Chapter 5

UNIQUENESS AND VALUE DISTRIBUTION OF CERTAIN DIFFERENCE POLYNOMIALS
5.1 UNIQUENESS OF ENTIRE FUNCTIONS OF CERTAIN DIFFERENCE POLYNOMIALS SHARING A SMALL FUNCTION

5.1.1 INTRODUCTION AND RESULTS

In this section, we study the uniqueness problems of difference polynomials of entire functions sharing a small function $\alpha$, using the concept of weakly weighted sharing and relaxed weighted sharing. The results obtained extends and generalizes the results due to Pulak Sahoo and Himadri Karmakar [52].

In recent years there has been an increasing interest in studying difference equations in the complex plane.

In 2014, C. Meng [49] proved the following results using the concept of weakly weighted sharing and relaxed weighted sharing.

**Theorem 5.1.A.** Let $f$ and $g$ be two transcendental entire functions of finite order and $\alpha(\neq 0, \infty)$ be a small function with respect to both $f$ and $g$. Suppose that $c$ is a non-zero complex constant and $n \geq 7$ is an integer. If $f^n(z) (f(z) - 1) f(z + c)$ and $g^n(z) (g(z) - 1) g(z + c)$ share "$(\alpha, 2)$", then $f \equiv g$.

**Theorem 5.1.B.** Let $f$ and $g$ be two transcendental entire functions of finite order and $\alpha(\neq 0, \infty)$ be a small function with respect to both $f$ and $g$. Suppose that $c$ is a non-zero complex constant and $n \geq 10$ is an integer. If $f^n(z) (f(z) - 1) f(z + c)$ and $g^n(z) (g(z) - 1) g(z + c)$ share $(\alpha, 2^*)$, then $f \equiv g$.

**Theorem 5.1.C.** Let $f$ and $g$ be two transcendental entire functions of finite order and $\alpha(\neq 0, \infty)$ be a small function with respect to both $f$ and $g$. Suppose that $c$ is a non-zero complex constant and $n \geq 16$ is an integer. If $E_2(\alpha(z), f^n(z) (f(z) - 1) f(z + c)) = E_2(\alpha(z), g^n(z) (g(z) - 1) g(z + c))$, then $f \equiv g$. 

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Recently, P. Sahoo [51] generalized the above theorems and obtained the following results.

**Theorem 5.1.D.** Let \( f \) and \( g \) be two transcendental entire functions of finite order and \( \alpha(\neq 0, \infty) \) be a small function with respect to both \( f \) and \( g \). Suppose that \( c \) is a non-zero complex constant, \( n \) and \( m \) (\( \geq 2 \)) are integers satisfying \( n + m \geq 10 \). If \( f^n(z)(f(z) - 1)^m f(z + c) \) and \( g^n(z)(g(z) - 1)^m g(z + c) \) share \( \alpha(2) \), then either \( f \equiv g \) or \( f \) and \( g \) satisfy the algebraic equation \( R(f, g) = 0 \), where \( R(f, g) \) is given by \( R(w_1, w_2) = w_1^n(w_1 - 1)^m w_1(z + c) - w_2^n(w_2 - 1)^m w_2(z + c) \).

**Theorem 5.1.E.** Let \( f \) and \( g \) be two transcendental entire functions of finite order and \( \alpha(\neq 0, \infty) \) be a small function with respect to both \( f \) and \( g \). Suppose that \( c \) is a non-zero complex constant, \( n \) and \( m \) (\( \geq 2 \)) are integers satisfying \( n + m \geq 13 \). If \( f^n(z)(f(z) - 1)^m f(z + c) \) and \( g^n(z)(g(z) - 1)^m g(z + c) \) share \( \alpha(2)^* \), then the conclusions of Theorem 5.1.D hold.

**Theorem 5.1.F.** Let \( f \) and \( g \) be two transcendental entire functions of finite order and \( \alpha(\neq 0, \infty) \) be a small function with respect to both \( f \) and \( g \). Suppose that \( c \) is a non-zero complex constant, \( n \) and \( m \) (\( \geq 2 \)) are integers satisfying \( n + m \geq 19 \). If \( \overline{E}_2(\alpha(z), f^n(z)(f(z) - 1)^m f(z + c)) = \overline{E}_2(\alpha(z), g^n(z)(g(z) - 1)^m g(z + c)) \), then the conclusions of Theorem 5.1.D hold.

Recently, P. Sahoo and H. Karmakar [52] extended the above theorems and proved the following results.

**Theorem 5.1.G.** Let \( f \) and \( g \) be two transcendental entire functions of finite order and \( \alpha(\neq 0) \) be a small function of both \( f \) and \( g \). Suppose that \( c \) is a non-zero complex constant, \( n(\geq 1) \), \( m(\geq 1) \) and \( k(\geq 0) \) are integers satisfying \( n \geq 2k + m + 6 \) when \( m \leq k + 1 \) and \( n \geq 4k - m + 10 \) when \( m > k + 1 \). If \( f^n(z)(f(z) - 1)^m f(z + c))^{(k)} \) and \( g^n(z)(g(z) - 1)^m g(z + c))^{(k)} \) share \( \alpha(2) \), then either \( f \equiv g \) or \( f \) and \( g \) satisfy the algebraic equation \( R(f, g) = 0 \), where \( R(f, g) \) is given by
\[
R(w_1, w_2) = w_1^n(w_1 - 1)^m w_1(z + c) - w_2^n(w_2 - 1)^m w_2(z + c).
\]
Theorem 5.1.H. Let $f$ and $g$ be two transcendental entire functions of finite order and $\alpha(\neq 0)$ be a small function of both $f$ and $g$. Suppose that $c$ is a non-zero complex constant, $n(\geq 1)$, $m(\geq 1)$ and $k \ (\geq 0)$ are integers satisfying $n \geq 3k + 2m + 8$ when $m \leq k + 1$ and $n \geq 6k - m + 13$ when $m > k + 1$. If $(f^n(z)(f(z) - 1)^m f(z + c))^{(k)}$ and $(g^n(z)(g(z) - 1)^m g(z + c))^{(k)}$ share $(\alpha, 2)^*$, then the conclusions of Theorem 5.1.G hold.

Theorem 5.1.I. Let $f$ and $g$ be two transcendental entire functions of finite order and $\alpha(\neq 0)$ be a small function of both $f$ and $g$. Suppose that $c$ is a non-zero complex constant, $n(\geq 1)$, $m(\geq 1)$ and $k \ (\geq 0)$ are integers satisfying $n \geq 5k + 4m + 12$ when $m \leq k + 1$ and $n \geq 10k - m + 19$ when $m > k + 1$. If $\overline{E}_2(\alpha(z), (f^n(z)(f(z) - 1)^m f(z + c))^{(k)}) = \overline{E}_2(\alpha(z), (g^n(z)(g(z) - 1)^m g(z + c))^{(k)})$, then the conclusions of Theorem 5.1.G hold.

In this section, we assume $c_j \in \mathbb{C}\{0\} \ (j = 1, 2, \ldots, d)$ are constants, $n(\geq 1)$, $m(\geq 1)$ and $k \ (\geq 0)$ are integers, $s_j(j = 1, 2, \ldots, d)$ are non-negative integers, $\lambda = \sum_{j=1}^{d} s_j = s_1 + s_2 + \cdots + s_d$. With these assumptions, we study the uniqueness problems of difference polynomials sharing a small function of more general form

$$F(z) = (f(z)^n(f(z) - 1)^m \prod_{j=1}^{d} f(z + c_j)^{s_j})^{(k)}$$

and hence extend and generalize the results obtained by P. Sahoo and H. Karmakar [52].

5.1.2 LEMMAS

Let $H$ be defined as in (2.3.1).

Lemma 5.1.1. ([14]) Let $f$ be meromorphic function of order $\rho(f) < \infty$ and let $c$ be a non-zero complex constant. Then, for each $\varepsilon > 0$, we have

$$T(r, f(z + c)) = T(r, f) + O\left\{r^{\rho(f) - 1 + \varepsilon}\right\} + O\{\log r\}.$$  

Lemma 5.1.2. ([19]) Let $f$ be meromorphic function of finite order and $c$ be a non-zero
complex constant. Then,

\[ m \left( r, \frac{f(z + c)}{f(z)} \right) + m \left( r, \frac{f(z)}{f(z + c)} \right) = O\{r^{\rho(f) - 1 + \varepsilon}\}. \]

**Lemma 5.1.3.** Let \( f \) be an entire function of order \( \rho(f) < \infty \) and \( F(z) = f^n(z)(f(z) - 1)^m \prod_{j=1}^{d} f(z + c_j)^{s_j} \) where \( n \geq 1 \), \( m \geq 1 \) and \( k \geq 0 \) are integers. Then,

\[ T(r, F) = (n + m + \lambda)T(r, f) + O\{r^{\rho(f) - 1 + \varepsilon}\} + S(r, f), \]

for all \( r \) outside of a set of finite linear measure where \( \lambda = s_1 + s_2 + \ldots + s_d = \sum_{j=1}^{d} s_j. \)

