Chapter 4

APPLICATION OF SOFT DENSE SETS TO SOFT CONTINUITY (SOFT UNIFORM CONTINUITY) AND SOFT EQUICONTINUITY (SOFT UNIFORM EQUICONTINUITY)

4.1 Introduction

In this chapter, we continue the study of soft sets from our previous chapter. Das and Samanta in [10], introduced the notion of soft real sets, soft real numbers. In [11], they introduced the concept of soft metric spaces by using the notion of soft points and investigated some basic properties of these spaces. In [50], maps between metric spaces and dense subsets in metric spaces are studied. Also, some map gluing theorems which describe the continuity (uniform continuity) of maps between metric spaces in terms of dense subsets of the domain were obtained. Characterizations of continuity of a map between metric spaces, in terms of convergent sequences taken from dense subset of the domain of the map and in terms of restriction of the map to a dense subset of the domain are given. Also, concept of equicontinuity of maps have been studied in the literature.

The above study of maps between metric spaces lead us to study of soft maps and soft dense subsets in soft metric spaces. In this chapter, we give characterization of soft continuity (soft uniform continuity) and soft equicontinuity (soft uniform equicontinuity) in terms of soft dense subsets. We discuss soft pointwise convergence of sequence of soft maps which are soft pointwise equicontinuous when co-domain is soft complete space. We also discuss soft uniform convergence of soft maps and prove that limit of soft uniformly convergent sequence of soft continuous maps is soft continuous.

In section 2, we first define soft dense subsets and give characterization of soft continuous maps between soft metric spaces in terms of soft dense subsets
of domain space [Theorem 4.2.2 below]. By using this result, we get more map
for gluing theorems of soft continuity of soft maps between soft metric spaces
[Theorem 4.2.3 and Theorem 4.2.4 below]. We also give characterization of
soft continuity in terms of convergent sequence of soft points in soft dense
subset of domain of the soft map [Theorem 4.2.5 below]. As corollaries to
Theorem 4.2.5 we obtain (i) an extension theorem for a soft continuous map
which is defined on soft dense subsets of the domain [Corollary 4.2.1 below], (ii)
soft continuity of a soft map in terms of the soft continuity of its restrictions to
members of a soft cover of the domain having soft dense intersection [Corollary
4.2.2].

In section 3, We introduce the definition of soft equicontinuity at a soft
to and soft pointwise equicontinuity on a soft subset of an absolute soft
set \( \tilde{X} \). An example of soft equicontinuous set is also given [Example 4.3.1
below]. We define soft pointwise convergence of sequence of soft mappings
from \( \tilde{X} \) to \( \tilde{Y} \) with an example [Definition 4.3.2 and Example 4.3.2 below].
We discuss soft continuity of soft pointwise limit of a sequence of soft maps
when the family of the given soft map is soft equicontinuous [Theorem 4.3.1
below] and soft pointwise convergence of a sequence of soft maps which are
soft pointwise equicontinuous, when codomain space is soft complete [Theorem
4.3.2 below]. Finally, we give characterization of soft pointwise equicontinuity
of soft map in terms of convergence sequence of soft points in soft dense subset
[Theorem 4.3.3 below]. From Theorem 4.3.3 and Theorem 4.2.5 we get another
characterization of soft continuous functions [Theorem 4.3.4 below].

In section 4, we give characterization of soft uniformly continuous maps
in terms of sequences of soft points [Theorem 4.4.1 below] and in terms of its
restriction to soft sequentially compact soft subsets of domain [Theorem 4.4.2
below]. Next, we obtain a map gluing theorem on soft uniformly continuous
maps in terms of soft dense subsets [Theorem 4.4.3 below]. We give another
characterizations of soft uniform continuity of a soft map [Theorem 4.4.4 and
4.4.5 Theorem below]. We obtain analogs of Corollary and for soft uniform
continuous map [Corollary 4.4.2 and Corollary 4.4.3 below].

