2.1 Introduction

In the present chapter we are going to discuss the phase transition of strongly interacting hadronic matter at finite density. The phase diagram of strongly interacting matter has been at the center of attention for quite long time. At very high density we also expect a hadron-quark phase transition and a new state of matter called Quark Gluon Plasma (QGP) may be formed [1]. In nature, the highest densities of matter are reached in central regions of compact stars where the temperature is relatively low. The density might be as large as 10 times normal nuclear saturation density. It is possible that baryonic matter is deconfined under such conditions [2, 3, 4]. So an understanding of the physics of strongly interacting matter at such environmental conditions would have important cosmological and astrophysical significance.

In the laboratory such conditions of large temperatures and densities can be created by the collision of heavy ions at high energies. Presently the strongly interacting matter at high temperature and close to zero baryon densities – a scenario relevant for early universe – is being explored at Relativistic Heavy Ion Collider (RHIC) at BNL and the Large Hadron Collider (LHC) at CERN. A wealth of information has been obtained from RHIC, and a lot more is expected from both the future runs there as well as from LHC. More recently
a variety of energy scans at RHIC and the upcoming facility (FAIR) at GSI, are expected to give us a glimpse of matter in the baryon-rich environments - the so-called *compressed baryonic matter* (CBM). These experiments will also be useful in the search for signatures of critical phenomena associated with a second order critical end point (CEP).

At the same time, observational data are being collected by a large number of telescopes and satellites [5] such as the radio telescopes at the Arecibo, Parkes, Jodrell Bank, and Green Bank Observatories, the Hubble Space Telescope, European Space Agency’s International Gamma Ray Astrophysics Laboratory (INTEGRAL) satellite, Very Large Telescope (VLT) of the European Southern Observatory, the X-ray satellites Chandra, XMM-Newton and NASA’s Rossi X-ray Timing Explorer and the Swift satellite. Observations from these facilities are supposed to tell us about the properties of strongly interacting matter at high densities relevant for the astrophysics of compact stars.

Thus on one hand the laboratory experiments are expected to scan the phase space temperature and various conserved quantum number densities of strongly interacting matter. On the other hand the astrophysical observations are expected to uncover the physics for high baryon number density region of the phase diagram. It should be noted here that the physical characteristics of the matter under consideration may be quite different in the two cases. The time-scale of the dynamics of heavy-ion experiments is so small that only strong interactions may equilibrate thermodynamically. While the dynamics in the astrophysical scenario is slow enough to allow even weak interactions may equilibrate. Thus a question naturally arises – to what extent can laboratory experiments be used to infer about the compact star interiors? The aim of this chapter is to address this question at a preliminary level from the characteristics of the "β-equilibrated" phase diagrams.

One should be able to study the properties of systems described above from first principles using Quantum Chromodynamics (QCD), which is *the* theory of strong interactions. However, QCD is highly non-perturbative in the region of temperature and density that we are interested in. The most reliable way to analyze the physics in this region of interest is to perform a numerical computation of the lattice version of QCD (Lattice QCD). The scheme is robust but numerically costly. Moreover, there are problems in applying this scheme for the systems having finite baryon density. Thus it has become a common practice to study the physics of strongly interacting matter under the given conditions using various QCD inspired effective models. Generally
the QCD inspired phenomenological models are much easier to handle compared to Lattice or the perturbative QCD calculations. But in all these models despite their simplicity, the absence of a proper order parameter for deconfinement transition adds to the uncertainties inherent in such studies and hence reduces the predictive power of such models. To investigate the properties of dense quark matter it is natural to start with a model for dense matter which is built from the quark level. Until now various quark models, such as, different versions of the MIT bag model [6, 7], the color-dielectric model [8, 9] and different formulations of the NJL model [10, 11] have been used to study the NS structure. Despite the similarity of the results on the value of the maximum NS mass, the predictions on the NS configurations can differ substantially from model to model. The most striking difference is in the quark matter content of the NS, which can be extremely large in the case of EOS related to the MIT bag model or the color-dielectric model, but it is vanishingly small in the case of the original version of the NJL model [10, 12]. In the case of NJL model it turns out that, as soon as quark matter appears at increasing NS mass, the star becomes unstable, with only the possibility of a small central region with a mixed phase of nucleonic and quark matter. This may be a result of the lack of confinement in NJL model. In fact an indirect relationship between confinement and NS stability has been found in a study using NJL model with density dependent cut-off [13]. Hence it is important to study the EOS from the Polyakov–Nambu–Jona-Lasinio (PNJL) model [14, 15, 16], where a better description of confinement has been incorporated through Polyakov loop mechanism. Moreover, a comparison with NJL model might be helpful in understanding the role of Polyakov loop at high chemical potential.

A detailed study of 2+1 flavor strong interactions has already been done using the PNJL model. The general thermodynamic properties along with the phase diagram [17], as well as details of fluctuation and correlations of various conserved charges [18] have been reported. Here we extend the work by including $\beta$–equilibrium into the picture. In the context of NJL model such a study was done earlier in [19, 20]. In Ref.[21, 22] the properties of pseudoscalar and neutral mesons have been studied in finite density region within the framework of 2+1 flavor NJL model in $\beta$–equilibrium.

We investigate and compare different properties of the NJL and PNJL models in the $T-\mu_B$ plane. The specialization of these studies to the possible dynamical evolution of NS and/or CBM created in heavy-ion collisions will be
kept as a future exercise.

2.2 Formalism

The supermassive compact objects like neutron stars are born in the aftermath of supernova explosions. The initial temperature of a new born NS can be as high as $T \sim 100$ MeV. For about one minute following its birth, the star stays in a special proto-neutron star state: hot, opaque to neutrinos, and larger than an ordinary NS (see, e.g., [23, 24] and references therein). Later the star becomes transparent to neutrinos generated in its interior. It cools down gradually, initially through neutrino emission ($t \leq 10^5$ years) and then through the emission of photons ($t \geq 10^5$ years) [25], and transforms into an ordinary NS. The weak interaction responsible for the emission of these neutrinos eventually drive the stars to the state of $\beta$-equilibrium along with the imposed condition of charge neutrality.