**Proof.** Since \( f \) is an entire function of finite order, from Lemma 5.1.2 and the standard Valiron-Mohon’ko theorem, we have

\[
(n + m + \lambda)T(r, f(z)) = T(r, f^{n+\lambda}(z)(f(z) - 1)^m) + S(r, f)
\]

\[
= m \left( r, \frac{f^{n+\lambda}(z)(f(z) - 1)^m}{F(z)} \right) + m(r, F(z)) + S(r, f)
\]

\[
\leq m \left( r, \frac{f^{n+\lambda}(z)(f(z) - 1)^m}{F(z)} \right) + m(r, F(z)) + S(r, f)
\]

\[
\leq T(r, F(z)) + O\{r^{\rho(f) - 1 + \varepsilon}\} + S(r, f). \tag{5.1.1}
\]

On the other hand, from Lemma 5.1.1, we have

\[
T(r, F(z)) \leq m(r, f^n(z)) + m(r, (f(z) - 1)^m) + m \left( r, f^{\lambda}(z) \cdot \prod_{j=1}^{d} \frac{f(z + c_j)^{s_j}}{f(z)^{s_j}} \right) + S(r, f)
\]

\[
\leq (n + m) m(r, f(z)) + \lambda m(r, f(z)) + \sum_{j=1}^{d} s_j \cdot m \left( r, \frac{f(z + c_j)}{f(z)} \right) + S(r, f)
\]

\[
\leq (n + m + \lambda) m(r, f(z)) + O\{r^{\rho(f) - 1 + \varepsilon}\} + S(r, f)
\]

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\[ \leq (n + m + \lambda) T(r, f(z)) + O\{r^{\rho(f)-1+\varepsilon}\} + S(r, f). \]  
(5.1.2)

From (5.1.1) and (5.1.2), we can prove this lemma easily. \( \square \)

**Lemma 5.1.4.** Let \( f \) and \( g \) be entire functions, \( n(\geq 1), m(\geq 1) \) and \( k(\geq 0) \) be integers, and let

\[ F(z) = \left( f^n(z)(f(z) - 1)^m \prod_{j=1}^{d} (f(z + c_j)^{s_j}) \right)^{(k)}, \quad G(z) = \left( g^n(z)(g(z) - 1)^m \prod_{j=1}^{d} g(z + c_j)^{s_j} \right)^{(k)}. \]

If there exists the non-zero constants \( b_1 \) and \( b_2 \) such that \( \overline{N}(r, b_1; F) = \overline{N}(r, 0; G) \) and \( \overline{N}(r, b_2; G) = \overline{N}(r, 0; F) \), then \( n \leq 2k + m + \lambda + 2 \) when \( m \leq k + 1 \) and \( n \leq 4k - m + \lambda + 4 \) when \( m > k + 1 \).

**Proof.** Let \( F_1(z) = f^n(z)(f(z) - 1)^m \prod_{j=1}^{d} (f(z + c_j)^{s_j}) \) and \( G_1(z) = g^n(z)(g(z) - 1)^m \prod_{j=1}^{d} g(z + c_j)^{s_j} \).

From Lemma 5.1.3, we have

\[ T(r, F_1) = (n + m + \lambda) T(r, f) + O\{r^{\rho(f)-1+\varepsilon}\} + S(r, f), \]  
(5.1.3)

\[ T(r, G_1) = (n + m + \lambda) T(r, g) + O\{r^{\rho(g)-1+\varepsilon}\} + S(r, g). \]  
(5.1.4)

By second fundamental theorem and by the hypothesis, we have

\[ T(r, F) \leq \overline{N}(r, 0; F) + \overline{N}(r, c_1; F) + S(r, F) \]
\[ \leq \overline{N}(r, 0; F') + \overline{N}(r, 0; G) + S(r, F'). \]  
(5.1.5)

Using (2.3.2), (2.3.3), (5.1.3) and (5.1.5), we have

\[ (n + m + \lambda) T(r, f) \leq T(r, F) - \overline{N}(r, 0; F) + N_{k+1}(r, 0; F_1) + S(r, f) \]
\[ \leq \overline{N}(r, 0; G) + N_{k+1}(r, 0; F_1) + S(r, f) \]

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\[ \leq N_{k+1}(r, 0; F_1) + \overline{N}_{k+1}(r, 0; G_1) + S(r, f) + S(r, g). \]  

(5.1.6)

When \( m \leq k + 1 \), using (5.1.6) and Lemma 5.1.1, we see that

\[ (n + m + \lambda) T(r, f) \leq (k + m + \lambda + 1) (T(r, f) + T(r, g)) + O\{r^{\rho(f)-1+\varepsilon}\} \]
\[ + O\{r^{\rho(g)-1+\varepsilon}\} + S(r, f) + S(r, g). \]  

(5.1.7)

Similarly,

\[ (n + m + \lambda) T(r, g) \leq (k + m + \lambda + 1) (T(r, f) + T(r, g)) + O\{r^{\rho(f)-1+\varepsilon}\} \]
\[ + O\{r^{\rho(g)-1+\varepsilon}\} + S(r, f) + S(r, g). \]  

(5.1.8)

From (5.1.7) and (5.1.8), we have

\[ (n - 2k - m - \lambda - 2) (T(r, f) + T(r, g)) \leq O\{r^{\rho(f)-1+\varepsilon}\} + O\{r^{\rho(g)-1+\varepsilon}\} \]
\[ + S(r, f) + S(r, g), \]

which gives \( n \leq 2k + m + \lambda + 2 \).

When \( m > k + 1 \), using (5.1.6) and Lemma 5.1.1, we have

\[ (n + m + \lambda) T(r, f) \leq (2k + \lambda + 2) (T(r, f) + T(r, g)) + O\{r^{\rho(f)-1+\varepsilon}\} \]
\[ + O\{r^{\rho(g)-1+\varepsilon}\} + S(r, f) + S(r, g). \]  

(5.1.9)

Similarly,

\[ (n + m + \lambda) T(r, g) \leq (2k + \lambda + 2) (T(r, f) + T(r, g)) + O\{r^{\rho(f)-1+\varepsilon}\} \]
\[ + O\{r^{\rho(g)-1+\varepsilon}\} + S(r, f) + S(r, g). \]  

(5.1.10)
From (5.1.9) and (5.1.10), we have

\[(n - 4k + m - \lambda - 4) (T(r, f) + T(r, g)) \leq O\{r^{\rho(f)-1+\varepsilon}\} + O\{r^{\rho(g)-1+\varepsilon}\} + S(r, f) + S(r, g),\]

which gives \(n \leq 4k - m + \lambda + 4\). This proves the lemma. \(\square\)

**Lemma 5.1.5.** ([52]) Let \(F\) and \(G\) be non-constant entire functions and \(p \geq 2\) be an integer. If \(\overline{E}_p(1, F) = \overline{E}_p(1, G)\) and \(H \neq 0\), then

\[T(r, F) \leq N_2(r, 0; F) + N_2(r, 0; G) + 2 \overline{N}(r, 0; F) + \overline{N}(r, 0; G) + S(r, F) + S(r, G),\]

and the same inequality hold for \(T(r, G)\).

### 5.1.3 STATEMENT AND PROOF OF MAIN RESULTS

**Theorem 5.1.1.** Let \(f\) and \(g\) be two transcendental entire functions of finite order and \(\alpha(\neq 0)\) be a small function of both \(f\) and \(g\). Let \(c_j\) (\(j = 1, 2, \ldots, d\)) be complex constants, \(s_j\) (\(j = 1, 2, \ldots, d\)) be non-negative integers. Suppose \(n \geq 1\), \(m \geq 1\) and \(k \geq 0\) are integers satisfying \(n \geq 2k + m + \lambda + 5\) when \(m \leq k + 1\) and \(n \geq 4k - m + \lambda + 9\) when \(m > k + 1\). If \((f^n(z)(f(z)-1)^m \prod_{j=1}^{d} f(z+c_j)^{s_j})^{(k)}\) and \((g^n(z)(g(z)-1)^m \prod_{j=1}^{d} g(z+c_j)^{s_j})^{(k)}\) share \("(\alpha, 2)"\), then either \(f \equiv g\) or \(f\) and \(g\) satisfy the algebraic equation \(R(f, g) = 0\), where \(R(f, g)\) is given by \(R(w_1, w_2) = w_1^n(w_1 - 1)^m \prod_{j=1}^{d} w_1(z+c_j)^{s_j} - w_2^n(w_2 - 1)^m \prod_{j=1}^{d} w_2(z+c_j)^{s_j}\).

**Proof.** Let \(F = \frac{F_1^{(k)}}{\alpha}\) and \(G = \frac{G_1^{(k)}}{\alpha}\) where

\[F_1(z) = f^n(z)(f(z)-1)^m \prod_{j=1}^{d} f(z+c_j)^{s_j}\]
\[G_1(z) = g^n(z)(g(z)-1)^m \prod_{j=1}^{d} g(z+c_j)^{s_j}.\]

Then \(F\) and \(G\) are transcendental meromorphic functions that share \("(1, 2)"\) except the zeros and poles of \(\alpha(z)\).