In section 5, we introduce the definition of soft uniformly equicontinuous
family of soft maps. In Theorem 4.5.1 below, we show that limit of uniformly
convergent sequence of soft continuous maps is soft continuous and in Theorem
4.5.2 below, we show that a soft equicontinuous family of soft maps with soft
sequentially compact domain is soft uniformly equicontinuous. Theorem 4.5.3
below is the analogous result of Theorem 4.3.3 for soft uniformly equicontinuity.

In section 6, We give an example to show that converse of the Theorem
4.3.1 need not be true [Example 4.6.1 below]. We also give an example to show
that assumption of soft completeness in Theorem 4.3.2 cannot be dropped [Example 4.6.2 below].

We begin with the following preliminaries.

Let $\mathbb{R}$ be the set of real numbers and $\mathcal{B}(\mathbb{R})$ be the collection of all non-empty bounded subsets of $\mathbb{R}$ and $A$ be the set of parameters. Then a mapping $F : A \rightarrow \mathcal{B}(\mathbb{R})$ is called a soft real set. It is denoted by $(F,A)$. If specifically $(F,A)$ is a singleton soft set i.e. there exists $x \in \mathbb{R}$ such that $F(a) \subseteq \{x\}$ for all $a$ in $A$, then after identifying $(F,A)$ with the corresponding soft element, it will be called a soft real number.

We use notations $\tilde{r}, \tilde{s}, \tilde{t}$ to denote soft real numbers whereas $\bar{r}, \bar{s}, \bar{t}$ will denote a particular type of soft real numbers such that $\bar{r}(\lambda) = \{r\}$, for all $\lambda \in A$. In such a case we also use $\tilde{r}(\lambda) = r$ instead of $\bar{r}(\lambda) = \{r\}$. For example, $\bar{0}$ is the soft real number where $\bar{0}(\lambda) = 0$, for all $\lambda \in A$.

Definition 4.1.1. [11] For two soft real numbers $\tilde{r}, \tilde{s}$ we define,

1. $\tilde{r} \leq \tilde{s}$ if $\tilde{r}(\lambda) \leq \tilde{s}(\lambda)$, for all $\lambda \in A$;
2. $\tilde{r} \geq \tilde{s}$ if $\tilde{r}(\lambda) \geq \tilde{s}(\lambda)$, for all $\lambda \in A$;
3. $\tilde{r} < \tilde{s}$ if $\tilde{r}(\lambda) < \tilde{s}(\lambda)$, for all $\lambda \in A$;
4. $\tilde{r} > \tilde{s}$ if $\tilde{r}(\lambda) > \tilde{s}(\lambda)$, for all $\lambda \in A$.

Let $\tilde{X}$ be the absolute soft set. Let $SP(\tilde{X})$ be the collection of all soft points of $\tilde{X}$ and $R(A)^*$ denote the set of all non-negative soft real numbers.

Definition 4.1.2. [11]

1. A mapping $\tilde{d} : SP(\tilde{X}) \times SP(\tilde{X}) \rightarrow R(A)^*$ is said to be a soft metric on the soft set $\tilde{X}$ if $\tilde{d}$ satisfies the following conditions:
   
   (a) $\tilde{d}(P_x^b, P_a^x) \geq 0$, for all $P_x^b, P_a^x \subseteq \tilde{X}$.
   (b) $\tilde{d}(P_x^b, P_a^x) = 0$ if and only if $P_x^b = P_a^x$.
   (c) $\tilde{d}(P_x^b, P_a^x) = \tilde{d}(P_a^x, P_x^b)$ for all $P_x^b, P_a^x \subseteq \tilde{X}$.
   (d) For all $P_a^x, P_x^b, P_c^z \in \tilde{X}$, $\tilde{d}(P_a^x, P_c^z) \leq \tilde{d}(P_a^x, P_x^b) + \tilde{d}(P_x^b, P_c^z)$.

The soft set $\tilde{X}$ with a soft metric $\tilde{d}$ on $\tilde{X}$ is called a soft metric space and denoted by $(\tilde{X}, \tilde{d}, A)$.

