradiction or $|f(v_2)| = m - 1$

If $|f(v_2)| = m - 1$, then, $f(v_2) \subset f(u_1)$, $f(v_2) \subset f(u_2)$ and $f(v_2) \subset f(u_3)$. Thus, either $f(u_3) \subset f(u_1)$ or $f(u_3) \subset f(u_2)$, both contradict the injectivity of $f$, since $|f(u_1)| = |f(u_2)| = |f(u_3)|$.

**Subcase (ii):** If $|f(v_1)| = m - 1$, then, $f(v_1) \subset f(u_1)$, $f(v_1) \subset f(u_2)$ and $f(v_1) \subset f(u_3)$. Thus, either $f(u_3) \subset f(u_1)$ or $f(u_3) \subset f(u_2)$, both contradict the injectivity of $f$, since $|f(u_1)| = |f(u_2)| = |f(u_3)|$.

**Case 2:** $|f(u_1)| = |f(u_2)| = m$ and $|f(u_3)| = m - 2$ or $m + 2$.

**Subcase (i):** If $|f(u_3)| = m - 2$, then $f(u_3) \subset f(u_1)$ and $f(u_3) \subset f(u_2)$. Hence $|f(v_1)| = |f(v_2)| = |f(v_3)| = m - 1$. But, $f(v_1) \subset f(u_1)$ and $f(v_1) \subset f(u_2)$; $f(v_2) \subset f(u_1)$ and $f(v_2) \subset f(u_2)$; and $f(v_1) \supset f(u_3)$. By the same argument as that of Case (i), we reach a contradiction to the injectivity of $f$.

**Subcase (ii):** If $|f(u_3)| = m + 2$ then, $|f(v_1)| = |f(v_2)| = |f(v_3)| = m + 1$, which is, again as in Case (i), a contradiction to the fact that $f$ is injective. Hence, the proof is complete by method of contraposition. □

**Remark 2.5.4.** Because of Theorem 2.5.16, $K_{3,3}$ is not 1-uniform dcsl. Now, delete three independent edges of $K_{3,3}$ so that we get an
Figure 2.12: 1-uniform dcsl bipartite graph

even cycle which is 1-uniform dcsl. Hence, it is of interest to find the minimum number of edges, $\rho(m,n)$ to be deleted from a complete bipartite graph so that the resultant graph is 1-uniform dcsl. We have calculated the minimum number of edges to be deleted from $K_{3,3}$, $K_{3,4}$, $K_{4,4}$ and $K_{4,5}$ so as to obtain a 1-uniform dcsl-graph as 2, 4, 6, and 9 respectively. However, in general, the calculation of the minimum number $\rho(m,n)$ of edges to be deleted from a complete bipartite graph $K_{m,n}$ so that the resulting graph is a bipartite 1-uniform dcsl-graph is under further investigation.

**Problem 5.** Find out $\rho(m,n)$ for complete bipartite graph $K_{m,n}$.

**Theorem 2.5.17.** The $n$-dimensional cube $Q_n$ admits a 1-uniform dcsl.
Proof. Consider the standard labeling of the vertices of $Q_n$ with binary $n$-vectors, which has the property that there is an edge in $Q_n$ if and only if the corresponding $n$-tuples of 0’s and 1’s differ in exactly one coordinate. As well known, this labeling has the property that the distance between any two vertices in $Q_n$ is equal to the Hamming distance between the corresponding $n$-tuples of 0’s and 1’s which is defined as the number of coordinates in which they differ. If we consider the ground set $X$ as consisting of the labels $x_1, x_2, \ldots, x_n$, then for any subset $A$ of $X$ we have $x_i \in A$ if and only if the $i$-th coordinate of the $n$-vector equals 1. With this 1–1 correspondence, it follows that the subsets of $X$ are assigned to the vertices of $Q_n$ in such way that their corresponding characteristic vectors satisfy the condition that there is an edge $uv$ in $Q_n$ if and only if the symmetric difference of $A_i$ and $A_j$ corresponding to $u$ and $v$ respectively, consists of exactly that number of elements, which is equal to the Hamming distance between the characteristic vectors corresponding to $A_i$ and $A_j$. Thus, the labeling turns out to be equivalent to the 1-uniform dcsl of $Q_n$. 

See Figure 2.13 for the cases $n = 1, 2$ and 3.

**Theorem 2.5.18.** The line graph $L(G)$ of a graph $G$ of order $\geq 4$
is 1-uniform dcs1-graph if and only if $G \cong C_{n(even)}$ or $G \cong P_n$.

Proof. As we already know, $L(C_n) = C_n$ and $L(P_n) = P_{n-1}$ and hence by Theorem 2.5.10 and by Theorem 2.5.7, $L(G)$ is 1-uniform dcs1 if $G \cong C_{n(even)}$ or $G \cong P_n$.

Conversely, suppose that $G$ is neither an even cycle nor a path. Then, there exists at least one vertex of degree greater than two, say $v$ in $G$, whence $v$ is adjacent to at least three other vertices. That is, three edges are incident at the vertex $v$ and hence the corresponding vertices in $L(G)$ are pairwise adjacent and hence form a triangle. By Proposition
2.5.3. $L(G)$ is not 1-uniform dcsl.

2.5.2 1-Uniform distance-compatible set-labelings of trees

The main aim of this section is to present an account of 1-uniform dcsl trees. In fact, we prove that all trees admit a 1-uniform dcsl. We have already proved that all finite stars are 1-uniform dcsl-graphs. In this section we consider different classes of 1-uniform dcsl-trees and develops an algorithmic method of labeling to establish that all trees are 1-uniform dcsl.