Suppose that \(H \neq 0\).
Using (2.3.2), (5.1.3) and Lemma 5.1.3, we have

\[ N_2(r, 0; F) \leq N_2(r, 0; F_1^{(k)}) + S(r, f) \]
\[ \leq T(r, F_1^{(k)}) - (n + m + \lambda)T(r, f) + N_{k+2}(r, 0; F_1) + S(r, f) \]
\[ \leq T(r, F) - (n + m + \lambda)T(r, f) + N_{k+2}(r, 0; F_1) + S(r, f). \]

From this, we get

\[ (n + m + \lambda)T(r, f) \leq T(r, F) - N_2(r, 0; F) + N_{k+2}(r, 0; F_1) + S(r, f). \] (5.1.11)

Also by (2.3.3), we obtain

\[ N_2(r, 0; F) \leq N_2(r, 0; F_1^{(k)}) + S(r, f) \]
\[ \leq N_{k+2}(r, 0; F_1) + S(r, f). \] (5.1.12)

Similarly,

\[ N_2(r, 0; G) \leq N_{k+2}(r, 0; G_1) + S(r, g). \] (5.1.13)

Using (5.1.13) and Lemma 2.3.5 in (5.1.11), we have

\[ (n + m + \lambda) T(r, f) \leq N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + N_{k+2}(r, 0; F_1) \]
\[ + S(r, f) + S(r, g) \]
\[ \leq N_{k+2}(r, 0; F_1) + N_{k+2}(r, 0; G_1) + S(r, f) + S(r, g). \] (5.1.14)

Suppose that \( m \leq k + 1 \), then from (5.1.14), we have

\[ (n + m + \lambda) T(r, f) \leq (k + m + \lambda + 2) (T(r, f) + T(r, g)) + O\{r^{\rho(f)-1+\varepsilon}\} \]
\[ + O\{r^{\rho(g)-1+\varepsilon}\} + S(r, f) + S(r, g). \] (5.1.15)
Similarly,

\[
(n + m + \lambda) T(r, g) \leq (k + m + \lambda + 2) (T(r, f) + T(r, g)) + O\{r^\rho(f)^{-1+\varepsilon}\} + O\{r^\rho(g)^{-1+\varepsilon}\} + S(r, f) + S(r, g).
\]

From (5.1.15) and (5.1.16), we have

\[
(n-2k-m-\lambda-4) (T(r, f) + T(r, g)) \leq O\{r^\rho(f)^{-1+\varepsilon}\} + O\{r^\rho(g)^{-1+\varepsilon}\} + S(r, f) + S(r, g),
\]

which contradicts the assumption that \(n \geq 2k + m + \lambda + 5\).

Next, assume that \(m > k + 1\). From (5.1.14), we have

\[
(n + m + \lambda) T(r, f) \leq (2k + \lambda + 4) (T(r, f) + T(r, g)) + O\{r^\rho(f)^{-1+\varepsilon}\} + O\{r^\rho(g)^{-1+\varepsilon}\} + S(r, f) + S(r, g).
\]

Similarly,

\[
(n + m + \lambda) T(r, g) \leq (2k + \lambda + 4) (T(r, f) + T(r, g)) + O\{r^\rho(f)^{-1+\varepsilon}\} + O\{r^\rho(g)^{-1+\varepsilon}\} + S(r, f) + S(r, g).
\]

From (5.1.17) and (5.1.18), we have

\[
(n+m-4k-\lambda-8) (T(r, f) + T(r, g)) \leq O\{r^\rho(f)^{-1+\varepsilon}\} + O\{r^\rho(g)^{-1+\varepsilon}\} + S(r, f) + S(r, g),
\]

a contradiction, since \(n \geq 4k - m + \lambda + 9\). Therefore, we have \(H = 0\).

\[
\Rightarrow \left( \frac{F''}{F'} - \frac{2F'^2}{F - 1} \right) - \left( \frac{G''}{G'} - \frac{2G'^2}{G - 1} \right) = 0.
\]
Integrating twice, we get
\[
\frac{1}{F - 1} = \frac{A}{G - 1} + B, \tag{5.1.19}
\]

From (5.1.19), \( F \) and \( G \) share \( 1 \) CM and hence they share \( "(1, 2)" \). Therefore \( n \geq 2k + m + \lambda + 5 \) if \( m \leq k + 1 \) and \( n \geq 4k - m + \lambda + 9 \) if \( m > k + 1 \).

Next, we discuss the following three cases.

**Case 1.** Suppose that \( B \neq 0 \) and \( A = B \), then from (5.1.19), we have
\[
\frac{1}{F - 1} = \frac{BG}{G - 1}. \tag{5.1.20}
\]

If \( B = -1 \), then from (5.1.20), we have \( FG = 1 \). Then
\[
(f^n(z)(f(z) - 1)^m \prod_{j=1}^{d} f(z + c_j)^{s_j})^{(k)} \cdot (g^n(z)(g(z) - 1)^m \prod_{j=1}^{d} g(z + c_j)^{s_j})^{(k)} = \alpha^2.
\]

It follows that \( N(r, 0; f) = S(r, f) \) and \( N(r, 1; f) = S(r, f) \). Thus, we have
\[
\delta(0, f) + \delta(1, f) + \delta(\infty, f) = 3,
\]

which is not possible.

If \( B \neq -1 \), then from (5.1.20), we have \( \frac{1}{F} = \frac{BG}{(1 + B)G - 1} \), hence \( N \left( r, \frac{1}{1 + B}; G \right) = \overline{N}(r, 0; F) \).

Using (2.3.2), (2.3.3), (5.1.4) and the second fundamental theorem of Nevanlinna, we deduce that
\[
T(r, G) \leq \overline{N}(r, 0; G) + \overline{N} \left( r, \frac{1}{1 + B}; G \right) + \overline{N}(r, \infty; G) + S(r, G)
\]
\[
\leq \overline{N}(r, 0; F) + \overline{N}(r, 0; G) + \overline{N}(r, \infty; G) + S(r, G)
\]

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\[\leq N_{k+1}(r, 0; F_1) + T(r, G) + N_{k+1}(r, 0; G_1) - (n + m + \lambda)T(r, g) + S(r, g).\]

(5.1.21)

If \(m \leq k + 1\), then from (5.1.21), we have

\[(n + m + \lambda)T(r, g) \leq (k + m + \lambda + 1)(T(r, f) + T(r, g)) + O\{r^{\rho(f)-1+\varepsilon}\} + O\{r^{\rho(g)-1+\varepsilon}\} + S(r, f) + S(r, g).\]

Hence,

\[(n-2k-m-\lambda-2)(T(r, f) + T(r, g)) \leq O\{r^{\rho(f)-1+\varepsilon}\} + O\{r^{\rho(g)-1+\varepsilon}\} + S(r, f) + S(r, g),\]

a contradiction since \(n \geq 2k + m + \lambda + 5\).

If \(m > k + 1\), then from (5.1.21), we have

\[(n + m + \lambda)T(r, g) \leq (2k + \lambda + 2)(T(r, f) + T(r, g)) + O\{r^{\rho(f)-1+\varepsilon}\} + O\{r^{\rho(g)-1+\varepsilon}\} + S(r, f) + S(r, g).\]

Hence,

\[(n-4k+m-\lambda-4)(T(r, f) + T(r, g)) \leq O\{r^{\rho(f)-1+\varepsilon}\} + O\{r^{\rho(g)-1+\varepsilon}\} + S(r, f) + S(r, g),\]

a contradiction since \(n \geq 4k - m + \lambda + 9\).

**Case 2.** Let \(B \neq 0\) and \(A \neq B\). From (5.1.19), we have

\[F' = \frac{(B + 1)G - (R - A + 1)}{BG + (A - B)}\]

and hence

\[\overline{N}\left(r, \frac{B - A + 1}{B + 1}; G\right) = \overline{N}(r, 0; F').\]

Proceeding as in case 1, we get a contradiction.

**Case 3.** Let \(B = 0\) and \(A \neq 0\), then from (5.1.19), we have

\[F - \frac{G + A - 1}{A}\] and \[G - AF - (A - 1).\]
If $A \neq 1$, it follows that
\[
\overline{N}(r, \frac{A-1}{A}; F) = \overline{N}(r, 0; G) \quad \text{and} \quad \overline{N}(r, 1 - A; G) = \overline{N}(r, 0; F).
\]

By applying Lemma 5.1.4, we arrive at a contradiction. Therefore $A = 1$ and hence $F = G$.

\[
\Rightarrow (f^n(z)(f(z) - 1)^m \prod_{j=1}^{d} f(z + c_j)^{s_j})^{(k-1)} = (g^n(z)(g(z) - 1)^m \prod_{j=1}^{d} g(z + c_j)^{s_j})^{(k-1)} + c_{k-1},
\]

where $c_{k-1}$ is a constant. If $c_{k-1} \neq 0$, by Lemma 5.1.4, we get $n \leq 2k + m + \lambda$ when $m \leq k + 1$ and $n \leq 4k - m + \lambda$ when $m > k + 1$, which contradicts the hypothesis. Hence, $c_{k-1} = 0$.

Repeating the same process $k - 1$ times, we get
\[
f^n(z)(f(z) - 1)^m \prod_{j=1}^{d} f(z + c_j)^{s_j} = g^n(z)(g(z) - 1)^m \prod_{j=1}^{d} g(z + c_j)^{s_j}. \quad (5.122)
\]

Set $h = f/g$. If $h$ is a constant, then substituting $f = gh$ in (5.1.22), we have
\[
g^n \prod_{j=1}^{d} g(z + c_j)^{s_j} \left[ g^m(h^{n+m+\lambda} - 1) - mC_1 g^{m-1}(h^{n+m+\lambda-1} - 1) + \cdots + (-1)^m(h^{n+\lambda} - 1) \right] \equiv 0.
\]

(5.1.23)

Since $g$ is a transcendental entire function, we have $g^n \prod_{j=1}^{d} g(z + c_j)^{s_j} \neq 0$.