2. Let $(\tilde{X}, \tilde{d}, A)$ be a soft metric space and $(Y, A)$ be a non-null soft subset of $\tilde{X}$. Then the mapping $\tilde{d}_Y : SP(Y, A) \times SP(Y, A) \rightarrow R(A)^*$ given by
\[ \tilde{d}_Y(P^y_a, P^z_a) = \tilde{d}(P^y_b, P^z_a) \text{ for all } P^y_b, P^z_a \in (Y, A) \] is a soft metric on \((Y, A)\).

This metric \(\tilde{d}_Y\) is known as the relative metric induced on \((Y, A)\) by \(\tilde{d}\). The soft metric \((\tilde{Y}, \tilde{d}_Y, A)\) is called a metric subspace or simply a subspace of the soft metric space \((\tilde{X}, \tilde{d}, A)\).

3. Let \((\tilde{X}, \tilde{d}, A)\) be a soft metric space and \(\tilde{r}\) be a non-negative soft real number. For any \(P^x_a \in \tilde{X}\), by an open ball with centre \(P^x_a\) and radius \(\tilde{r}\), we mean the collection of soft points of \(\tilde{X}\) satisfying \(\tilde{d}(P^x_a, P^y_b) \lesssim \tilde{r}\).

The open ball with centre \(P^x_a\) and \(\tilde{r}\) is denoted by \(B(P^x_a, \tilde{r})\).

Thus \(B(P^x_a, \tilde{r}) = \{ P^y_b \in \tilde{X} : \tilde{d}(P^x_a, P^y_b) \lesssim \tilde{r}\} \subset SP(\tilde{X})\).

\(SS(B(P^x_a, \tilde{r}))\) will be called a soft open ball with centre \(P^x_a\) and radius \(\tilde{r}\).

4. Let \((\tilde{X}, \tilde{d}, A)\) be a soft metric space and \(\tilde{r}\) be a non-negative soft real number. For any \(P^x_a \in \tilde{X}\), by a closed ball with centre \(P^x_a\) and radius \(\tilde{r}\), we mean the collection of soft points of \(\tilde{X}\) satisfying \(\tilde{d}(P^x_a, P^y_b) \lesssim \tilde{r}\).

The closed ball with centre \(P^x_a\) and \(\tilde{r}\) is denoted by \(B[P^x_a, \tilde{r}]\).

Thus \(B[P^x_a, \tilde{r}] = \{ P^y_b \in \tilde{X} : \tilde{d}(P^x_a, P^y_b) \lesssim \tilde{r}\} \subset SP(\tilde{X})\).

\(SS(B[P^x_a, \tilde{r}])\) will be called a soft closed ball with centre \(P^x_a\) and radius \(\tilde{r}\).

5. Let \((\tilde{X}, \tilde{d}, A)\) be a soft metric space and \((Y, A)\) be a non-null soft subset of \(\tilde{X}\). A soft point \(P^x_a \in \tilde{X}\) is said to be a soft limit point of \((Y, A)\), if and only if every soft open ball \(SS(B(P^x_a, \tilde{r}))\) containing \(P^x_a\) in \((\tilde{X}, \tilde{d}, A)\) contains at least one soft point of \((Y, A)\) other than \(P^x_a\). A soft limit point of a set \((Y, A)\) may or may not belong to the soft set \((Y, A)\).

The soft set generated by the collection of all soft points of \((Y, A)\) and soft limit points of \((Y, A)\) in \((\tilde{X}, \tilde{d}, A)\) is said to be soft closure of \((Y, A)\) in \((\tilde{X}, \tilde{d}, A)\). It is denoted by \((\overline{Y}, \overline{A})\).

6. Let \(\{P^x_{an}\}_n\) be a sequence of soft points in a soft metric space \((\tilde{X}, \tilde{d}, A)\).

The soft sequence \(\{P^x_{an}\}_n\) is called as convergent soft sequence in \((\tilde{X}, \tilde{d}, A)\) if there is a soft point \(P^x_a \in \tilde{X}\) such that \(\tilde{d}(P^x_{an}, P^x_a) \to \tilde{0}\) as \(n \to \infty\), that is for given \(\tilde{\varepsilon} > \tilde{0}\), there is a natural number \(N\) such that \(\tilde{0} \lesssim \tilde{d}(P^x_{an}, P^x_a) \lesssim \tilde{\varepsilon}\), whenever \(n \geq N\).