**Theorem 2.5.19.** Let $G$ be a tree of diameter $\leq 3$. Then $G$ is a 1-uniform dcsl-graph.

**Proof.** If $G$ is a tree of diameter less than or equal to three, then $G$ is isomorphic to $K_1$, $K_2$, $K_{1,n}$ or $S_{m,n}$. By Figure 2.7, and by Theorem 2.5.8, $K_1$, $K_2$ and $K_{1,n}$ are 1-uniform dcsl-graphs. We will prove that $S_{m,n}$ is also 1-uniform dcsl.

$$S_{m,n} = \overline{K}_m + K_1 + K_1 + \overline{K}_n.$$

Let $u_1, u_2, u_3, \ldots, u_m$ be the $m$ vertices of $\overline{K}_m$, $u$ and $v$ are the vertices of $K_2$ and let $v_1, v_2, v_3, \ldots, v_n$ are the $n$ vertices of $\overline{K}_n$ such that $u$ is adjacent to $u_i, 1 \leq i \leq m$ and $v$ is adjacent to $v_j, 1 \leq j \leq n$. Let
$X = \{1, 2, 3, \ldots, m+n, m+n+1\}$.

Define $f : V(S_{m,n}) \to 2^X$ defined by

$f(u) = \{1, 2\};$

$f(u_1) = \{1\};$

$f(u_2) = \{2\};$

$f(u_i) = \{1, 2, i\}, 3 \leq i \leq m;$

$f(v) = \{1, 2, m+1\};$

$f(v_j) = f(v) \cup \{m + 1 + j\}, 1 \leq j \leq n.$

Then, clearly $S_{m,n}$ is 1-uniform dcsl. Thus, all trees of diameter $\leq 3$ is 1-uniform dcsl trees.

Figure 2.14 gives a 1-uniform dcsl of $S_{7,7}$.

Figure 2.14: 1-uniform dcsl of $S_{7,7}$

There exists 1-uniform dcsl trees with diameter greater than three.
Given a graph $G$, we denote by $G^+$ the graph obtained from $G$ by augmenting a new vertex $v'$ for each vertex $v$ of $G$ and augmenting a new edge $vv'$. In particular, $P_n^+$ is called a comb.

**Theorem 2.5.20.** The comb, $P_n^+$ admits a 1-uniform dcsl for all finite $n$.

*Proof.* Consider the comb $G$ of $2n$ vertices. Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of its stem $P_n$ and $u_1, u_2, u_3, \ldots, u_n$ are the pendant vertices.

Let $X = \{1, 2, 3, \ldots, 2n - 1\}$

Define $f : V(P_n^+) \rightarrow 2^X$ defined by

$f(u_1) = \emptyset$;

$f(v_1) = \{1\}$;

$f(v_i) = f(v_{i-1}) \cup \{2(i-1)\}, \ 2 \leq i \leq n$;

$f(u_j) = f(v_j) \cup \{2j-1\}, \ 2 \leq j \leq n$. Then the set-labeling $f$ is indeed a 1-uniform dcsl of $G$.

The graph in Figure 2.15 displays a comb together with one of its 1-uniform dcsl. Note that the dscl-labeling defined in Theorem 2.5.20 is a set-indexer.

Given a graph $G = (V, E)$, we denote by $G^{+k}$, the graph obtained from $G$ by augmenting $k$ isolated vertices $v_1^1, v_2^2, \ldots, v_j^j$ and adjoining
Figure 2.15: 1-uniform dcsl of comb

\[ v_i v_j, \ 1 \leq j \leq k \] to \( G \). In particular when \( k = 1 \), we get \( G^+ = G^{+1} \).

Also, \( P_n^{+k} \) is called the \( k \)-uniform caterpillar.

Figure 2.16: 1-uniform dcsl of caterpillar

**Theorem 2.5.21.** Every caterpillars admits a 1-uniform dcsl-labeling.

**Proof.** Let \( G \) be a caterpillar with \( m \) vertices. Let \( v_1, v_2, v_3, \ldots, v_n \) be the internal vertices of \( G \) and let \( p_i \) \( (1 \leq i \leq n) \) be the number of
pendant vertices adjacent to $v_i$. Let $v^j_i$ $(1 \leq j \leq p_i)$ be the pendant vertices adjacent to $v_i$ $(1 \leq i \leq n)$.

Let $X = \{1, 2, 3, \ldots, m\}$

Define $f : V(G) \to 2^X$ by

$f(v_1) = \{1\}$;

$f(v_i) = f(v_{i-1}) \cup \{max(f(v^i_{i-1}) + 1\}, 2 \leq i \leq n$;

$f(v^i_i) = f(v_i) \cup \{max(f(v_i) + j\}, 1 \leq j \leq p_i, 1 \leq i \leq n$.

Now,

$| f^\oplus(v_i v_j) | = | f(v_i) \oplus f(v_j) | = j - i = d(v_i, v_j), 1 \leq i < j \leq n$.

Also,

$| f^\oplus(v_i v^j_i) | = | f(v_i) \oplus f(v^j_i) | = 1 = d(v_i, v^j_i), 1 \leq i \leq n, 1 \leq j \leq p_i$.

Again,

$| f^\oplus(v^j_i v^k_i) | = | f(v^j_i) \oplus f(v^k_i) | = | f(v_i) \cup \{maxf(v_i) + j\} \oplus f(v_i) \cup \{maxf(v_i) + k\} = 2 = d(v^j_i, v^k_i), 1 \leq i \leq n, 1 \leq j < k \leq p_i$.

Again,

$| f^\oplus(v^j_i v^k_k) | = | f(v^j_i) \oplus f(v^k_k) | = | f(v_i) \cup \{maxf(v_i) + j\} \oplus f(v_k) \cup \{maxf(v_k) + l\} | = k - i + 2 = d(v^j_i, v^l_k), 1 \leq i < k \leq n, 1 \leq j \leq p_i, 1 \leq l \leq p_k$.

Thus, $f$ is a 1-uniform dcs1. □
A 1-uniform dcsl of a caterpillar is depicted in Figure 2.16.