Then, from (5.1.23), we get
\[
g^m(h^{n+m+\lambda} - 1) - mC_1 g^{m-1}(h^{n+m+\lambda-1} - 1) + \cdots + (-1)^m(h^{n+\lambda} - 1) \equiv 0,
\]

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which implies \( h = 1 \) and hence \( f \equiv g \).

If \( h \) is not constant, then from (5.1.22), we find that \( f \) and \( g \) satisfy the algebraic equation

\[
R(f, g) = 0, \quad R(w_1, w_2) = w_1^n(w_1 - 1)^m \prod_{j=1}^{d} w_1(z + c_j)^{s_j} - w_2^n(w_2 - 1)^m \prod_{j=1}^{d} w_2(z + c_j)^{s_j}.
\]

Hence the proof of Theorem 5.1.1. \( \square \)

**Theorem 5.1.2.** Let \( f \) and \( g \) be two transcendental entire functions of finite order and \( \alpha (\neq 0) \) be a small function of both \( f \) and \( g \). Let \( c_j \) \((j = 1, 2, \ldots, d)\) be complex constants, \( s_j(j = 1, 2, \ldots, d)\) be non-negative integers. Suppose \( n \geq 1 \), \( m \geq 1 \) and \( k \geq 0 \) are integers satisfying \( n \geq 3k+2m+2\lambda+6 \) when \( m \leq k+1 \) and \( n \geq 6k-m+2\lambda+11 \) when \( m > k+1 \). If \( f^n(z)(f(z)-1)^m \prod_{j=1}^d f(z+c_j)^{s_j})^{(k)} \) and \( g^n(z)(g(z)-1)^m \prod_{j=1}^d g(z+c_j)^{s_j})^{(k)} \) share \((\alpha, 2)^*\), then the conclusions of Theorem 5.1.1 hold.

**Proof.** Let \( F, \ G, \ F_1(z) \) and \( G_1(z) \) be defined as in Theorem 5.1.1.

Then, \( F \) and \( G \) are transcendental meromorphic functions that share \((1, 2)^*\) except the zeros and poles of \( \alpha(z) \).

Let \( H \neq 0 \). Then using (2.3.3) for \( p = 1 \), (5.1.13) and Lemma 2.3.6 in (5.1.11), we get

\[
(n + m + \lambda) T(r, f) \leq N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G)
+ \bar{N}(r, 0; F) + \bar{N}(r, \infty; F) + N_{k+2}(r, 0; F_1) + S(r, f) + S(r, g)
\]
\[
\leq N_{k+2}(r, 0; F_1) + \bar{N}_{k+2}(r, 0; G_1) + N_{k+1}(r, 0; F_1) + S(r, f) + S(r, g). \tag{5.1.24}
\]

If \( m \leq k+1 \), from (5.1.24), we obtain

\[
(n + m + \lambda) T(r, f) \leq (2k + 2m + 2\lambda + 3) T(r, f) + (k + m + \lambda + 2) T(r, g)
+ O\{r^{\rho(f)-1+\varepsilon}\} + O\{r^{\rho(g)-1+\varepsilon}\} + S(r, f) + S(r, g). \tag{5.1.25}
\]
Similarly,

\[(n + m + \lambda) T(r, g) \leq (2k + 2m + 2\lambda + 3) T(r, g) + (k + m + \lambda + 2) T(r, f)
+ O\{r^{\rho(f) - 1+\varepsilon}\} + O\{r^{\rho(g) - 1+\varepsilon}\} + S(r, f) + S(r, g). \]  

(5.1.26)

From (5.1.25) and (5.1.26), we get

\[(n-3k-2m-2\lambda-5) (T(r, f)+T(r, g)) \leq O\{r^{\rho(f) - 1+\varepsilon}\}+O\{r^{\rho(g) - 1+\varepsilon}\}+S(r, f)+S(r, g),\]

contradicting the fact that \(n \geq 3k + 2m + 2\lambda + 6\).

If \(m > k + 1\), then from (5.1.24), we obtain

\[(n + m + \lambda) T(r, f) \leq (4k + 2\lambda + 6) T(r, f) + (2k + \lambda + 4) T(r, g) + O\{r^{\rho(f) - 1+\varepsilon}\}
+ O\{r^{\rho(g) - 1+\varepsilon}\} + S(r, f) + S(r, g). \]  

(5.1.27)

Similarly,

\[(n + m + \lambda) T(r, g) \leq (4k + 2\lambda + 6) T(r, g) + (2k + \lambda + 4) T(r, f) + O\{r^{\rho(f) - 1+\varepsilon}\}
+ O\{r^{\rho(g) - 1+\varepsilon}\} + S(r, f) + S(r, g). \]  

(5.1.28)

From (5.1.27) and (5.1.28), we get

\[(n-6k+m-2\lambda-10) (T(r, f)+T(r, g)) \leq O\{r^{\rho(f) - 1+\varepsilon}\}+O\{r^{\rho(g) - 1+\varepsilon}\}+S(r, f)+S(r, g),\]

contradicting the fact that \(n \geq 6k - m + 2\lambda + 11\).

Thus, \(II = 0\) and the rest of the theorem follows from the proof of Theorem 5.1.1.

Hence the proof of Theorem 5.1.2.

\[\square\]

**Theorem 5.1.3.** Let \(f\) and \(g\) be two transcendental entire functions of finite order and
$\alpha(\neq 0)$ be a small function of both $f$ and $g$. Let $c_j$ ($j = 1, 2, \ldots, d$) be complex constants, $s_j$ ($j = 1, 2, \ldots, d$) be non-negative integers. Suppose $n (\geq 1)$, $m (\geq 1)$ and $k (\geq 0)$ are integers satisfying $n \geq 5k + 4m + 4\lambda + 8$ when $m \leq k + 1$ and $n \geq 10k - m + 4\lambda + 15$ when $m > k + 1$. If 
\[ \overline{E}_{2j}(\alpha(z), (f^n(z)(f(z) - 1)^m \prod_{j=1}^{d} f(z + c_j s_j)^{(k)}) = \overline{E}_{2j}(\alpha(z), (g^n(z)(g(z) - 1)^m \prod_{j=1}^{d} g(z + c_j s_j)^{(k)}) \right. \] 
then the conclusions of Theorem 5.1.1 hold.

**Proof.** Let $F$, $G$, $F_1(z)$ and $G_1(z)$ be defined as in Theorem 5.1.1.

Then, $F$ and $G$ are transcendental meromorphic functions such that $\overline{E}_{2j}(1, F) = \overline{E}_{2j}(1, G)$ except the zeros and poles of $\alpha(z)$.

Let $H \neq 0$. Then, by (2.3.3), (5.1.13) and Lemma 5.1.5 in (5.1.11), we get

\[(n + m + \lambda) T(r, f) \leq N_2(r, 0; G) + 2\overline{N}(r, 0; F) + \overline{N}(r, 0; G) + N_{k+2}(r, 0; F_1) + S(r, f) + S(r, g) \]
\[\quad \leq N_{k+2}(r, 0; F_1) + N_{k+2}(r, 0; G_1) + 2N_{k+1}(r, 0; F_1) + N_{k+1}(r, 0; G_1) + S(r, f) + S(r, g). \tag{5.1.29} \]

If $m \leq k + 1$, then from (5.1.29), we obtain

\[(n + m + \lambda) T(r, f) \leq (3k + 3m + 3\lambda + 4) T(r, f) + (2k + 2m + 2\lambda + 3) T(r, g) + O\{r^{\rho(f) - 1 + \varepsilon}\} + O\{r^{\rho(g) - 1 + \varepsilon}\} + S(r, f) + S(r, g). \tag{5.1.30} \]

Similarly,

\[(n + m + \lambda) T(r, g) \leq (3k + 3m + 3\lambda + 4) T(r, g) + (2k + 2m + 2\lambda + 3) T(r, f) + O\{r^{\rho(f) - 1 + \varepsilon}\} + O\{r^{\rho(g) - 1 + \varepsilon}\} + S(r, f) + S(r, g). \tag{5.1.31} \]

From (5.1.30) and (5.1.31), we get

\[(n - 5k - 4m - 4\lambda - 7) (T(r, f) + T(r, g)) \leq O\{r^{\rho(f) - 1 + \varepsilon}\} + O\{r^{\rho(g) - 1 + \varepsilon}\} + S(r, f) + S(r, g), \]
contradicting the fact that \( n \geq 5k + 4m + 4\lambda + 8 \).

If \( m > k + 1 \), then from (5.1.29), we obtain

\[
(n + m + \lambda) T(r, f) \leq (6k + 3\lambda + 8) T(r, f) + (4k + 2\lambda + 6) T(r, g) + O\{r^{\rho(f)-1+\varepsilon}\} + O\{r^{\rho(g)-1+\varepsilon}\} + S(r, f) + S(r, g).
\]  

(5.1.32)

Similarly,

\[
(n + m + \lambda) T(r, g) \leq (6k + 3\lambda + 8) T(r, g) + (4k + 2\lambda + 6) T(r, f) + O\{r^{\rho(f)-1+\varepsilon}\} + O\{r^{\rho(g)-1+\varepsilon}\} + S(r, f) + S(r, g).
\]  

(5.1.33)

From (5.1.32) and (5.1.33), we get

\[
(n-10k+m-4\lambda-14) (T(r, f)+T(r, g)) \leq O\{r^{\rho(f)-1+\varepsilon}\}+O\{r^{\rho(g)-1+\varepsilon}\}+S(r, f)+S(r, g),
\]

contradicting the fact that \( n \geq 10k - m + 4\lambda + 15 \).