The mass, radius and other characteristics of such a star depend on the equation of state (EOS), which in turn, is determined by the composition of the star [26]. The possible central density of a compact star may be high enough for the usual neutron-proton matter to undergo a phase transition to some exotic forms of strongly interacting matter. Some of the suggested exotic forms of strongly interacting matter are the hyperonic matter, the quark matter, the superconducting quark matter etc. If there is a hadron to quark phase transition inside the NS, then all the characteristics of the NS will depend on the nature of the phase transition [27, 28].

Furthermore, there have been suggestions that the strange quark matter, containing almost equal numbers of u, d and s quarks, may be the ground state of strongly interacting matter (see [29] and references therein). If such a conjecture is true, then there is a possibility of the existence of self-bound pure strange stars as well. In fact, the conversion of NS to strange star may really be a two step process [30]. The first process involves the deconfinement of nuclear to two-flavor quark matter; the second process deals with the conversion of excess down quarks to strange quarks resulting into a $\beta$-equilibrated charge neutral strange quark matter. There are several mechanisms by which the conversion of strange quark may be triggered at the center of the star [31, 32]. The dominant reaction mechanism by which the strange quark production in quark matter occurs [33] is the non-leptonic weak interaction pro-
cess.

\[ u_1 + d \leftrightarrow u_2 + s \]  \hspace{1cm} (2.1)

Initially when the quark matter is formed, \( \mu_d > \mu_s \), and the above reaction converts excess d quarks to s quarks. But in order to produce chemical equilibrium the semileptonic interactions,

\[ d(s) \rightarrow u + e^- + \bar{\nu}_e \]  \hspace{1cm} (2.2)

\[ u + e^- \rightarrow d(s) + \nu_e \]  \hspace{1cm} (2.3)

play important role along with the above non-leptonic interactions. These imply the \( \beta \)-equilibrium condition \( \mu_d = \mu_u + \mu_e + \mu_{\nu} \); and \( \mu_s = \mu_d \).

Actually, the only conserved charges in the system are the baryon number \( n_B \) and the electric charge \( n_Q \). Since we are assuming neutrinos to leave the system, lepton number is not conserved [10]. Strange chemical potential \( \mu_S \) is zero because strangeness is not conserved. So two of the four chemical potentials (\( \mu_u \), \( \mu_d \), \( \mu_s \) and \( \mu_e \)) are independent. In terms of the baryon chemical potential (\( \mu_B \)), which is equivalent to the quark chemical potential (\( \mu_q = \mu_B / 3 \)), and the charge chemical potential (\( \mu_Q \)) these can be expressed as, \( \mu_u = \mu_q + \frac{2}{3} \mu_Q \); \( \mu_d = \mu_q - \frac{1}{3} \mu_Q \); \( \mu_s = \mu_q - \frac{1}{3} \mu_Q \); \( \mu_e = - \mu_Q \). These conditions are put as constraints in the description of the thermodynamics of a given system through the NJL and PNJL models which will be discussed in the next section.

### 2.2.1 NJL Model

Nambu and Jona-Lasinio (NJL) model is one of the widely used effective models that mimics QCD within certain limits [34]. In its original version, the NJL model was a model of interacting nucleons. The (approximate) chiral symmetry implies (almost) massless fermions on the Lagrangian level. There was no proper explanation of the large nucleon mass without destroying the symmetry. Nambu and Jona-Lasinio first proposed the idea that the mass gap in the Dirac spectrum of the nucleon can be generated in the same way as energy gap of a superconductor in BCS theory. They introduce the Lagrangian with a point like four-fermi interaction as:

\[
\mathcal{L}_{NJL} = \bar{\psi}_N (i \overleftrightarrow{\partial} - m + \mu \gamma^0) \psi_N + \frac{G}{2} [ (\bar{\psi}_N \psi_N)^2 + (\bar{\psi}_N i \gamma_5 \tau \psi_N)^2 ] \]  \hspace{1cm} (2.4)
Here $\Psi_N$ is the nucleon field, $m$ is the small bare mass of the nucleon, $\tau$ is a Pauli matrix acting in isospin space, and $G$ a dimensionful coupling constant. The self energy of the nucleons induced by the interaction actually gives rise to a large mass of the nucleons, even if their bare mass is zero (chiral symmetric limit).

Reinterpretation of NJL model as a quark model was done in [35]. After this reinterpretation the nucleon field $\Psi_N$ is replaced by the quark field $\Psi$ in the NJL Lagrangian. The quark self energy is calculated in Hartree or Hartree-Fock approximation, and this gives a constant shift in the quark mass.

\begin{equation}
M = m_0 + 2iG \int \frac{d^4 p}{(2\pi)^4} \text{Tr} S(p)
\end{equation}

(2.5)

where, $m_0$ is the bare quark mass, $M$ is quark constituent mass and $S(p)$ is the dressed quark propagator.

\begin{equation}
S(p) = (\phi - M)^{-1}
\end{equation}

(2.6)

Restricting the interaction term in scalar and pseudo scalar-isovector channels only, the NJL Lagrangian in terms of quark field $\Psi$ is now

\begin{equation}
\mathcal{L}_{NJL} = \bar{\psi}(i\partial^\mu - m + \mu \gamma^0)\psi + \frac{G}{2} [(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5 \vec{\tau}\psi)^2]
\end{equation}

(2.7)

Under mean field approximation we get the pion condensate $\langle \bar{\psi}i\gamma_5 \vec{\tau}\psi \rangle = 0$ and the Lagrangian can be rewritten in terms of the chiral condensate $\sigma = \langle \bar{\psi}\psi \rangle$ as:

\begin{equation}
\mathcal{L}_{MF} = \bar{\psi}(i\partial^\mu - m_0 + \gamma_0 \mu + G\sigma)\psi - \frac{G}{2} \sigma^2
\end{equation}

(2.8)

where $\sigma$ is the mean field which is basically the trace of the modified fermionic propagator $(S(p))$ corresponding to the modified quark mass $M = m_0 - G\sigma$ i.e.