![Figure 2.17: 1-uniform dcsl of olive tree](image)

**Theorem 2.5.22.** Every olive tree admits a 1-uniform dcsl.

**Proof.** Let $T$ be an olive tree and $v$ be the root vertex. Let $d(v) = m \geq 2$, and the vertices on its $j^{th}$ level, $i^{th}$ position be denoted by $v_j^i$, $1 \leq j \leq m$, $1 \leq i \leq m - j + 1$. Let $X = \{1, 2, \ldots, n\}$ where $n$ is the order of $T$. Now, define $f : V(G) \to 2^X$, defined by

- $f(v) = \{1\}$;
- $f(v_1^i) = \{1, i+1\}$, $1 \leq i \leq m$;
- $f(v_j^1) = f(v_{j-1}^2) \cup \{\max(f(v_{j-1}^{m-j+2}))\}$, $2 \leq j \leq m$;
- $f(v_j^i) = f(v_{j-1}^{i+1}) \cup \{\max(f(v_j^{i-1}))\}$, $2 \leq i \leq m - j + 1$, $1 \leq j \leq m$. 
Clearly $f$ is a 1-uniform dcsl of the olive tree.

\[ \text{Theorem 2.5.22 is illustrated in Figure 2.17.} \]

**Theorem 2.5.23.** Every binary tree admits a 1-uniform dcsl.

**Proof.** Let $T_b$ be a binary tree with $n$ levels and $p$ vertices. Let $v_0$ be the root vertex of $T_b$ and $v^j_i$ ($1 \leq i \leq n$, $1 \leq j \leq 2^i$) be the vertices of the binary tree with $v_0$ as the root vertex. Let $v^j_i$ be the parent vertex of $v^j_{i+1}$ and $v^j_{i+1}$ and let $m_{ij} = \text{max} f(v^j_i)$. Let $X = \{1, 2, 3, \ldots, p\}$. Define $f : V(T_b) \rightarrow 2^X$, defined by

\[
\begin{align*}
f(v_0) &= \{1\}; \\
f(v^j_{i+1}) &= f(v^j_i) \cup \{2m_{ij}\}; \\
f(v^j_{i+1}) &= f(v^j_i) \cup \{2m_{ij} + 1\}, 1 \leq i \leq n, 1 \leq j \leq 2^i.
\end{align*}
\]

Clearly, $f$ is a 1-uniform dcsl of the binary tree. \[ \square \]

Figure [2.18](#) illustrates our construction of the dcsl of a binary tree described in the proof of the Theorem 2.5.23. Now, we are in a position to prove that all trees admit 1-uniform dcsl.

**Theorem 2.5.24.** Every tree admits a 1-uniform dcsl.

**Proof.** Let $T$ be a tree of order $n$. Choose any vertex say $v_1$, and apply BFS algorithm to the tree $T$, with $v_1$ as the root.
Define $f : V(G) \to 2^X$ defined as given below.

$v_1$ is the vertex at the 0th level. Define $f(v_1) = \{1\}$.

Let $v_2, v_3, \ldots, v_{n_1}$ be the $n_1 - 1$ vertices in the first level. Assign $f(v_i) = f(v_1) \cup \{i\}$, $2 \leq i \leq n_1 - 1$.

In general, if $v_j$ is the $j^{th}$ vertex of $T$, which is on the $m^{th}$ level, define $f(v_j) = f(v_j^p) \cup \{j\}$, where $v_j^p$ is the parent of $v_j$ on the $m - 1^{th}$ level.

The assignment $f$ is one-one, since at each vertex we have included a new distinct vertex. Also, if $v_i$ is on the $l^{th}$ level then, $|f(v_i)| = l + 1$, $1 \leq i \leq n$.

Also, since at each level, at each vertex, we are including exactly one distinct element of the underlying set $X$, $d(v_1, v_j) = l - 1 = |f(v_1) \oplus f(v_j)|$, $2 \leq j \leq n$, where $v_j$ is a vertex on the $l^{th}$ level.
\[ d(v_h, v_k) = l + m - 2h = | f(v_h) \oplus f(v_k) |, \quad 1 \leq h \leq k \leq n, \]

where \( v_h \) is any vertex at \( l \)-th level, \( v_k \) any vertex at \( m \)-th level and \( h \) is nothing but the number of level (or generation) at which, \( v_h \) and \( v_k \) have the first common forefather \( v_i \). Hence we conclude that \( f \) is a 1-uniform dcsl for the tree \( T \).

An arbitrary tree and its corresponding 1-uniform dcsl are shown in Figure 2.19 and in Figure 2.20.

![Figure 2.19: An arbitrary tree](image)

We have established that all trees are 1-uniform dcsl-graphs. So the natural question is to see whether a unicyclic graph which is obtained from a tree by adding one backedge is 1-uniform dcsl. Next theorem shows that a unicyclic graph is a 1-uniform dcsl-graph if and only if the cycle is even.
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Theorem 2.5.25. Unicyclic graphs are 1-uniform dcsl-graphs if and only if the cycle is even.

Proof. Let $G$ be a graph with unique even cycle, say $C_n$. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$. Let $T_i$ be the tree attached to $v_i$ in the sense, $v_i$ can be taken as the root vertex of $T_i$. Let $v_{ij}^k$ be the vertices of the rooted tree $T_i$, $2 \leq j \leq l_i$ where $l_i$ is the number of levels of $T_i$ and $v_{i1}^0 = v_i$. Take $X = \{1, 2, \ldots, |V(G)|\}$. Let $f$ be a 1-uniform dcsl of $C_n$, so that the subset, $X_i$ of $X$ is assigned to the vertex $v_i$ of $C_n$, such that $X_i \neq X_j$ if $v_i \neq v_j$. Now, extend the dcsl $f$ to the whole graph by defining $f(v_{i2}^1) = f(v_{i1}^1) \cup \max\{\max (X_i) + 1\}$, where $i$ varies over all vertices which are root vertex of some $T_i$. Also, define $f(v_{ij}^k) = f(v_{ij}^{k-1}) \cup \max\{f(v_{ij}^{k-1}) + j\}$, $k \geq 2, 2 \leq j \leq l_i$. One can easily verify
that $f$ is a dcsl of $G$.

Conversely, if $G$ contains a unique odd cycle then, by Theorem 2.5.13, $G$ cannot admit a dcsl.