Thus \( H \equiv 0 \) and rest of the theorem follows from the proof of Theorem 5.1.1.

Hence the proof of Theorem 5.1.3. \( \square \)

**Remark 5.1.1.** For \( j = 1, 2, \ldots, d, \) if \( (s_j = 0 \text{ for } j \neq 1) \) and \( (c_j = c, \ s_j = 1 \text{ for } j = 1) \) (i.e., \( \lambda = 1 \)) in Theorems 5.1.1 – 5.1.3, we obtain Theorems 5.1.G – 5.1.I respectively.

**Remark 5.1.2.** For \( j = 1, 2, \ldots, d, \) if \( (s_j = 0 \text{ for } j \neq 1) \) and \( (c_j = c, \ s_j = 1 \text{ for } j = 1) \) (i.e., \( \lambda = 1 \)) also \( k = 0 \) in Theorems 5.1.1 – 5.1.3, we obtain Theorems 5.1.D – 5.1.F respectively.

**Remark 5.1.3.** For \( j = 1, 2, \ldots, d, \) if \( (s_j = 0 \text{ for } j \neq 1) \) and \( (c_j = c, \ s_j = 1 \text{ for } j = 1) \) (i.e., \( \lambda = 1 \)) also \( m = 1, \ k = 0 \) in Theorems 5.1.1 – 5.1.3, we obtain Theorems 5.1.A – 5.1.C respectively.
5.2 UNIQUENESS AND VALUE DISTRIBUTION OF
q-SHIFT LINEAR DIFFERENCE POLYNOMIALS

5.2.1 INTRODUCTION AND RESULTS

An exception to the standard notation is that $S(r, f)$ is defined to be any quantity of the
growth $o(T(r, f))$ as $r \to \infty$, outside of an exceptional set of finite logarithmic measure. This differs from the usual convention, where the exceptional set is assumed to be of finite linear measure. For convenience, we assume that $S(f)$ includes all constant functions and $S = S(f) \cup \{\infty\}$.

Difference Nevanlinna theory has emerged as a result of recent interest on value distribution and growth of meromorphic solutions of difference equations.

Recently, there has been an increasing interest in studying difference equations, the difference product and the q-difference in complex plane $\mathbb{C}$ (see [31], [41]).

Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ be a non-zero polynomial, where $a_n(\neq 0), a_{n-1}, \cdots, a_0$ are complex constants. Denote $\Gamma_1, \Gamma_2$ by $\Gamma_1 = m_1 + m_2, \Gamma_2 = m_1 + 2m_2$ respectively, where $m_1$ is the number of simple zeros of $P(z)$ and $m_2$ is the number of multiple zeros of $P(z)$. Also, denote $d = \gcd(\lambda_0, \lambda_1, \cdots, \lambda_n)$ where $\lambda_i = n + 1$ if $a_i = 0, \lambda_i = i + 1$ if $a_i \neq 0$.

In 2015, Xu, Liu and Cao [57] investigated the value distributions for q-shift meromorphic functions and obtained the following results.

**Theorem 5.2.A.** Let $f$ be a zero order transcendental meromorphic (resp. entire) function, $q \in \mathbb{C} \setminus \{0\}, \eta \in \mathbb{C}$. Then for $n > \Gamma_1 + 4$ (resp. $n > \Gamma_1$), $P(f) f(qz + \eta) - \alpha(z)$ has infinitely many solutions, where $\alpha(z) \in S(f) \setminus \{0\}$.

**Theorem 5.2.B.** Let $f$ and $g$ be two transcendental entire functions of zero order and let $q \in \mathbb{C} \setminus \{0\}, \eta \in \mathbb{C}$. If $P(f) f(qz + \eta)$ and $P(g) g(qz + \eta)$ share $1$ CM and $n > 2\Gamma_2 + 1$ be
an integer, then one of the following results hold:

(i) \( f \equiv tg \) for a constant \( t \) such that \( t^d = 1 \);

(ii) \( f \) and \( g \) satisfy the algebraic equation \( R(f, g) = 0 \), where
\[
R(w_1, w_2) = P(w_1)w_1(qz + \eta) - P(w_2)w_2(qz + \eta);
\]

(iii) \( fg \equiv \mu \), where \( \mu \) is a complex constant satisfying \( \alpha^2_\mu \mu^{n+1} \equiv 1 \).

**Theorem 5.2.C.** Let \( f \) and \( g \) be two transcendental entire functions of zero order and let \( q \in \mathbb{C} \setminus \{0\}, \eta \in \mathbb{C} \). If \( E_l(1; P(f)f(qz + \eta)) = E_l(1; P(g)g(qz + \eta)) \) and \( l, m, n \) are integers satisfying one of the following conditions:

(i) \( l \geq 3, n > 2\Gamma_2 + 1 \);

(ii) \( l = 2, n > \Gamma_1 + 2\Gamma_2 + 2 - \lambda \);

(iii) \( l = 1, n > 2\Gamma_1 + 2\Gamma_2 + 3 - 2\lambda \);

(iv) \( l = 0, n > 3\Gamma_1 + 2\Gamma_2 + 4 - 3\lambda \).

Then the conclusions of Theorem 5.2.B hold, where \( \lambda = \min\{\Theta(0, f), \Theta(0, g)\} \).

Recently, P. Sahoo and G. Biswas [50] extended Theorems 5.2.A, 5.2.B and 5.2.C by obtaining the following results:

**Theorem 5.2.D.** Let \( f \) be a transcendental entire function of zero order and \( \alpha(z) \in \mathcal{S}(f) \setminus \{0\} \). Suppose that \( \eta \) is a non-zero complex constant, \( n \) and \( k \) are positive integers. Then for \( n > \Gamma_1 + km_2, (P(f)f(qz + \eta))^{(k)}(z) - \alpha(z) = 0 \) has infinitely many solutions.

**Theorem 5.2.E.** Let \( f \) and \( g \) be two transcendental entire functions of zero order and let \( q \in \mathbb{C} \setminus \{0\}, \eta \in \mathbb{C} \). If \( E_l(1; (P(f)f(qz + \eta))^{(k)}) = E_l(1; (P(g)g(qz + \eta))^{(k)}) \) and \( l, m, n \) are integers satisfying one of the following conditions:

(i) \( l \geq 2, n > 2\Gamma_2 + 2km_2 + 1 \);

(ii) \( l = 1, n > \frac{1}{2}(\Gamma_1 + 4\Gamma_2 + 5km_2 + 3) \);

(iii) \( l = 0, n > 3\Gamma_1 + 2\Gamma_2 + 5km_2 + 4 \).

Then the following results hold:

\( f \equiv tg \) for a constant \( t \) such that \( t^d = 1 \);

(ii) \( f \) and \( g \) satisfy the algebraic equation \( R(f, g) = 0 \), where
\[ R(w_1, w_2) = P(w_1)w_1(qz + \eta) - P(w_2)w_2(qz + \eta); \]

(iii) \(fg \equiv \mu\), where \(\mu\) is a complex constant satisfying \(\alpha^n \mu^{n+1} \equiv 1\).

In this section, we investigate the value distribution of entire and meromorphic function \(f(z)\) and its linear difference polynomial

\[ L(f) = \sum_{j=1}^{s} b_j f(q_j z + c_j) \neq 0, \quad (5.2.1) \]

where \(q_j \in \mathbb{C} \setminus \{0\} \ (j = 1, 2, \cdots, s)\), \(b_j\) and \(c_j \in \mathbb{C}\) are constants. We also study the uniqueness problems when two difference products of entire functions share one value with finite weight. Our results generalizes the previous theorems of P. Sahoo and G. Biswas [50].

### 5.2.2 LEMMAS

**Lemma 5.2.1.** ([41]) Let \(f\) and \(g\) be two non-constant meromorphic functions. If \(E_2(1; f) = E_2(1; g)\), then one of the following cases hold:

(i) \(T(r) \leq N_2(r, 0; f) + N_2(r, 0; g) + N_2(r, \infty; f) + N_2(r, \infty; g) + S(r)\),

(ii) \(f = g\),

(iii) \(fg = 1\),

where \(T(r) = \max\{T(r, f), T(r, g)\}\) and \(S(r) = o\{T(r)\}\).

**Lemma 5.2.2.** ([2]) Let \(F\) and \(G\) be two non-constant meromorphic functions. If \(E_1(1; F) = E_1(1; G)\) and \(H \neq 0\) then

\[
T(r, F) \leq N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + \frac{1}{2} \overline{N}(r, 0; F) + \frac{1}{2} \overline{N}(r, \infty; F) + S(r, F) + S(r, G);
\]

the same inequality hold for \(T(r, G)\).
Lemma 5.2.3. ([2]) Let $F$ and $G$ be two non-constant meromorphic functions sharing 1 IM and $H \neq 0$. Then

\[ T(r, F) \leq N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + 2\overline{N}(r, 0; F) + \overline{N}(r, 0; G) \\
+ 2\overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + S(r, F) + S(r, G); \]

the same inequality hold for $T(r, G)$.