\begin{equation}
\sigma = -i \int \frac{d^4 p}{(2\pi)^4} \text{Tr} S(p)
\end{equation}

(2.9)

As we see from Eq.(2.5) or Eq.(2.9) the integrals are divergent. When thermodynamic variables are calculated from this model generally a three momentum cut-off $\Lambda$ is used in most of the cases. Thus, NJL model has three parameters i.e. the bare quark mass $m_0$, the three momentum cut-off $\Lambda$ and the coupling constant $G$ which are fixed by fitting the values of pion mass, pion decay constant and the quark condensate.
2.2. FORMALISM

One of the important aspects of QCD namely the chiral symmetry breaking is successfully realized in this model. The massless Lagrangian of NJL model is chirally symmetric, but depending on the coupling strength, quark mass can be generated dynamically by the formation of non zero chiral condensate. Thus the chiral symmetry is broken i.e. $SU(2)_R \times SU(2)_L \rightarrow SU(2)_V$. However, this model does not describe the confinement property of QCD because of the absence of gluon dynamics.

2.2.2 PNJL Model

The incorporation of effect of confinement in NJL model is done by coupling the model with the Polyakov loop. The Polyakov loop extended Nambu Jona-Lasinio (PNJL) model was first introduced in Ref.[36, 37, 38]. In Refs.[39, 14] the model is extended by the inclusion of the Polyakov loop effective potential [40, 41]. While the NJL part is supposed to give the correct chiral properties, the Polyakov loop part should simulate the deconfinement physics. With the success of the Polyakov loop model, people were encouraged to include the dynamical fermions along with the gauge degrees of freedom inside a single theoretical framework. The PNJL model is the result of this endeavor. The initial motivation for the PNJL model was to understand the coincidence of chiral symmetry restoration and deconfinement transition observed in LQCD simulation (see discussions in Ref.[42]).

The Lagrangian of PNJL model consists of two parts: the conventional NJL part and an effective potential for gluons expressed in terms of the traced Polyakov loop. The ordinary derivative in the NJL Lagrangian is replaced by the covariant derivative incorporating the coupling between the gauge field and the quark fields.

The explicit expression for the Lagrangian density for two flavor quark matter is given by [14]:

$$L_{PNJL} = \bar{\psi} (i\gamma_\mu D^\mu - \bar{m}_0 + \mu \gamma^0) \psi + \frac{G}{2} [ (\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5 \vec{\tau}\psi)^2 ] - U(\Phi, \bar{\Phi}, T)$$  \hspace{1cm} (2.10)

where,

\begin{align*}
D^\mu &= \partial^\mu - iA^\mu \\
A_\mu &= \delta_\mu^0 A^0
\end{align*}  \hspace{1cm} (2.11)
2.2. FORMALISM

$A^{\mu}$ is given by $A^{\mu} = g A^{\mu}_{a} \lambda^{a}_{2}$ with gauge coupling constant $g$. $A^{\mu}_{a}$ is the SU(3) gauge field and $\lambda^{a}_{2}$ are the Gell-Mann matrices. $\hat{m}_{0}$ = current quark matrix whose diagonal components are the up and down quark masses, $\psi$ is the quark field and $G$ is the effective coupling constant of the scalar-pseudo scalar four-point interaction of the quark fields.

The effective potential $U(\Phi, \bar{\Phi}, T)$ is expressed in terms of the traced Polyakov loop $\Phi = (\text{Tr}_{c} L) / N_{c}$ and its (charge) conjugate $\bar{\Phi} = (\text{Tr}_{c} L^{\dagger}) / N_{c}$, where $L$ is a matrix in color space given by, $L(\vec{x}) = \mathcal{P} \exp \left[ -i \int_{0}^{\beta} d\tau A_{4}(\vec{x}, \tau) \right]$, $\beta = 1 / T$ is the inverse temperature and $A_{4} = A_{a}^{\mu} \lambda^{a}_{2}$, $A_{a}^{\mu}$ being the temporal component of the Euclidian gluon field. Assuming a constant $A_{a}^{4}$ and the $A_{i}$’s to be zero for $(i = 1, 2, 3)$, $\Phi$ and its conjugate $\bar{\Phi}$, are treated as classical field variables in PNJL model. The choice of $U(\Phi, \bar{\Phi}, T)$ differs among different versions of the model. But, its form is chosen to reproduce the Lattice data.

Following the bosonization procedure the contact interaction term can be written in terms of bosonized $\sigma$ and $\vec{\pi}$ fields and an effective Lagrangian can be constructed [14]:

$$L_{\text{eff}} = - \frac{\sigma^{2} + \vec{\pi}^{2}}{2G} - U(\Phi, \bar{\Phi}, T) - i Tr \ln S^{-1},$$

(2.12)

where $S^{-1}$ is the inverse quark propagator given by,

$$S^{-1} = i \gamma_{\mu} \partial^{\mu} - \gamma_{0} A^{0} - \hat{M}$$

(2.13)

with $\sigma$ and $\vec{\pi}$ are the chiral condensate and pion condensate respectively. For two flavor quark matter in isospin symmetric limit ($m_{u} = m_{d}$) the chiral condensate of u quark and d quark are same, and we write it as: $\sigma_{u} = \sigma_{d} = \sigma$. The constituent quark mass is;

$$\hat{M} = \hat{m}_{0} - \sigma - i \gamma_{5} \vec{\tau} \cdot \vec{\pi}.$$

(2.14)

Under mean field approximation we get the $\vec{\pi}$ condensate equals to zero and the Lagrangian can be rewritten only in terms of the chiral condensate $\sigma$. All the thermodynamic quantities can be calculated from the thermodynamic potential which is obtained from the effective Lagrangian [39, 14]).

Since gluon dynamics in this model is limited to spatially constant temporal background field expressed in terms of Polyakov loop, the model cannot be
applied to high temperature regime where transverse degrees of freedom starts to contribute significantly.

In this particular work we have considered a system of u, d and s quark matter in finite density. When strange quark is included in the model we have to deal with the 2+1 flavor PNJL model Lagrangian. Here, \( m_u = m_d \neq m_s \).