**Remark 2.5.5.** In the construction of a unicyclic graph from a tree by adding a backedge, we cannot join the vertices of the tree on the same level or vertices on the levels $i$ and $i + j$ where $j$ is even. Since, otherwise it produce an odd cycle.

**Remark 2.5.6.** It is interesting to note that we can construct as many number of 1-uniform dcsl-graphs by attaching trees of different order to the vertices of a 1-uniform dcsl cycle. We can relabel the vertices of the resulting graph or we can give suitable labels to the vertices of trees such that the new graph is 1-uniform dcsl-graph.

We have seen that paths admit 1-uniform dcsl. We shall now prove that the cartesian product of two paths also admit 1-uniform dcsl.

**Theorem 2.5.26.** The cartesian product $P_m \times P_n$ of two paths $P_m$ and $P_n$ is a dcsl-graph.

**Proof.** The product graph of $P_m$ and $P_n$ is a grid of $m$ rows and $n$ columns. Let $v_{1j}, v_{2j}, \ldots, v_{mj}$, $1 \leq j \leq n$ are the vertices of $m$ rows of the grid $P_m \times P_n$. Take $X = \{1, 2, \ldots, m + n - 1\}$. 
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Define $f : V(P_m \times P_n) \rightarrow 2^X$ by

$$f(v_{11}) = \{1\}$$

$$f(v_{1j}) = f(v_{1j-1}) \cup \{j\}, \quad 1 \leq j \leq n$$

$$f(v_{ij}) = f(v_{i-1j}) \cup \{n + j - 1\}, \quad 1 \leq i \leq m, \quad 1 \leq j \leq n$$

Clearly $f$ is a dcs1 for $P_m \times P_n$ and hence the theorem.

\[\square\]

Figure 2.21: A 1-uniform dcs1 product graph

\textbf{Theorem 2.5.27.} The cartesian product of a star $K_{1,n}$ and a path $P_m$, $K_{1,n} \times P_m$ is a dcs1-graph.

\textit{Proof.} Let $X = \{1, 2, \ldots, m + n\}$ and $V(K_{1,n}) = \{u_i\}, \quad 1 \leq i \leq n + 1$ and $V(P_m) = \{v_j\}, \quad 1 \leq j \leq m + 1$. Define $f : V(K_{1,n} \times P_m) \rightarrow 2^X$ defined by

$$f(u_1v_1) = \{1\};$$
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\[ f(u_1v_2) = \{1, n + 2\}; \]
\[ f(u_1v_j) = \{1, n + 2, n + 3, \ldots, n + j\} \quad 3 \leq j \leq m + 1 \]
\[ f(u_i v_j) = \{1, i, n + 2, n + 3, \ldots, n + j\} \quad 2 \leq i \leq n + 1, \quad 3 \leq j \leq m + 1\}.

Clearly, \( f \) is one-one and \( f \) is a 1-uniform dcsl.

\[ \Box \]

**Theorem 2.5.28.** The cartesian product \( K_{1,n} \times K_{1,m} \) is a dcsl-graph.

**Proof.** Let \( X = \{1, 2, \ldots, m + n + 1\} \) and \( V(K_{1,n}) = \{u_i, \ 1 \leq i \leq n + 1\} \) and \( V(P_m) = \{v_j, \ 1 \leq j \leq m\} \). Let \( u_1 \) is the central vertex of \( K_{1,n} \) and \( v_1 \) is the central vertex of \( K_{1,m} \). A labeling for \( V(K_{1,n} \times K_{m,1}) \) is given in the table 2.22.

<table>
<thead>
<tr>
<th>( v_1 )</th>
<th>( v_2 )</th>
<th>( v_3 )</th>
<th>( v_4 )</th>
<th>( v_5 )</th>
<th>( \ldots )</th>
<th>( v_m )</th>
<th>( v_{m+1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u_1 )</td>
<td>{1}</td>
<td>{1,n+2}</td>
<td>{1,n+3}</td>
<td>{1,n+4}</td>
<td>{1,n+5}</td>
<td>( \ldots )</td>
<td>{1,n+m}</td>
</tr>
<tr>
<td>( u_2 )</td>
<td>{1,2}</td>
<td>{1,2,n+2}</td>
<td>{1,2,n+3}</td>
<td>{1,2,n+4}</td>
<td>{1,2,n+5}</td>
<td>( \ldots )</td>
<td>{1,2,n+m}</td>
</tr>
<tr>
<td>( u_3 )</td>
<td>{1,3}</td>
<td>{1,3,n+2}</td>
<td>{1,3,n+3}</td>
<td>{1,3,n+4}</td>
<td>{1,3,n+5}</td>
<td>( \ldots )</td>
<td>{1,3,n+m}</td>
</tr>
<tr>
<td>( u_4 )</td>
<td>{1,4}</td>
<td>{1,4,n+2}</td>
<td>{1,4,n+3}</td>
<td>{1,4,n+4}</td>
<td>{1,4,n+5}</td>
<td>( \ldots )</td>
<td>{1,4,n+m}</td>
</tr>
<tr>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( u_n )</td>
<td>{1,n}</td>
<td>{1,n,n+2}</td>
<td>{1,n,n+3}</td>
<td>{1,n,n+4}</td>
<td>{1,n,n+5}</td>
<td>( \ldots )</td>
<td>{1,n,n+m}</td>
</tr>
<tr>
<td>( u_{m+1} )</td>
<td>{1,n+1}</td>
<td>{1,n+1,n+2}</td>
<td>{1,n+1,n+3}</td>
<td>{1,n+1,n+4}</td>
<td>{1,n+1,n+5}</td>
<td>( \ldots )</td>
<td>{1,n+1,n+m}</td>
</tr>
</tbody>
</table>

*Figure 2.22: A 1-uniform dcsl of \( K_{1,n} \times K_{1,m} \)*

One can easily check that \( f \) is one-one and \( f \) is a 1-uniform dcsl.  \( \Box \)
Theorem 2.23 is illustrated in Figure 2.23

Remark 2.5.7. It can be easily shown that the product of two 1-uniform dcsl-graphs is again 1-uniform dcsl.