Lemma 5.2.4. ([57]) Let $f$ be a transcendental meromorphic function of zero order and $q, \eta$ be two non-zero complex constants. Then

\[ T(r, f(qz + \eta)) = T(r, f(z)) + S(r, f), \]
\[ N(r, \infty; f(qz + \eta)) \leq N(r, \infty; f(z)) + S(r, f), \]
\[ N(r, 0; f(qz + \eta)) \leq N(r, 0; f(z)) + S(r, f), \]
\[ \overline{N}(r, \infty; f(qz + \eta)) \leq \overline{N}(r, \infty; f(z)) + S(r, f), \]
\[ \overline{N}(r, 0; f(qz + \eta)) \leq \overline{N}(r, 0; f(z)) + S(r, f). \]

Lemma 5.2.5. ([36]) Let $f$ be a non-constant meromorphic function with hyper order less than 1 and $L(f)(\neq 0)$ be defined as in (5.2.1). Then

\[ N(r, 0; L(f)) \leq T(r, L(f)) - T(r, f) + N(r, 0; f) + S(r, f), \]
\[ N(r, 0; L(f)) \leq N(r, 0; f) + (s - 1)N(r, \infty; f) + S(r, f), \]

Also, $T(r, L(f)) \leq T(r, f) + (s - 1)N(r, \infty; f) + S(r, f)$.

Lemma 5.2.6. Let $f$ be a transcendental meromorphic function of zero order and $q_j(\neq 0), c_j \in \mathbb{C}$ ($j = 1, 2, \cdots, s$). Then

\[ (n - s)T(r, f) + S(r, f) \leq T(r, P(f)L(f)) \leq (n + s)T(r, f) + S(r, f). \]
In addition, if $f$ is a transcendental entire function of zero order, then

$$T(r, P(f)L(f)) = T(r, P(f) \sum_{j=1}^{s} f(z)) + S(r, f) = (n + s)T(r, f) + S(r, f).$$

**Proof.** If $f(z)$ is a meromorphic function of order zero, then from Lemmas 2.1.4, 5.2.4 and 5.2.5, we have

$$T(r, P(f)L(f)) \leq nT(r, f) + T(r, L(f)) + S(r, f) \leq (n + s)T(r, f) + S(r, f). \quad (5.2.2)$$

On the other hand, from Lemmas 2.1.4, 5.2.4 and 5.2.5, we have

$$nT(r, f) \leq T(r, P(f)) + S(r, f) \leq T(r, P(f) L(f)) + T(r, 1/L(f)) + S(r, f) \leq T(r, P(f) L(f)) + T(r, L(f)) + S(r, f)$$

i.e. $(n - s)T(r, f) + S(r, f) \leq T(r, P(f)L(f))$. \quad (5.2.3)

From (5.2.2) and (5.2.3), we have

$$(n - s)T(r, f) + S(r, f) \leq T(r, P(f)L(f)) \leq (n + s)T(r, f) + S(r, f).$$

\[
\square
\]

**Lemma 5.2.7.** Let $f$ and $g$ be two entire functions, $n$, $k$ and $s$ be three positive integers $q_{ij} \neq 0, c_{ij} \in \mathbb{C}$ ($j = 1, 2, \cdots, s$). Let $F = (P(f)L(f))^{(k)}, G = (P(g)L(g))^{(k)}$. If there exists two non-zero constants $c_1$ and $c_2$ such that $N(r, c_1; F) = \overline{N}(r, 0; G)$ and $N(r, c_2; G) = \overline{N}(r, 0; F)$, then $n \leq 2\Gamma + 2km + s$.

**Proof.** Let $F_1 = P(f)L(f)$ and $G_1 = P(g)L(g)$. 

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By second fundamental theorem of Nevanlinna, we have

\[ T(r, F) \leq \overline{N}(r, 0; F) + \overline{N}(r, c_1; F) + S(r, F) \]
\[ \leq \overline{N}(r, 0; F) + \overline{N}(r, 0; G) + S(r, F). \]  \hspace{1cm} (5.2.4)

Using (2.3.2), (2.3.3), (5.2.4), Lemmas 2.1.4, 5.2.4 and 5.2.6, we get

\[ (n + s)T(r, f) \leq T(r, F) - \overline{N}(r, 0; F) + N_{k+1}(r, 0; F_1) + S(r, f) \]
\[ \leq \overline{N}(r, 0; G) + N_{k+1}(r, 0; F_1) + S(r, f) \]
\[ \leq N_{k+1}(r, 0; F_1) + \overline{N}_{k+1}(r, 0; G_1) + S(r, f) + S(r, g) \]
\[ \leq N_{k+1}(r, 0; P(f)) + N(r, 0; L(f)) + N_{k+1}(r, 0; P(g)) + N(r, 0; L(g)) \]
\[ + S(r, f) + S(r, g) \]
\[ \leq (m_1 + m_2 + km_2 + s)(T(r, f) + T(r, g)) + S(r, f) + S(r, g). \]  \hspace{1cm} (5.2.5)

Similarly,

\[ (n + s)T(r, g) \leq (m_1 + m_2 + km_2 + s)(T(r, f) + T(r, g)) + S(r, f) + S(r, g). \]  \hspace{1cm} (5.2.6)

From (5.2.5) and (5.2.6), we have

\[ (n - 2m_1 - 2m_2 - 2km_2 - s)(T(r, f) + T(r, g)) \leq S(r, f) + S(r, g), \]

which gives \( n \leq 2\Gamma_1 + 2km_2 + s \).

Hence the proof. \( \Box \)

**Lemma 5.2.8.** ([44]) Let \( f \) be a non-constant zero order meromorphic function and \( g \in \mathbb{C} \setminus \{0\} \). Then \( m \left( r, \frac{f(qz+q)}{f(z)} \right) = S(r, f) \), on a set of logarithmic density 1.

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5.2.3 STATEMENT AND PROOF OF MAIN RESULTS

Theorem 5.2.1. Let $f$ be a transcendental meromorphic (resp. entire) function of zero order and $\alpha(z) \in S(f) \setminus \{0\}$. Let $q_j \in \mathbb{C} \setminus \{0\}$ ($j = 1, 2, \cdots, s$), $b_j$ and $c_j \in \mathbb{C}$ be constants such that $L(f) = \sum_{j=1}^{s} b_j f(q_j z + c_j) \neq 0$. Suppose $n$ and $k$ are positive integers. Then for $n > \Gamma_1 + km_2 + 2s + 1$ (resp. $n > \Gamma_1 + km_2$), $(P(f)L(f))^{(k)} - \alpha(z) = 0$ has infinitely many solutions.

Proof. Let $F$, $G$, $F_1$ and $G_1$ be defined as in Lemma 5.2.7.

Case 1. Suppose $f$ is transcendental meromorphic function of zero order.

Assume that $F_1^{(k)} - \alpha(z)$ has only finitely many zeros. Then,

$$\overline{N}(r, 0; F_1^{(k)}) = O\{log r\} = S(r, f).$$  \hfill (5.2.7)

By the second fundamental theorem for three small functions and the Valiron-Mohon’ko lemma, we have

$$T(r, F) \leq \overline{N}(r, 0; F_1^{(k)}) + \overline{N}(r, \alpha; F_1^{(k)}) + \overline{N}(r, \infty; F_1^{(k)}) + S(r, f)$$

$$\leq T(r, F_1^{(k)}) - T(r, F_1) + N_{k+1}(r, 0; F_1) + \overline{N}(r, \infty; f) + S(r, f).$$  \hfill (5.2.8)

By Lemma 5.2.6 and (5.2.8), we get

$$(n - s) T(r, f) \leq N_{k+1}(r, 0; F_1) + \overline{N}(r, \infty; f) + S(r, f)$$

$$\leq N_{k+1}(r, 0; P(f)) + N(r, 0; L(f)) + \overline{N}(r, \infty; f) + S(r, f)$$

$$\leq (m_1 + m_2 + km_2 + s + 1)T(r, f) + S(r, f).$$

Hence,

$$(n - m_1 - m_2 - km_2 - 2s - 1)T(r, f) \leq S(r, f),$$

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which contradicts the assumption $n > \Gamma_1 + km_2 + 2s + 1$.

**Case 2.** If $f$ is transcendental entire function.

Suppose $F_1^{(k)} - \alpha(z)$ has only finitely many zeros. By using the same arguments as in Case 1 and Lemma 5.2.6, we get

\[(n - m_1 - m_2 - km_2)T(r, f) \leq S(r, f),\]

which contradicts the assumption $n > \Gamma_1 + km_2$.

This proves the theorem. $\square$

**Corollary 5.2.1.** Let $p(z), \; q(z)$ be non-zero polynomials. Then the $q$-shift difference equation $(P(f)L(f))^{(k)} - p(z) = q(z)$ has no transcendental meromorphic solution of zero order, provided that $n \geq s + 1$.