The thermodynamic potential of 2+1 flavor PNJL model for non-zero quark chemical potential is

\[
\Omega = U'(\Phi, \bar{\Phi}, T) + 2g_S \sum_{f=u,d,s} \sigma_f^2 - \frac{g_D}{2} \sigma_u \sigma_d \sigma_s - 6 \sum_{f=u,d,s} \int_0^\Lambda \frac{d^3p}{(2\pi)^3} E_f \Theta(\Lambda - |\vec{p}|)
\]

\[
- 2T \sum_{f=u,d,s} \int_0^\infty \frac{d^3p}{(2\pi)^3} \ln \left[ 1 + 3(\Phi + \bar{\Phi}) e^{-\frac{(E_f - \mu_f)}{T}} \right] e^{-\frac{(E_f - \mu_f)}{T}} + e^{-3\frac{(E_f - \mu_f)}{T}}
\]

\[
- 2T \sum_{f=u,d,s} \int_0^\infty \frac{d^3p}{(2\pi)^3} \ln \left[ 1 + 3(\Phi + \bar{\Phi}) e^{-\frac{(E_f + \mu_f)}{T}} \right] e^{-\frac{(E_f + \mu_f)}{T}} + e^{-3\frac{(E_f + \mu_f)}{T}}
\]

(2.15)

where, \( \sigma_f = \langle \bar{\psi}_f \psi_f \rangle \) and \( E_f = \sqrt{\vec{p}^2 + M_f^2} \) with, \( M_f = m_f - 2g_S \sigma_f + \frac{g_D}{2} \sigma_f \sigma_f \).

\( U'(\Phi, \bar{\Phi}, T) \) is so chosen to have exact \( Z(3) \) center symmetry and is given by,

\[
\frac{U'(\Phi, \bar{\Phi})}{T^4} = \frac{U(\Phi, \bar{\Phi})}{T^4} - \kappa \ln [J(\Phi, \bar{\Phi})],
\]

(2.16)

where,

\[
\frac{U(\Phi, \bar{\Phi})}{T^4} = -\frac{b_2(T)}{2} \Phi \bar{\Phi} - \frac{b_3}{6} (\Phi^3 + \bar{\Phi}^3) + \frac{b_4}{4} (\Phi \bar{\Phi})^2
\]

(2.17)

with \( b_2(T) = a_0 + a_1 \left( \frac{T}{T_0} \right) + a_2 \left( \frac{T}{T_0} \right)^2 + a_3 \left( \frac{T}{T_0} \right)^3 \), and \( J[\Phi, \bar{\Phi}] = (27/24\pi^2)(1 - 6 \Phi \bar{\Phi} + 4(\Phi^3 + \bar{\Phi}^3) - 3(\Phi \bar{\Phi})^2) \) is the Vandermonde determinant. A fit of the coefficients \( a_i \), \( b_i \) is performed to reproduce the pure-gauge Lattice data and \( T_0 = 270 \) MeV is adopted in our work. Finally \( \kappa = 0.2 \) is used which gives reasonable values for pressure for the temperature range used here at zero baryon density as compared to full Lattice QCD computations. In the present work we have considered a system of u, d, s quarks with electrons. For simplicity, electrons are considered as free non-interacting fermions [10] and the corresponding thermodynamic potential is,

\[
\Omega_e = -\frac{\mu_e^4}{12\pi^2} + \frac{\mu_e^2 T^2}{6} + \frac{7\pi^2 T^4}{180}
\]

(2.18)

where, \( \mu_e \) is the electron chemical potential.
2.3 Results and Discussions

![Graph showing constituent quark masses as functions of $\mu_q$ for (a) $\mu_e = 0$ MeV and (b) $\mu_e = 40$ MeV, at $T = 50$ MeV.](image_url)

Figure 2.1: Constituent quark masses as functions of $\mu_q$ for (a) $\mu_e = 0$ MeV and (b) $\mu_e = 40$ MeV, at $T = 50$ MeV.
2.3. RESULTS AND DISCUSSIONS

The thermodynamic potential $\Omega$ is extremised with respect to the scalar fields under the condition $\mu_d = \mu_u + \mu_e$ and $\mu_s = \mu_d$. The equations of motions for the mean fields $\sigma_u$, $\sigma_d$, $\sigma_s$, $\Phi$ and $\bar{\Phi}$ for any given values of temperature $T$, quark chemical potential $\mu_q$ and electron chemical potential $\mu_e$ are determined through the coupled equations,

$$\frac{\partial \Omega}{\partial \sigma_u} = 0, \quad \frac{\partial \Omega}{\partial \sigma_d} = 0, \quad \frac{\partial \Omega}{\partial \sigma_s} = 0, \quad \frac{\partial \Omega}{\partial \Phi} = 0, \quad \frac{\partial \Omega}{\partial \bar{\Phi}} = 0. \quad (2.19)$$

In Fig. 2.1, we show the typical variation of constituent quark masses as a function of $\mu_q$, for two representative values of electron chemical potential $\mu_e = 0$ MeV and $\mu_e = 40$ MeV, with a fixed temperature $T = 50$ MeV. At this temperature, both $m_u$ and $m_s$ in the PNJL model, show a discontinuous jump at around $\mu_q = 350$ MeV indicating a first order phase transition. The jump in $m_s$ is smaller, and is actually a manifestation of chiral transition in the two flavor sector, arising due to the coupling of the strange condensate to the light flavor condensates. On the other hand in the NJL model the quark masses show a smooth variation at this temperature, indicating a crossover. It is important to note that the constituent mass of the strange quark goes down to the current mass at a larger $\mu_q$ in both the models, leading to sort of a second crossover at around $\mu_q = 500$ MeV. This will have important implications for some of the thermodynamic observables as we discuss below.

2.3.1 Phase Diagram

The phase diagrams for NJL and PNJL models are obtained from the behavior of the mean fields, and are shown in Fig. 2.2(a) and Fig. 2.2(b) for $\mu_e = 0$ MeV and $\mu_e = 40$ MeV respectively.