Recall that the minimum cardinality of the underlying set $X$ such that $G$ admits a 1-uniform dcsl is called the 1-uniform dcsl index $\delta_d$ of $G$. Now, we calculate the dcsl index of certain 1-uniform dcsl-graphs

**Proposition 2.5.29.** For any 1-uniform dcsl-graph $G$, $\delta_d(G) \geq \text{diam}(G)$.

**Proof.** $\text{diam}(G) = \max \{d(v_i, v_j), v_i, v_j \in V(G)\}$. Now, if $X_i, X_j$ are subsets of $X$ such that $f(v_i) = X_i$, $f(v_j) = X_j$, then $|X_i \oplus X_j| = d(v_i, v_j)$, only if $|X| \geq d(v_i, v_j), \forall i, j$. Thus, $|X| \geq \text{diam}(G)$. Therefore, $\delta_d(G) \geq \text{diam}(G)$. \hfill $\square$
The natural question under investigation is whether the bound for $\delta_d(G)$ obtained in Proposition 2.5.29 is attainable. The following lemma answers this question; affirmatively.

**Lemma 2.5.30.** $\delta_d(P_n) = n - 1$, $n > 2$

**Proof.** Suppose $P_n$ is 1-uniform dcsl with a set $X$ of cardinality $n - 2$. Then, $|X_1 \oplus X_n| \leq n - 2$, where $f(v_1) = X_1$ and $f(v_n) = X_n$. But, $d(v_1, v_n) = n - 1$, a contradiction. Hence, $\delta_d(P_n) \geq n - 1$. Let $X = \{1, 2, \ldots, n - 1\}$. Consider the labeling $f : V(G) \rightarrow 2^X$ defined by $f(v_1) = \{1\}$, $f(v_2) = \{1, 2\}$, $f(v_3) = \{2\}$ and $f(v_i) = \{2, 3, \ldots, i - 1\}$, $4 \leq i \leq n$. Then, $|f^\oplus(v_1v_2)| = 1$, $|f^\oplus(v_1v_3)| = 2$ and $|f^\oplus(v_2v_3)| = 1$. Now, $f^\oplus(v_iv_j) = f(v_i) \oplus f(v_j) = \{2, 3, \ldots, i - 1\} \oplus \{2, 3, \ldots, j - 1\}$, $(i \leq j) = \{i, i + 1, i + 2, \ldots, j - 1\}$, which implies $|f^\oplus(v_iv_j)| = j - i = d(v_i, v_j)$. Also, $d(v_j, v_i)(i \geq 4) = i - j = |f^\oplus(v_jv_i)| (j = 1, 2, 3)$. Thus, $f$ is a 1-uniform dcsl of $P_n$, which is unique up to the cardinalities of the sets in the set-labeling $f$. □

**Theorem 2.5.31.** $\delta_d(K_{1,n}) = n$.

**Proof.** Suppose $K_{1,n}$ has a 1-uniform dcsl with respect to the ground set $X = \{1, 2, \ldots, n - 1\}$. Let $v_1, v_2, v_3, \ldots, v_{n+1}$ be the vertices of $K_{1,n}$ with $v_1$ as the central vertex.

**Case i:** $|f(v_1)| = 1$. Then, it is necessary that $|f(v_i)| = 2$ for all $i$, $2 \leq i \leq n + 1$ and $f(v_1) \subset f(v_2)$. But, the number of subsets of $X$ with cardinality 2 containing $f(v_1)$ is $n - 2$, which in turn implies one vertex remains unlabeled, a contradiction.
Case ii: \(| f(v_1) | = k, k > 1\). Then, \(| f(v_i) | = k + 1 \) or \( k - 1 \), 
\( 2 \leq i \leq n + 1 \).

(a) If \(| f(v_i) | = k + 1 \forall i\), then \( f(v_1) \subset f(v_i) \forall i \). But, the number of such subsets is less than \( n \), again a contradiction as in Case(i).

(b) If \(| f(v_i) | = k - 1 \), \( k > 2 \forall i\), then, \( f(v_i) \subset f(v_1) \forall i \) and, the number of such subsets is less than \( n \), again a contradiction.

If \(| f(v_i) | = k + 1\), for some \( i \) and \( k - 1 \) for some \( i \) then, some vertices would remain unlabeled, again a contradiction. Hence, we conclude that \( \delta_d(K_{1,n}) = n \).

It is a tedious task to find the dcsl index of an arbitrary tree. We strongly believe that it may be an NP-complete problem. However, we find the 1-uniform dcsl index \( \delta_d \) of the classes of trees with less than 6 vertices and of diameters less than or equal to three. We denote a tree on \( n \) vertices with diameter \( d \) by \( T_n^d \).

**Proposition 2.5.32.** 1-uniform dcsl index \( \delta_d(T_n^d) \leq n - 1 \) for \( n \leq 6 \) and \( d \leq 3 \).

We consider each classes seperately as follows.

1. \( \delta_d(T_2^1) = 1 \).

Let \( v_1 \) and \( v_2 \) are the two vertices of \( T_2^1 \). Let \( X = \{1\} \). Define \( f(v_1) = \emptyset \) and \( f(v_2) = \{1\} \). Then, this is a 1-uniform dcsl with dcsl set \( X \). Therefore \( \delta_d(T_2^1) \leq 1 \). Also by Theorem 2.5.29 \( \delta_d(T_2^1) \geq 1 \). Hence \( \delta_d(T_2^1) = 1 \).
2. \( \delta_d(T^2_3) = 2 \).

Let \( v_1, v_2 \) and \( v_3 \) are the three vertices of \( T^2_3 \). Define \( f(v_1) = \{1\}, f(v_2) = \{1, 2\} \) and \( f(v_3) = \{2\} \). Then, \( f \) is a 1-uniform dcsl with dcsl set \( X = \{1, 2\} \). Therefore \( \delta_d(T^2_3) \leq 2 \). Also by Theorem 2.5.29 \( \delta_d(T^2_3) \geq 2 \). Hence, \( \delta_d(T^2_3) = 2 \).