**Proof.** Assume that $f$ is a transcendental meromorphic solution of zero order of

\[(P(f)L(f))^{(k)} - p(z) = q(z).\]

i.e., \[(P(f)L(f))^{(k)} = p(z) + q(z).\]

Integrating above equation $k$ times, we get

\[
\frac{H(z)}{P(f)f^s} = \frac{L(f)}{f^s},
\]

where $H(z)$ is a polynomial. From Lemma 5.2.8 and since $f$ is a meromorphic function of zero order, we have

\[
(n + s)T(r, f) = T \left( r, \frac{L(f)}{f^s} \right) + S(r, f)
\]

\[
= N \left( r, \frac{L(f)}{f^s} \right) + S(r, f)
\]

\[
\leq 2sT(r, f) + S(r, f),
\]

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which contradicts with \( n \geq s + 1 \). Hence the proof.

**Theorem 5.2.2.** Let \( f \) and \( g \) be two transcendental entire functions of zero order. Let \( a_j \in \mathbb{C} \setminus \{0\} \) \((j = 1, 2, \cdots, s)\), \( b_j \) and \( c_j \in \mathbb{C} \). If \( E_t(1; (P(f)L(f))^{(k)}) = E_t(1; (P(g)L(f))^{(k)}) \) and \( l, m, n \) and \( s \) are integers satisfying one of the following conditions:

(i) \( l \geq 2, n > 2\Gamma_2 + 2km_2 + s \);

(ii) \( l = 1, n > \frac{1}{2}(\Gamma_1 + 4\Gamma_2 + 5km_2 + 3s) \);

(iii) \( l = 0, n > 3\Gamma_1 + 2\Gamma_2 + 5km_2 + 4s \),

then one the following results hold:

(i) \( f \equiv tg \) for a constant \( t \) such that \( t^l = 1 \);

(ii) \( f \) and \( g \) satisfy the algebraic equation \( R(f, g) = 0 \), where

\[
R(w_1, w_2) = P(w_1)L(w_1) - P(w_2)L(w_2);
\]

(iii) \( fg \equiv \mu \), where \( \mu \) is a complex constant satisfying \( \alpha_n^2\beta^2\mu^{n+1} = 1 \).

**Proof.** Let \( F, G, F_1 \) and \( G_1 \) be defined as in Lemma 5.2.7.

i.e. \( F = F_1^{(k)} \); \( G = G_1^{(k)} \). Then \( F \) and \( G \) are transcendental entire functions satisfying \( E_t(1; F) = E_t(1; G) \).

Using (2.3.2) and Lemma 5.2.6, we get

\[
N_2(r, 0; F) \leq N_2(r, 0; F_1^{(k)}) + S(r, f) \\
\leq T(r, F_1^{(k)}) - T(r, F_1) + N_{k+2}(r, 0; F_1) + S(r, f) \\
= T(r, F) - (n + s)T(r, f) + N_{k+2}(r, 0; F_1) + S(r, f).
\]

Hence,

\[(n + s)T(r, f) \leq T(r, F) - N_2(r, 0; F) + N_{k+2}(r, 0; F_1) + S(r, f). \quad (5.2.9)\]
Also from (2.3.3), we have

\[ N_2(r, 0; F) \leq N_2(r, 0; F_1^{(k)}) + S(r, f) \leq N_{k+2}(r, 0; F_1) + S(r, f). \tag{5.2.10} \]

We consider the following three cases:

**Case 1.** Let \( l \geq 2 \).

Suppose that \( F \) and \( G \) satisfy Lemma 5.2.1 (i).

From (5.2.9) and (5.2.10), we get

\[
(n + s)T(r, f) \leq N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + N_{k+2}(r, 0; F_1) \\
+ S(r, f) + S(r, g) \\
\leq N_{k+2}(r, 0; F_1) + N_{k+2}(r, 0; G_1) + S(r, f) + S(r, g) \\
\leq (m_1 + 2m_2 + km_2 + s)(T(r, f) + T(r, g)) + S(r, f) + S(r, g).
\tag{5.2.11} \]

Similarly,

\[
(n + s)T(r, g) \leq (m_1 + 2m_2 + km_2 + s)(T(r, f) + T(r, g)) + S(r, f) + S(r, g).
\tag{5.2.12} \]

From (5.2.11) and (5.2.12), we get

\[
(n - 2m_1 - 4m_2 - 2km_2 - s)(T(r, f) + T(r, g)) < S(r, f) + S(r, g),
\]

which contradicts the assumption \( n > 2l_2 + 2km_2 + s \). Therefore, by Lemma 5.2.1, we have either \( FG = 1 \) or \( F = G \).
Let $FG = 1$. Then

$$
(P(f)L(f))^{(k)}(P(g)L(g))^{(k)} = 1. \tag{5.2.13}
$$

If possible, we assume that $P(z) = 0$ has $m$ roots $u_1, u_2, \ldots, u_m$ with multiplicities $s_1, s_2, \ldots, s_m$. Then, $s_1 + s_2 + \cdots + s_m = n$. Then

$$
\left[ a_n(f - u_1)^{s_1}(f - u_2)^{s_2} \cdots (f - u_m)^{s_m} \sum_{j=1}^{s} b_j f(q_j z + c_j) \right]^{(k)}
	imes \left[ a_n(g - u_1)^{s_1}(g - u_2)^{s_2} \cdots (g - u_m)^{s_m} \sum_{j=1}^{s} b_j g(q_j z + c_j) \right]^{(k)} = 1. \tag{5.2.14}
$$

Since $f$ and $g$ are entire functions, from (5.2.14), we get $u_1 = u_2 = u_2 = \ldots = u_m = 0$. Also, from (5.2.14), we have $u_1, u_2, \ldots, u_m$ are the Picard exceptional values. If $m \geq 2$ and $u_i \neq 0 (i = 1, 2, \ldots, m)$, by Picard's theorem of entire functions, we can see that the Picard exceptional values of $f$ are atleast three, which is a contradiction.

Next, we assume that $P(z) = 0$ has only one root. Then $P(f) = a_n(f - a)^n$ and $P(g) = a_n(g - a)^n$, where $a$ is any complex constant. From (5.2.13), we get

$$
\left[ a_n(f - a)^n \sum_{j=1}^{s} b_j f(q_j z + c_j) \right]^{(k)} \left[ a_n(g - a)^n \sum_{j=1}^{s} b_j g(q_j z + c_j) \right]^{(k)} = 1. \tag{5.2.15}
$$

Since $f$ and $g$ are transcendental entire functions, by Picard's theorem, we can get that $f - a = 0$ and $g - a = 0$ do not have any zeros. Then, we obtain that $f(z) = e^{\alpha(z)} + a$ and $g(z) = e^{\beta(z)} + a$, where $\alpha(z)$ and $\beta(z)$ are non-constant polynomials. From (5.2.15), we also have that $\sum_{j=1}^{s} b_j f(q_j z + c_j)^{(k)} \neq 0$ and $\sum_{j=1}^{s} b_j g(q_j z + c_j)^{(k)} \neq 0$ and therefore $a = 0$.

Thus, $f(z) = e^{\alpha(z)}$ and $g(z) = e^{\beta(z)}$, $P(z) = a_n z^n$ and

$$
\left[ a_n f^n \sum_{j=1}^{s} b_j f(q_j z + c_j) \right]^{(k)} \left[ a_n g^n \sum_{j=1}^{s} b_j g(q_j z + c_j) \right]^{(k)} = 1. \tag{5.2.16}
$$
If $k = 0$, then from (5.2.16), we get

$$a_n f^n \sum_{j=1}^{s} b_j f(q_j z + c_j) a_n g^n \sum_{j=1}^{s} b_j g(q_j z + c_j) = 1. \quad (5.2.17)$$

Set $M(z) = f(z)g(z)$. If $M(z)$ is non-constant, then from (5.2.17), we get

$$a_n^2 M(z)^n \sum_{j=1}^{s} b_j^2 M(q_j z + c_j) = 1,$$

i.e.,

$$a_n^2 M(z)^n = \frac{1}{\sum_{j=1}^{s} b_j^2 M(q_j z + c_j)}. \quad (5.2.18)$$

Since $f$ and $g$ are transcendental entire functions of zero order, from (5.2.18), Lemma 5.2.4 and $n > s$, we get a contradiction. Hence, $M$ is a constant. From (5.2.18), we get $f(z)g(z) = \mu$, where $\mu$ is a constant satisfying $a_n^2 b^2 \mu^{n+1} = 1$ and $b = \sum_{j=1}^{s} b_j$.

If $k \geq 1$ then from (5.2.16), we have

$$[a_n e^{n\alpha(z)+\alpha(q_1 z+c_1)+\alpha(q_2 z+c_2)+...+\alpha(q_s z+c_s)}]^{(k)} [a_n e^{n\beta(z)+\beta(q_1 z+c_1)+\beta(q_2 z+c_2)+...+\beta(q_s z+c_s)}]^{(k)} = 1.$$

i.e.,

$$[a_n e^{n\alpha(z)+\alpha(q_1 z+c_1)+\alpha(q_2 z+c_2)+...+\alpha(q_s z+c_s)}]^{(k)}$$

$$= a_n e^{n(\alpha(z)+\alpha(q_1 z+c_1)+\alpha(q_2 z+c_2)+...+\alpha(q_s z+c_s)} P(\alpha', \alpha_{c_1}', ..., \alpha_{c_s}', ..., \alpha', \alpha_{c_1}, ..., \alpha_{c_s}), \alpha_{c_1}, ..., \alpha_{c_s}) \quad (5.2.19)$$

where $\alpha_{c_j} = \alpha(q_j z+c_j)$. Obviously, $P(\alpha', \alpha_{c_1}', ..., \alpha_{c_s}', ..., \alpha', \alpha_{c_1}, ..., \alpha_{c_s})$ has infinitely many zeros which contradicts (5.2.16).