As is evident from the figures, the broad features of the phase diagrams remain same in all cases. The difference between the NJL and PNJL models arise mainly due to the Polyakov loop, whose presence is primarily responsible for raising the transition/crossover temperature in the PNJL model. Thus the CEP for PNJL model occurs at slightly higher $T$ and lower $\mu_q$ compared to NJL model. Note that the phase diagram with $\mu_e = 0$ MeV is identical to the case without $\beta-$equilibrium [17]. This is because the minimization conditions (2.19) are independent of the electrons except through the $\beta-$equilibrium conditions. However this is true only so far as the phase diagram is concerned. Various other physical quantities are found to differ even for $\mu_e = 0$ as dis-
2.3. RESULTS AND DISCUSSIONS

Figure 2.2: Comparison of phase diagram in NJL and PNJL model at \( \beta \)-equilibrium for (a) \( \mu_e=0 \); (b) \( \mu_e=40 \). The solid circle and square represent the CEP for NJL and PNJL model respectively.
cussed below. For non-zero $\mu_e$ we find a slight lowering of the temperature for the CEP by about 10 MeV. This is an important quantitative difference between the physics of neutron stars and that of compressed baryonic matter created in the laboratory. In this context we would like to mention that in Ref. [43] QCD phase diagram has been studied both for isospin asymmetric and symmetric situations. The authors considered a two equation of state model where non-linear walecka model was used to describe hadronic sector and (P)NJL model for quark sector. It has been shown in [43] that CEP remain unaffected by the isospin asymmetry.

### 2.3.2 Equation of State

The system under investigation can be characterized primarily by the behavior of the EOS. Generally for a many body system, increase in pressure at large densities is indicative of a repulsive behavior of the interaction at large densities (large $\mu_q$) or short distances and an attractive nature at larger distances or lower densities [44, 45]. Consequentially the energy density will show similar behavior. The resulting EOS given by the variation of pressure $P = -\Omega$ with energy density $\epsilon = -T^2 \frac{\partial(\Omega/T)}{\partial T}$, is shown in Fig. 2.3(a) at $T = 50$ MeV, for both NJL and PNJL models, for the two representative electron chemical potentials. Here again for the PNJL model there exists a discontinuity due to a first order nature of the transition, whereas for NJL model the EOS is smooth. Beyond this region a smaller steepening in $\epsilon$ is visible, that occurs due to second crossover feature noted above as the strange quark condensate starts to melt. A possible implication for this small surge may be that in a strange quark star, at a given central density, the pressure would be somewhat lesser than the situation without this surge.

Generally, the EOS can be used to study the dynamics of neutron star and that of heavy-ion collisions through the respective flow equations. The main differences would be due to the presence of $\beta$–equilibrium and the back reaction of the non-trivial space-time metric on the EOS for neutron stars. Such a comprehensive comparative study will be taken up in a later work.

In Fig. 2.3(b), the variation of the isentropic speed of sound squared $c_s^2 = \frac{\partial P}{\partial \epsilon}$ is plotted against $\mu_q$ at $T = 50$ MeV. In the NJL model the $c_s^2$ starts from a non-zero value, steadily decreases and then shows a sharp fall around the crossover region at $\mu_q \sim 320$ MeV. This is followed by a sharp rise, a dip and then approaches the ideal gas value of 1/3. In contrast the $c_s^2$ in the PNJL
model starting from the ideal gas value remains almost constant up to $\mu_q \sim 200$ MeV and then falls sharply to almost zero. This is followed by a discontinuous
Figure 2.4: Comparison of pressures quark matter with and without strangeness at $\mu_e = 0$; (a) $T=50$ MeV and (b) $T=100$ MeV in PNJL model.

jump, a similar dip at $\mu_q \sim 500$ MeV and a gradual approach to a non-zero value quite different from the ideal gas limit.
2.3. RESULTS AND DISCUSSIONS

The difference at $\mu_q = 0$ MeV occurs specifically due to the Polyakov loop which suppresses any quark-like quasi-particles. As a result the $c_s^2$ is completely determined by the ideal electron gas. On the other hand those quasi-particles with heavy constituent masses tend to lower the $c_s^2$ in the NJL model. The difference at the transition region is again mainly due to the discontinuous phase transition in PNJL model which leads to $c_s^2$ almost going down to zero, and a crossover in the NJL model where $c_s^2$ is small but non-zero. In [46] it was noted that for two conserved charges, pressure is not constant any more in the mixed phase, rather its variation becomes slower, resulting in a smaller but non-zero speed of sound. In our computation though we do not find $c_s^2$ exactly equal to zero, but to confirm such an effect we need a full space-time simulation of the mixed phase through the process of bubble nucleation which is beyond the scope of the present work.

In both the models the dip around $\mu_q = 500$ MeV arises due to the behavior of the strange quark condensate as discussed earlier. If it were possible to achieve such extremely high densities in heavy-ion experiments, then such a dip would slow down the flow and would result in a larger fire ball life time. At even higher $\mu_q$ the $c_s^2$ in NJL model approaches the free field limit quite fast but in the PNJL model it still remains quite low due to the non-trivial interaction brought in by the Polyakov loop. It would be interesting to study the implication of slow speed of sound inside the core of a neutron star.

If Witten’s conjecture is true, then the free energy should be minimum in a strange quark matter compared to the matter without strangeness. We have checked by comparing the pressures of a 2+1 flavor and 2 flavor quark matter separately for two different temperatures in PNJL model at $\mu_e = 0$ as shown in Fig.2.4. As we see the pressure of 2+1 flavor is larger than that of 2 flavor quark matter. Since pressure is negative of thermodynamic potential (free energy), we can say that 2+1 flavor matter is more stable.