3. \( \delta_d(T^3_4) = 3 \).

Let \( V(T^3_4) = \{v_1, v_2, v_3, v_4\} \). Let \( v_1 \) and \( v_4 \) are the antipodal vertices of \( T^3_4 \) and \( v_2 \) and \( v_3 \) are the internal vertices. Now, \( d((T^3_4)) = 3 \), implies \( \delta_d(T^3_4) \geq 3 \). Let \( X = \{1, 2, 3\} \). Define \( f : V(T^3_4) \rightarrow 2^X \) defined by, \( f(v_1) = \{1\}, f(v_2) = \{1, 3\}, f(v_3) = \{3\} \) and \( f(v_4) = \{2, 3\} \). Then, \( f \) is a 1-uniform dcsl and hence \( \delta_d(T^3_4) = 3 \).

4. \( \delta_d(T^3_5) = 4 \).

Figure 2.24: 1-uniform dcsl tree, \( T^3_5 \)
Consider, the tree given in Figure 2.24. Since, $T_5^3$ contains $T_4^3$ as an induced subgraph $\delta_d(T_5^3) \geq \delta_d(T_4^3)$. We already proved that $\delta_d(T_4^3) = 3$. Now, consider $x = \{1, 2, 3\}$. There are two different diametral path in $T_5^3$ and hence we cannot label the antipodal vertices with the subsets of $X$ such that the resulting graph is a 1-uniform dcsl-graph. It is easy to see that the labeling given in Figure 2.24 is a 1-uniform dcsl. Hence, we can conclude that $\delta_d(T_5^3) = 4$.

5. $\delta_d(T_6^3) = 5$.

Consider the graphs shown in Figure 2.25. $T_6^3$ is isomorphic to one of these graphs.

The labeling given in Figure 2.25 is a 1-uniform dcsl. Also, if we take $X = \{1, 2, 3, 4\}$, we cannot label any pendant vertex by
$X$, since there are more than two diametral paths. Therefore, 
\[ \delta_d(T_6^3) = 5. \]

**Remark 2.5.8.** We strongly believe that 1-uniform dcsl index of an arbitrary tree of order $n$ is $n - 1$; which we pose as a Conjecture.

**Conjecture 3.** $\delta_d(T) = n - 1$, where $T$ is a tree of order $n$.

**Theorem 2.5.33.** 1-uniform dcsl index of an even cycle $C_n$ is $\frac{n}{2} + 1$.

**Proof.** Let $C_n$ be the even cycle with vertex set \{v_1, v_2, \ldots, v_n\}. Let $X = \{1, 2, \ldots, \frac{n}{2} + 1\}$. Define $f : V(C_n) \to 2^X$ defined by 
\[
\begin{align*}
    f(v_1) &= \{1\}; \\
    f(v_2) &= \{1, 2\}; \\
    f(v_n) &= \{1, 3\}; \\
    f(v_3) &= \{1, 2, 4\}; \\
    f(v_{n-1}) &= \{1, 3, 5\}; \\
    & \vvdots
\end{align*}
\]
\[
\begin{align*}
    f(v_{\frac{n}{2}+1}) &= \{1, 2, \ldots, \frac{n}{2} + 1\}. 
\end{align*}
\]
Then, this is a 1-uniform dcsl. Also, note that diameter of $C_n$ is $\frac{n}{2}$, so that the minimum cardinality of $X$ should necessarily be $\frac{n}{2} + 1$. Thus, 1-uniform dcsl index of an even cycle $C_n$ is $\frac{n}{2} + 1$.

\qed
2.6 *k*-Uniform dcsl-graphs

This section is an attempt to generalize the concept of 1-uniform dcsl-graphs to *k*-uniform dcsl-graphs for an arbitrary value of *k*. A dcsl *f* of a graph *G* = (*V*, *E*) is *k*-uniform if all the constants of proportionality with respect to *f* are equal to *k*, and if *G* admits such a *k*-uniform dcsl for some positive integer *k*, then *G* is a *k*-uniform dcsl-graph.

**Remark 2.6.1.** The constant of proportionality in a dcsl-graph need not be an integer and also need not be uniform. If it is uniform then it should be an integer. For suppose *G* is a *k*-uniform dcsl-graph with the labeling *f*. Let *u* and *v* are any two adjacent vertices of *G*. Then by the definition of *k*-uniform dcsl-graph | *f*(*u*) ⊕ *f*(*v*) | = *k*(*d*(*u*, *v*)) = *k*, since *d*(*u*, *v*) = 1. Thus *k* is necessarily be an integer.

We already proved that paths are *k*-uniform dcsl-graphs for *k* = 1. It is interesting to check whether it is true for higher values of *k*. Here we prove that paths are 2-uniform dcsl-graphs too.

**Theorem 2.6.1.** All paths are 2-dcsl-graphs.

**Proof.** Let *P*ₙ be a path on *n* vertices. Let *X* = {1, 2, ..., 2*n*}. Define *f* : *V*(*P*ₙ) → 2*X* defined by *f*(*v*ᵢ) = {1, 2, ..., 2*i*}, 1 ≤ *i* ≤ *n*. Then, for all *n* ≥ *j* > *i* ≥ 1,

\[ f^\oplus({v_i}{v_j}) = \{1, 2, \ldots, 2j\} - \{1, 2, \ldots, 2i\} = \{2i + 1, 2i + 2, \ldots, 2j\}. \]

Therefore, | *f*^\oplus\(^{\oplus}\)({v_i}{v_j}) | = 2*j* − 2*i* = 2(*j* − *i*) = 2(*d*(*v_i*, *v_j*)).

Thus, paths are 2-dcsl-graphs. □
Figure 2.26 gives a 2-dcsl of $P_4$.

![Diagram of 2-DCSL of $P_4$]

Figure 2.26: A 2-dcsl of $P_4$

Now, we prove that paths are arbitrarily $k$-uniform dcsl in the sense that paths are $k$-uniform dcsl for all integer values of $k$.