Next, we assume that $F = G$. Then

$$\left(P(f) L(f)\right)^{(k)} = \left(P(g) L(g)\right)^{(k)}.$$

Integrating once, we get

$$\left(P(f) L(f)\right)^{(k-1)} = \left(P(g) L(g)\right)^{(k-1)} + \eta_{k-1},$$

where $\eta_{k-1}$ is a constant. If $\eta_{k-1} \neq 0$, from
Lemma 5.2.7, it follows that \( n \leq 2\Gamma_2 + 2km_2 + s \), contrary with the fact that \( n > 2\Gamma_2 + 2km_2 + s \) and \( \Gamma_2 > 1 \). Hence, \( \eta_{k-1} = 0 \). Repeating the integration \((k - 1)\) times, we deduce that

\[
P(f)L(f) = P(g)L(g). \tag{5.2.20}
\]

Set \( h = \frac{f}{g} \). If \( h \) is not a constant, from (5.2.20), we see that \( f \) and \( g \) satisfy the algebraic equation \( R(f,g) = 0 \), where \( R(w_1,w_2) = P(w_1)L(w_1) - P(w_2)L(w_2) \).

If \( h \) is a constant, then put \( f = gh \) in (5.2.20), we get

\[
\left( \sum_{j=1}^{s} b_j g(q_j z + c_j) h \right) \left( a_n g^n h^n + a_{n-1} g^{n-1} h^{n-1} + \cdots + a_0 \right) \\
= \left( \sum_{j=1}^{s} b_j g(q_j z + c_j) \right) \left( a_n g^n + a_{n-1} g^{n-1} + \cdots + a_0 \right),
\]

where \( a_n \neq 0, a_{n-1}, \ldots, a_0 \) are constants. Hence,

\[
\left( \sum_{j=1}^{s} b_j g(q_j z + c_j) \right) \left( a_n g^n(h^{n+1} - 1) + a_{n-1} g^{n-1} (h^n - 1) + \cdots + a_0 (h - 1) \right) = 0. \tag{5.2.21}
\]

Since \( g \) is transcendental entire function, we have \( \sum_{j=1}^{s} b_j g(q_j z + c_j) \neq 0 \).

Then, from (5.2.21), we have

\[
a_n g^n(h^{n+1} - 1) + a_{n-1} g^{n-1} (h^n - 1) + \cdots + a_0 (h - 1) = 0. \tag{5.2.22}
\]

If \( a_n \neq 0, a_{n-1} = a_{n-2} = \cdots = a_0 = 0 \), then we get \( h^{n+1} = 1 \).

If \( a_n \neq 0 \) and there exists \( a_i \neq 0 \) \((i \in 0,1,\ldots,n-1)\). Suppose that \( h^{n+1} \neq 1 \) then from (5.2.22), we have \( T(r,g) = S(r,g) \) which contradicts with transcendental function \( g \).

Then, \( h^{n+1} = 1 \). Similarly, we have \( h^{j+1} = 1 \) when \( a_j \neq 0 \) for some \( j = 0,1,\ldots,n \).

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Thus, from definition of \( d \), we obtain that \( f = tg \) for a constant \( t \) such that \( t^d = 1, d = \gcd(\lambda_0, \lambda_1, \ldots, \lambda_n) \).

**Case 2.** Let \( l = 1 \) and \( H \neq 0 \).

Using Lemma 5.2.2, (5.2.9) and (5.2.10), we get

\[
(n + s)T(r, f) \leq N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + \frac{1}{2} N(r, 0; F)
\]

\[
+ \frac{1}{2} N(r, \infty; F) + N_k(r, 0; F_1) + S(r, F) + S(r, G)
\]

\[
\leq N_{k+2}(r, 0; F_1) + N_{k+2}(r, 0; G_1) + \frac{1}{2} N_{k+1}(r, 0; F_1) + S(r, f) + S(r, g)
\]

\[
\leq \frac{1}{2} [5m_1 + (5k + 9)m_2 + 5s] + S(r).
\]

Similarly,

\[
(n + s)T(r, g) \leq \frac{1}{2} [5m_1 + (5k + 9)m_2 + 5s] + S(r).
\]

Hence,

\[
\left( n - \frac{5m_1 + (5k + 9)m_2 + 3s}{2} \right) \leq S(r),
\]

which contradicts \( n > \frac{1}{2}(\Gamma_1 + 4\Gamma_2 + 5km_2 + 3s) \).

If \( H \equiv 0 \),

\[
\rightarrow \left( \frac{F''}{F'} - \frac{2F''}{F - 1} \right) - \left( \frac{G''}{G'} - \frac{2G'}{G - 1} \right) = 0.
\]

Integrating twice, we get

\[
\frac{1}{F - 1} = \frac{A}{G - 1} + B,
\]

where \( A \neq 0 \) and \( B \) are constants. From (5.2.24), \( F \) and \( G \) share 1 CM and hence they share (1, 2). Therefore, \( n > 2\Gamma_2 + 2km_2 + s \).
Next, we discuss the following three cases.

**Subcase 2.1.** Suppose that $B \neq 0$, and $A = B$. Then from (5.2.24), we have

$$\frac{1}{F - 1} = \frac{BG}{G - 1}. \quad (5.2.25)$$

If $B = -1$, then from (5.2.25), we have $FG = 1$, form this we get $f(z) = e^{\alpha(z)}$ and $g(z) = \mu e^{-\alpha(z)}$ where $\mu$ is a constant satisfying $a_n^2 b_2 \mu^{n+1} = 1$ as in Case 1.

If $B \neq -1$, from (5.2.25), we have

$$\frac{1}{F} = \frac{BG}{(1 + B)G - 1}.$$ Hence $\overline{N}(r, \frac{1}{1+B}; G) = \overline{N}(r, 0; F)$.

Using (2.3.2), (2.3.3) and the second fundamental theorem of Nevanlinna, we deduce that

$$T(r, G) \leq \overline{N}(r, 0; G) + \overline{N} \left( r, \frac{1}{1+B}; G \right) + \overline{N}(r, \infty; G) + S(r, G)$$

$$\leq \overline{N}(r, 0; F) + \overline{N}(r, 0; G) + \overline{N}(r, \infty; G) + S(r, G)$$

$$\leq N_{k+1}(r, 0; F_1) \mid T(r, G) \mid N_{k+1}(r, 0; G_1) \quad (n \mid s) T(r, g) \mid S(r, g).$$

Thus,

$$(n - 2m_1 - 2(k + 1)m_2 - s)\{T(r, f) + T(r, g)\} \leq S(r, f) + S(r, g),$$

which contradicts to $n > 2\Gamma_2 + 2km_2 + s$.

**Subcase 2.2.** Let $B \neq 0$ and $A \neq B$. From (5.2.24), we have

$$F = \frac{(B + 1)G - (B - A + 1)}{BG + (A - B)}$$

and hence

$$\overline{N} \left( r, \frac{B - A + 1}{B + 1}; G \right) = \overline{N}(r, 0; F).$$

Proceeding as in Subcase 1, we get a contradiction.

**Subcase 2.3.** Let $B = 0$ and $A \neq 0$.  

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From (5.2.24), we have \( F = \frac{G + A - 1}{A} \) and \( G = AF - (A - 1) \).

If \( A \neq 1 \), it follows that

\[
\mathcal{N} \left( r, \frac{A - 1}{A}; F \right) = \mathcal{N}(r, 0; G) \quad \text{and} \quad \mathcal{N}(r, 1 - A; G) = \mathcal{N}(r, 0; F)
\]

By applying Lemma 5.2.7, we arrive at a contradiction. Therefore \( A = 1 \) and hence \( F = G \) and rest of the theorem follows from the proof of Case 1.

**Case 3.** Let \( l = 0 \) and \( H \neq 0 \).

Using Lemma 5.2.3, (5.2.9) and (5.2.10), we get

\[
(n + s)T(r, f) \leq N_2(r, 0; G) + N_2(r, \infty; F) + 2N(r, 0; F) + \mathcal{N}(r, 0; G) + 2\mathcal{N}(r, \infty; F) + \mathcal{N}(r, \infty; G)
\]

\[
+ N_{k+2}(r, 0; F_1) + S(r, f) + S(r, g)
\]

\[
\leq N_{k+2}(r, 0; F_1) + N_{k+2}(r, 0; G_1) + N_{k+1}(r, 0; F_1) + N_{k+1}(r, 0; G_1)
\]

\[
+ S(r, f) + S(r, g)
\]

\[
\leq [5m_1 + (5k + 7)m_2 + 5s]T(r) + S(r).
\]

Similarly,

\[
(n + s)T(r, g) \leq [5m_1 + (5k + 7)m_2 + 5s]T(r) + S(r).
\]

Hence,

\[
(n - 5m_1 - (5k + 7)m_2 - 4s)T(r) \leq S(r),
\]

which contradicts our assumption \( n > 3\Gamma_1 + 2\Gamma_2 + 5km_2 + 4s \). Hence \( H = 0 \) and proceeding as in Case 2, we get the result. Hence the proof of Theorem 5.2.2. \( \square \)