2.3.3 Specific Heat and Compressibility

Specific heat and isothermal compressibility are two most important thermodynamic quantities which show critical behaviors near the phase boundary. They should reflect the large fluctuation near critical point because they are proportional to the fluctuations of the entropy and the density, respectively. Commensurate with the relative stiffening of the equation of state we find that the compressibility $\kappa = \frac{1}{n_q} \left( \frac{\partial n_q}{\partial \mu_q} \right)_T$, where $n_q$ is the quark number density, be-
Figure 2.5: (a) Variation of compressibility $\kappa$ with $\mu_q$. The peak around $\mu_q = 500$ MeV is shown in the inset where $\kappa$ represents the compressibility in both NJL and PNJL models. (b) Variation of specific heat scaled by its Stefan Boltzmann value.

haves accordingly. While $\kappa$ in the NJL model is found to be higher than that of the PNJL model in the hadronic phase, it is just the opposite in the partonic
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The variation of the specific heat $C_V = T \left( \frac{\partial s}{\partial T} \right)_V$, where $s = \left( \frac{\partial F}{\partial \mu} \right)$ is the entropy density of the system, is shown in Fig. 2.5(b). For a crossover (here in NJL model) the specific heat shows a peak. For a first order transition (here in the PNJL model) the $C_V$ is discontinuous. Also we see that the specific heat in the PNJL model is lower than that in the NJL model for a general variation of $\mu_q$ and $\mu_e$. A system described by the PNJL model is thus less susceptible to changing temperature than that described by the NJL model.

The variation of compressibility and specific heat shown here also captures the signature of a phase transition in the PNJL model and a crossover in the NJL model. Both compressibility as well as specific heat are second derivatives of $\Omega$ and represent respectively the quark number fluctuations and energy fluctuations [44]. Discontinuity in compressibility as well as specific heat indicates a first order phase transition for the PNJL model. At $\mu_q \sim 500$ MeV, both the models exhibit a small peak due to the onset of melting of the strange quark condensate.

2.3.4 Quark Number Densities and Charge Neutral Contours

We now consider the net charge density given by $n_Q = \frac{2}{3} n_u - \frac{1}{3} n_d - \frac{1}{3} n_s - n_e$, where the number density of individual quarks and electrons are obtained from the relations, $n_u = \frac{\partial \Omega}{\partial \mu_u}$, $n_d = \frac{\partial \Omega}{\partial \mu_d}$, $n_s = \frac{\partial \Omega}{\partial \mu_s}$, and $n_e = \frac{\partial \Omega}{\partial \mu_e}$. For $\mu_e = 0$, $n_e = 0$ and $\mu_u = \mu_d = \mu_S$. Since masses of light flavors (u and d) are equal, we have $n_u = n_d$ in the whole chemical potential range. However, $n_s$ is very small at low $\mu_q$ because of the large mass of strange quarks. At large $\mu_q$, the number density $n_s$ of strange quarks become almost equal to the light quark number densities as the constituent masses of strange quarks are reduced significantly. So the net charge density $n_Q$ will be close to zero and the system will become charge neutral asymptotically as shown in Fig. 2.6(a). At small $\mu_q$, $n_Q << 1$ as the individual number densities themselves are exceedingly small. In fact this feature continues till the transition region where the light constituent quark masses drop sharply giving rise to non-zero number densities. Therefore $n_Q$ shows a non-monotonic behavior, rising from almost zero it reaches a maxima at certain $\mu_q$ determined mainly by the melting of the strange quark condensate and thereafter decreases steadily towards zero.
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Figure 2.6: Total charge and quark number densities scaled by $T^3$ as a function of quark chemical potential in the PNJL model

For non-zero $\mu_e$, even at non-zero moderate values of $\mu_q$ one can expect charge neutral configuration. In the present study we have taken a constant

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value of the electron chemical potential so at a particular temperature the electron number density is fixed and it is negligible compared to the quark number densities at high $\mu_q$. For small $\mu_q$, it is the $n_e$ which dominates and
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keeps $n_Q$ negative. As soon as $n_u$ becomes large with increasing $\mu_q$, $n_Q$ goes through zero and becomes positive. Now since $\mu_s$ and $\mu_d$ are greater than $\mu_u$ due to $\beta$–equilibrium, both $n_s$ and $n_d$ start to grow faster with the increase of $\mu_q$. Finally at some $\mu_q$ the net charge becomes zero due to the mutual cancellation of $n_u$, $n_d$, and $n_s$, and thereafter it remains negative for higher $\mu_q$ as $d$ and $s$ quarks overwhelms the positively charged $u$ quark. So the quarks alone are responsible for the occurrence of charge neutrality at high $\mu_q$. The electron fraction ($\frac{n_e}{n_B}$) basically diminishes to zero as one goes to higher chemical potential as shown in Fig.2.7.

The behavior of $n_Q$ is similar for both PNJL and NJL model though the actual values of the various chemical potentials for the charge neutrality conditions vary.

Given that one may be interested in the charge neutral condition e.g. in the case of neutron stars, in Fig. 2.8 the charge neutral trajectories for NJL model are compared with those of PNJL model along with the phase diagrams. The trajectories are quite interesting in that they are closed ones pinned on to the $\mu_q$ axis. They start off close to $\mu_q = M_{\text{vac}}$, the constituent quark mass in the model in vacuum. They make an excursion in the $T - \mu_q$ plane and join back at a higher $\mu_q$. There is a maximum temperature $T_Q$ up to which the trajectory goes. Beyond this temperature no charge neutrality is possible. Below this temperature we have essentially two values of $\mu_q$ where charge neutrality occurs. There are significant differences between the contours of NJL and PNJL model in the hadronic phase. However beyond the transition and inside the deconfined region, the differences subside as the Polyakov loop relaxes the confining effect leading to the PNJL model behaving in a similar way to that of the NJL model.

The behavior of the charge neutral contour is highly dependent on $\mu_e$. With increasing $\mu_e$ the contour gradually closes in towards the transition line. For a given $T$ there are two $\mu_q$ values where charge neutrality is obtained – one on the hadronic side and one on the partonic side. As a result of the closing in of the contour, these two values come closer to the transition line from opposite sides with increasing $\mu_e$. Higher the $\mu_e$ closer we are to the transition region. Now suppose we are looking for an isothermal evolution of a system, or the isothermal configuration of a system such as the NS. Given the constraint of charge neutrality we would have a varying $\mu_e$ as the density profile changes. Similarly if $\mu_e$ is held constant then charge neutrality would not allow the temperature to remain fixed throughout and the evolution would take place
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Figure 2.8: Comparison of charge neutral trajectory in NJL and PNJL model at (a) $\mu_e=10$ ; (b) $\mu_e=40$

along the contours described above. So in general a combination of $T$ and $\mu_e$ is expected to maintain charge neutrality in a given system. A practical
picture of NS which has a profile of low density crust to gradual increase in
density to have a highly condensed core would be that there is a complex
profile for temperature and $\mu_e$ inside the NS. In fact if there exist a hadron-
parton boundary, it may be either with high temperature or high electron
density.