**Theorem 2.6.2.** Paths are $k$-uniform dcsl for all integer values of $k$

**Proof.** Let $P_n$ be a path on $n$ vertices $v_1, v_2, v_3, \ldots, v_n$. Let $X = \{1, 2, 3, \ldots, (n-1)k\}$. Define $f : V(P_n) \to 2^X$ defined by

\[
 f(v_1) = \emptyset, \\
 f(v_i) = \{1, 2, 3, \ldots, (i-1)k\}. 
\]

Then,

\[
| f(v_1) \oplus f(v_i) | = | \{1, 2, 3, \ldots, (i-1)k\} | \\
= (i-1)k = k(d(v_1, v_i)) \text{ for } 2 \leq i \leq n 
\]

and

\[
| f(v_i) \oplus f(v_j) | = | \{1, 2, 3, \ldots, (i-1)k\} \oplus \{1, 2, 3, \ldots, (j-1)k\} | \\
= (j-1)k - (i-1)k = (j-i)k \\
= k(d(v_i, v_j)) \text{ for } 2 \leq i < j \leq n. 
\]

Thus $P_n$ is $k$-uniform dcsl-graph for all integer values of $k$. \qed

**Remark 2.6.2.** In Theorem 2.5.8, we proved that stars $K_{1,n}$ with $n$ spokes admit 1-uniform dcsl. In the following theorem we prove that stars are arbitrarily $k$-uniform dcsl-graphs.
Theorem 2.6.3. \( K_{1,n} \) with \( n \) spokes admit \( k \)-uniform dcsl.

Proof. Let \( v_0, v_1, \ldots, v_n \) are the \( n + 1 \) vertices of \( K_{1,n} \) with \( v_0 \) as the central vertex. Let \( X = \{1, 2, \ldots, nk\} \)
Define \( f : V(K_{1,n}) \to 2^X \) defined by
\[
f(v_0) = \emptyset, \\
f(v_1) \equiv \{1, 2, \ldots, k\},
\]
and
\[
f(v_i) = \{(i - 1)k + 1, (i - 1)k + 2, \ldots, ik\}, \ 2 \leq i \leq n.
\]
Then,
\[
|f(v_0) \oplus f(v_i)| = k = k(d(v_0, v_i))
\]
and,
\[
|f(v_i) \oplus f(v_j)| = 2k = k(d(v_i, v_j)).
\]
Hence, stars are \( k \)-uniform dcsl-graphs.

Remark 2.6.3. Theorem 2.5.12 asserts that cycle \( C_n \) (\( n \geq 3 \)) is \( 1 \)-uniform dcsl if and only if \( n \) is even. The next two theorems, Theorem 2.6.4 and Theorem 2.6.5 establish that even cycles are arbitrarily \( k \)-uniform but, odd cycles admit \( k \)-uniform dcsl only when \( k \) is even.

Theorem 2.6.4. Every even cycle is \( k \)-uniform dcsl-graph.

Proof. Consider cycle \( C_n \), with \( n \) vertices where \( n \) is even. Let \( V(C_n) = \{v_1, v_2, v_3, \ldots, v_n\} \) and let \( X = \{1, 2, 3, \ldots, rk\} \) where \( r \) is the radius of the cycle.
Define \( f : V(C_n) \to 2^X \) defined by,
\[
f(v_1) = \emptyset; \\
f(v_i) = \{1, 2, 3, \ldots, (i - 1)k\}, \ 2 \leq i \leq \frac{n}{2} + 1;
\]
\[ f(v_{\frac{n}{2}+j}) = \{(j-1)k + 1, (j-1)k + 2, \ldots, rk\}, 2 \leq j \leq \frac{n}{2}. \]

Then \( |f(v_1) \oplus f(v_i)| = k(i-1) = k(d(v_1, v_i)), 2 \leq i \leq \frac{n}{2} + 1. \)

When \( i < j, \)
\[ |f(v_i) \oplus f(v_{\frac{n}{2}+j})| = |\{1, 2, 3, \ldots, (i-1)k\} \oplus \{(j-1)k + 1, (j-1)k + 2, \ldots, rk\}| = (j-1)k - (i-1)k = (j-i)k = k(d(v_i, v_{\frac{n}{2}+j})) \]
and, when \( i > j, \)
\[ |f(v_i) \oplus f(v_{\frac{n}{2}+j})| = (j-1)k + (r-(i-1))k = jk + rk - ik = (j-i+r)k = k(d(v_i, v_{\frac{n}{2}+j})). \]

Also, when \( i = j, \)
\[ |f(v_i) \oplus f(v_{\frac{n}{2}+j})| = |\{1, 2, 3, \ldots, (i-1)k\} \oplus \{(i-1)k+1, (i-1)k+2, \ldots, rk\}| = |\{1, 2, 3, \ldots, rk\}| = rk = k(d(v_i, v_{\frac{n}{2}+j})). \]

Thus, \( f \) is a \( k \)-uniform dcsl-graph.

\[ \square \]

**Theorem 2.6.5.** Odd cycles are \( k \)-uniform dcsl, then \( k \) is even.

**Proof.** Let \( C_n \) be an odd cycle with \( n \) vertices \( v_1, v_2, v_3, \ldots, v_n \). Suppose if possible \( C_n \) admits a \( k \)-uniform dcsl \( f \) with the dcsl set \( X \). Let \( f(v_i) = A_i, A_i \subseteq X, 1 \leq i \leq n. \)

**Case 1:** \( A_i = \emptyset \) for some \( i. \)

Without lose of generality, assume that \( A_1 = \emptyset \). Then, \( |A_2| = |A_n| = k, |A_3| = |A_{n-1}| = 2k, \ldots, |A_{\frac{n}{2}}| = |A_{\frac{n}{2}+1}| = \left\lceil \frac{n}{2} \right\rceil \cdot k \).