2.3.5 Constant Baryon Number Density ($\frac{n_B}{n_0}$) Contours

To contemplate this scenario in the light of the baryon densities achieved we
plot the contours for constant baryon densities, scaled by the normal nuclear
matter density ($n_0 = 0.15 \text{fm}^{-3}$) in Fig. 2.9 for $\mu_e = 40 \text{ MeV}$. The charge neutral
trajectories are also plotted along with the phase boundary. Obviously with
increasing baryon (quark) chemical potential baryon density would increase.
What is interesting is the fact that high densities can also occur for lower
chemical potential if the temperature is higher. For both NJL and PNJL model
at and above 3 times nuclear matter density the matter seems to be always in
the partonic phase. A little below this density matter may be in partonic phase
if it is at high temperature otherwise in the hadronic phase at low temperature.
Thus the actual trajectory on the phase diagram would determine whether
a hadron-parton boundary in the NS is in the mixed phase or in a state of
crossover. Within the range of the charge neutral contour we find the baryon
density increasing from a very small value to almost 10 times the normal
nuclear matter density. If $\mu_e$ is increased further the baryon densities would
also be much higher for a given $\mu_q$. So if we assume local charge neutrality as
well as isothermal profile along a hadron-parton phase boundary, the baryon
density close to the phase boundary may be too large. On the other hand
for reasonable densities close to the phase boundary it would be impossible
to maintain local charge neutrality along an isothermal curve. In that case it
may be possible that the charge neutrality condition takes the system around
the CEP to hold on to a reasonable density in the phase boundary region. This
leads us to speculate that the transition in a NS itself may also be a cross-over,
quite unlike the picture in most of the studies of NS.

2.3.6 Contour of Net Strangeness Fraction ($\frac{n_s}{n_B}$)

The net strangeness fraction ($n_s/n_B$) along with $n_B/n_0$ is shown in Fig. 2.10.
For a given temperature, there is a critical $\mu_q$ below which there is no net
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Figure 2.9: The contour of scaled baryon number density $n_B/n_0$; (scaled by normal nuclear matter density) along with phase diagram at $\mu_e=40$ for (a) NJL model and (b) for PNJL model; (From left $n_B/n_0 = 0.5, 1, 3, 5, 10$ respectively)
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Figure 2.10: The contour of net strangeness fraction \( n_s/n_B \) along with \( n_B/n_0 \) at \( \mu_e=40 \) for (a) NJL model and (b) PNJL model; the values of \( n_B/n_0 \) are 0.5, 1, 3, 5 and 10 (from left).

strangeness formation. At the critical \( \mu_q \) a non-zero \( n_s/n_B \) occurs depend-
ing on the $T$. This strangeness fraction continues to appear at lower $T$ for some higher $\mu_q$. So a given strangeness fraction can occur only upto a certain critical temperature. The intersection of lines of constant baryon density and strangeness fraction indicates the possibility of evolution of a system to higher (lower) strangeness fraction with increase (decrease) of $T$ at a constant density. In the range of 5 - 10 times nuclear matter density we see that the strangeness fraction is increasing significantly towards unity indicating a possibility of formation of quark matter with almost equal number of u, d and s quarks. Similar results have also been found in other model studies [8]. Chunks of matter with $n_s/n_B = 1$, called strangelets is expected to be stable (metastable up to weak decay) relative to nuclear matter in vacuum [47]. Investigation of these and various other properties of strange matter would be undertaken in future.

2.3.7 Isentropic Trajectories

Usually the hydrodynamic evolution of a system is expected to follow certain adiabat along which the entropy per baryon number ($s/n_B$) is a constant quantity. Among the various adiabats the system would choose one given its initial conditions. In the context of NS, a fixed entropy per baryon is expected in a proto-neutron star as well which is very different from a cold neutron star. It is usually hot and rich in leptons i.e. electrons and trapped neutrinos. Few seconds after birth, the matter in the core of a hot NS has almost constant lepton fraction (0.3 -0.4) and entropy per baryon (1 - 2, in units of Boltzmann constant) [48, 49]. The question as to whether the later evolution of the NS can be described to be one close to an adiabat is a matter of debate. On the other hand the commonly used approach of an isothermal evolution looks not quite favorable according to the above discussion on charge neutrality condition.

The behavior of $s/n_B$ in a plasma and in a hadron gas was analyzed within the framework of an extended Bag model by [50]. A case study of such adiabats was done in NJL model in [51]. It was found that unlike the prescription of adiabats meeting at the CEP given by [52], they meet close to the critical value of $\mu_q$ at $T = 0$ which is incidentally equal to the constituent quark mass $M_{vac}$ in the model in vacuum. It was argued in [51] that as $T \to 0$, $s \to 0$ by the third law of thermodynamics. Hence in order to keep $s/n_B$ constant, $n_B$ should go to zero. This condition is satisfied when $\mu_q = M_{vac}$ of the theory. These authors also found similar results for the linear sigma model. In the
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PNJL model the introduction of Polyakov loop produced a slight change in the configuration of the adiabats [53]. The constraint on the strangeness number to be zero also was found not to have a very significant effect [54].

![Diagram showing isentropic trajectories and phase diagrams](image)

Figure 2.11: The isentropic trajectories along with phase diagram for (a) at $\mu_e = 0$, 2+1-flavor PNJL, (b) at $\mu_e = 40$, 2+1-flavor PNJL, (c) at $\mu_e = 0$, 2-flavor PNJL and (d) at $\mu_e = 40$, 2+1-flavor NJL model. $s/n_B = 300, 100, 30, 10, 5, 3.5$ (from left).

The corresponding picture of isentropic trajectories with the condition of $\beta$–equilibrium is shown in Fig. 2.11. Four cases are depicted here. Fig. 2.11(a) and Fig. 2.11(b) show the cases with 2+1 PNJL model at $\mu_e = 0$ MeV and $\mu_e = 40$ MeV respectively. From these two figures we find that the electron density does not have a significant effect on the isentropic trajectories. This means that the quark degrees of freedom seem to have dominant effect in entropy over the electrons. The case with $n_s = 0$, i.e. effectively for a 2–flavor system is shown in Fig. 2.11(c). In general the situation is similar. For small $\mu_q$ there is almost no change in Fig. 2.11(c) and Fig. 2.11(a) as both the cases are identical to 2 flavors. At intermediate values strange quarks start to pop out.
Now the contours in Fig. 2.11(c) appear to be shifted and bent towards higher $\mu_q$. This is because for 2 flavors, a given baryon number density appears at a higher $\mu_q$ than that for 2+1 flavors. Hence to get a fixed $s/n_B$ the $\mu_q$ required is also higher. At even higher $\mu_q$ the thermal effects are negligible and hence $s/n_B$ become almost independent of the degrees of freedom. Thus again the contours become identical.

The results in the NJL model are significantly different from that of the PNJL model as can be seen by comparing the PNJL results with that of the NJL model shown in Fig. 2.11(d). Even for low $T$ and $\mu_q$ there is a significant entropy generation as there is no Polyakov loop to subdue the same. Similar differences continue to appear even in the partonic phase.

Considering a system that has been compressed to a few times the nuclear matter density it can try to relax back to lower densities along the adiabats. Interestingly the isentropic trajectories in the high density domain seem to behave as isothermals in the PNJL model. However as soon as the system converts into the hadronic phase, the adiabats drive it to a steep fall in temperature. We would like to mention that for a hadronic proto neutron star with beta-equilibrated nuclear matter with nucleons and leptons in the stellar core, the EOS evaluated in Bruckner–Bethe–Goldstone theory, was found to be similar for both isothermal and isentropic profiles [55].

In Ref.[56] isentropic trajectories were found in PNJL model for two different sets of parameters which were obtained by using the cutoff in the zero temperature integrals only (case I) and in all integrals (Case II). We in the present study followed the first method (case I) for regularisation and our result for PNJL model is quite similar to those studied in [56].

While the possibility that a neutron star can be described using adiabatic conditions is a point to be pondered about, we note here that an excursion of the phase diagram of a $\beta$–equilibrated matter is highly possible even in heavy-ion collisions to some extent. This is because both the isentropic lines as well as the characteristics of the phase boundary are quite similar for a wide variation of $\mu_e$ and $\mu_q$. At the same time one should remember that in the laboratory conditions $n_s$ is strictly zero. Anyway if a system is found to have travelled along an adiabat with $s/n_B \simeq 3$ to 4, it has most probably traversed close to the CEP. One can therefore try to correlate different observables like the enhancement of fluctuations of conserved charges and $s/n_B$ to be in the above range to study the approach towards the CEP in heavy-ion collisions.
2.4 Summary

In this chapter we have studied the 2+1 flavor strongly interacting matter under the condition of $\beta$–equilibrium. We have presented a comparative study of NJL versus PNJL model. The phase diagrams in these two models are broadly similar, but quantitatively somewhat different. The presence of the Polyakov loop delays the transition for larger values of temperature for a given quark chemical potential. As a result the CEP in the PNJL model is almost twice as hot as that in the NJL model. We have illustrated characteristics of the phase diagram with the behavior of some thermodynamic quantities like the constituent mass, compressibility, specific heat, speed of sound and the equation of state for $\mu_e = 0$ MeV and $\mu_e = 40$ MeV at $T=50$ MeV. We found striking differences between the NJL and PNJL model in terms of the softness of the equation of state in the hadronic and partonic phases.

The behavior of electric charge and baryon densities in the two models also differ in the hadronic phases to some extent. The differences become less with increasing electron density. We explained how the charge neutral trajectory is important in deciding the path along which the core of NS can change from hadronic to quark phase. For all values of $\mu_e$ we find that the contours are all closed ones and give a restricted range of temperature and densities that are allowed. We speculated a possible scenario in which the quark-hadron transition in a NS would be a crossover. Again the baryon density contours seemed to suggest that if a system has baryon density three times the nuclear matter density it is quite surely in the partonic phase. We also found that the strangeness fraction increases steadily with increasing baryon density implying a possibility of having a strange NS.

The isentropic trajectories were obtained along which a system in hydrodynamic equilibrium is expected to evolve. The adiabats flow down from high temperature and low density towards low temperature and $\mu_q = M_{\text{vac}}$, the constituent quark mass in vacuum. The adiabats then steeply rise along the transition line, thereafter goes towards higher densities with almost a constant slope. For small $s/n_B$ ratio the slope is so small that the isentropic trajectories almost become isothermal trajectories as well.

To summarize the scenario inside neutron stars we note that inside a newly born NS the temperature drops very quickly and gives rise to a system of low temperature nucleonic matter which may also be populated by hyperons and strange baryons due to high density near the core. The star is assumed to
be $\beta$–equilibrated and charge neutral. Now it is possible that due to some reason, e.g. sudden spin down, this nucleonic matter will start getting converted to predominantly two flavor quark matter within strong interaction time scale. This transition would start at the center and a conversion front moving outward will convert much of the central region of the star. Along the path of the conversion front, each point inside the star may lie on an isentropic trajectory. Gradually this system of predominantly 2 flavor quark matter will get converted to strange quark matter through weak interactions and finally a $\beta$–equilibrated charge neutral strange quark matter will be produced. The strangeness production occurs mainly through non-leptonic decay [33]. the system is expected to lie on a constant density line and move towards the point with highest strangeness possible at that density. Finally the semi-leptonic processes will take over and system will then evolve along a $\beta$–equilibrated charge neutral contour.

The natural extension of the work is to obtain the detailed evolution of a family of neutron stars starting with different initial conditions and gravity effects incorporated. We hope to study this in future. It would also be important to consider colored exotic states like diquarks [57] that may arise at high densities.

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