Suppose \( A_{\frac{n}{2}} \) and \( A_{\frac{n}{2}+1} \) have \( m \) entries in common. Then,
\[ |A_{\frac{n}{2}}| + |A_{\frac{n}{2}+1}| = k \Rightarrow \left\lceil \frac{n}{2} \right\rceil \cdot k - m + \left\lceil \frac{n}{2} \right\rceil \cdot k - m = k \Rightarrow 2 \cdot \left\lceil \frac{n}{2} \right\rceil \cdot k - 2m = k \Rightarrow (2 \cdot \left\lceil \frac{n}{2} \right\rceil - 1)k = 2m. \]
This is possible only when
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$k$ is even, since $2\lceil \frac{n}{2} \rceil - 1$ is odd. If $A_{\lceil \frac{n}{2} \rceil}$ and $A_{\lceil \frac{n}{2} \rceil+1}$ have no element in common then,

$$k = |A_{\lceil \frac{n}{2} \rceil} \oplus A_{\lceil \frac{n}{2} \rceil+1}| = \lceil \frac{n}{2} \rceil.k + \lceil \frac{n}{2} \rceil.k.$$ But, then $\lceil \frac{n}{2} \rceil.k = k \Rightarrow 2\lceil \frac{n}{2} \rceil = 1 \Rightarrow \lceil \frac{n}{2} \rceil = \frac{1}{2}$, which is absurd. Hence empty set cannot be assigned to any of the vertices of $C_n$.

**Case 2:** $A_i \neq \emptyset$ for all $i$, $1 \leq i \leq n$. By our assumption, $|A_i \oplus A_{i+1}| = k$, $1 \leq i \leq n - 1$ and, $|A_i \oplus A_{i+2}| = 2k$, $1 \leq i \leq n - 2$.

Suppose that, $|A_i| = C_i$ and, $A_i$ and $A_j$ have $l_i^j$ elements in common where, $1 \leq i < j \leq n - 1$. Then, by our assumption,

$$|A_1 \oplus A_3| = C_1 + C_3 - 2l_1^3 = 2k;$$

$$|A_3 \oplus A_5| = C_3 + C_5 - 2l_3^5 = 2k;$$

$$|A_5 \oplus A_7| = C_5 + C_7 - 2l_5^7 = 2k;$$

$$\ldots$$

$$\ldots$$

$$|A_{n-2} \oplus A_n| = C_{n-2} + C_n - 2l_{n-2}^n = 2k.$$ From the above equations, we get, $C_i + C_{i+2}$ is even, $1 \leq i \leq n - 2$, $i$ odd.

This is possible only when both $C_i$ and $C_{i+2}$ either both even or both odd. That is, either $C_{2i-1}$ is odd for all $i$, $1 \leq i \leq \frac{n+1}{2}$ or $C_{2i-1}$ is even for all $i$, $1 \leq i \leq \frac{n+1}{2}$.

**Subcase 1:** $C_{2i-1}$ is odd for all $i$, $1 \leq i \leq \frac{n+1}{2}$

In this case, $|A_1 \oplus A_n| = C_1 + C_n - 2l_1^n = k$. Now, since, $C_1 + C_n$ is even, $k$ is also an even number.

**Subcase 2:** $C_{2i-1}$ is even for all $i$, $1 \leq i \leq \frac{n+1}{2}$. 
We have, \(| A_1 \oplus A_n | = C_1 + C_n - 2l_1^n = k\). In this case also \(k\) is an even number since, \(C_1 + C_n\) is even. Thus the theorem is established.

**Corollary 2.6.6.** A non-bipartite graph \(G\) is \(k\)-uniform dcsl \((k > 1)\), then \(k\) is even.

**Proof.** Let \(G\) be a non-bipartite graph which admits a \(k\)-uniform dcsl. Then by Theorem 2.6.5 \(k\) should be an even number. Thus a non-bipartite graph \(G\) is \(k\)-uniform dcsl then, \(k\) is even.

**Remark 2.6.4.** In Theorem 2.5.2, we proved that the complete graph \(K_n\) admits 1-uniform dcsl if and only if \(n \in \{1, 2\}\). Hence it is interesting to find the values of \(k\) for which complete graph \(K_n\) admits \(k\)-uniform dcsl. The following theorem characterizes complete graph \(K_n(n > 3)\) which admits \(k\)-uniform dcsl.

**Theorem 2.6.7.** The complete graph \(K_n(n > 3)\) is \(k\)-uniform dcsl if and only if \(k\) is even.

**Proof.** Let \(K_n\) be the complete graph on \(n\) vertices \((n > 3)\), \(v_1, v_2, \ldots, v_n\). Let \(k\) be any arbitrary even number and, let \(X = \{1, 2, 3, \ldots, n^k/2\}\). Define \(f : V(K_n) \to 2^X\) defined by

\[ f(v_1) = \{1, 2, \ldots, \frac{k}{2}\} \text{ and, } f(v_i) = \{(i - 1)\frac{k}{2} + j, 1 \leq j \leq \frac{k}{2}\}, 2 \leq i \leq n. \]

Clearly, \(f\) is injective. Also, for any two arbitrarily chosen distinct vertices \(v_x, v_y\) of \(K_n\), we have, \(| f_\oplus(v_x, v_y) | = |f(v_x) \cup f(v_y) - (f(v_x) \cap f(v_y))| = |f(v_x)| + f(v_y)| = \frac{k}{2} + \frac{k}{2} = k\). Hence, \(f\) is a \(k\)-uniform-dcsl of \(K_n\).
Converse of the theorem follows from Corrolary 2.6.6, since $K_n$ is a non-bipartite graph.

Figure 2.27 gives the dcs1 of $K_4$ with $k = 2$.

![Figure 2.27: A 2-dcs1 of $K_4$](image)

**Remark 2.6.5.** Deletion of an edge from $k$-dcs1 complete graph does not violate the $k$-dcs1 property, it may be $k$-dcs1 with a distinct value of $k$ and with a different labeling.
2.7 References


16. F. Harary, Graph Theory, Addison Wesley, Reading Massachusetts, 1969.