We denote this by \(P^x_{an} \to P^x_a\) as \(n \to \infty\).

7. A sequence \(\{P^x_{an}\}_n\) of soft points in a soft metric space \((\tilde{X}, \tilde{d}, A)\) is considered as a cauchy sequence in \((\tilde{X}, \tilde{d}, A)\) if corresponding to every \(\tilde{\varepsilon} > \tilde{0}\), there exist \(n \in N\) such that \(\tilde{0} \lesssim \tilde{d}(P^x_{an}, P^x_{an}) \lesssim \tilde{\varepsilon}\).

Let \((\tilde{X}, \tilde{d}, A)\) and \((\tilde{Y}, \tilde{\rho}, B)\) be two soft metric spaces. The mapping \((\varphi, e) : (\tilde{X}, \tilde{d}, A) \to (\tilde{Y}, \tilde{\rho}, B)\) is a soft mapping defined in \([88]\), where \(\varphi : X \to Y\) and
e : A → B are two mappings

**Definition 4.1.3.** [11] A soft mapping \((\varphi, e)\) is soft continuous at the soft point \(P_a^x \in \tilde{X}\) if for every soft open ball \(SS(B((\varphi, e)(P_a^x), \tilde{e}))\) of \(\tilde{Y}\) there exists a soft open ball \(SS(B(P_a^x, \tilde{r}))\) of \((\tilde{X}, \tilde{d}, A)\) such that \((\varphi, e)(SS(B(P_a^x, \tilde{\delta}))) \subseteq SS(B((\varphi, e)(P_a^x), \tilde{e}))\).

This means that \((\varphi, e)\) is soft continuous at the soft point \(P_a^x \in \tilde{X}\) if for every \(\tilde{e} \geq 0\) there exists \(\tilde{\delta} \geq 0\) such that \(\tilde{d}(P_a^x, P_a^x) < \tilde{\delta}\) implies that \(\tilde{\rho}((\varphi, e)(P_a^y), (\varphi, e)(P_a^x)) < \tilde{e}\) for every \(P_a^y \in \tilde{X}\). If \((\varphi, e)\) is soft continuous at every soft point \(P_a^x\) of \((\tilde{X}, \tilde{d}, A)\), then it is said to be soft continuous on \((\tilde{X}, \tilde{d}, A)\).

**Definition 4.1.4.** [11] Let \((\tilde{X}, \tilde{d}, A)\) be a soft metric space. A mapping \((\varphi, e) : (\tilde{X}, \tilde{d}, A) \to (\tilde{Y}, \tilde{\rho}, B)\) is called a soft contraction mapping if there exists a soft real number \(\alpha \in \mathbb{R}, 0 \leq \alpha \leq 1\) such that \(\tilde{\rho}((\varphi, e)(P_a^y), (\varphi, e)(P_a^x)) \leq \alpha \cdot \tilde{d}(P_a^y, P_a^x)\) for all \(P_a^y, P_a^x \in \tilde{X}\).

**Definition 4.1.5.** [3] Let \((\tilde{X}, \tilde{d}, A)\) be a soft metric space. \((\tilde{X}, \tilde{d}, A)\) is called soft sequential compact metric space if every soft sequence has a soft subsequence that converges in \(\tilde{X}\).

**Definition 4.1.6.** [3] Let \((\tilde{X}, \tilde{d}, A)\) and \((\tilde{Y}, \tilde{\rho}, B)\) be two soft metric spaces. A soft mapping \((\varphi, e) : (\tilde{X}, \tilde{d}, A) \to (\tilde{Y}, \tilde{\rho}, B)\) is soft uniformly continuous if for given any \(\tilde{\epsilon} \geq 0\), there exists \(\tilde{\delta} \geq 0\) (depending only on \(\tilde{\epsilon}\)) such that for any soft points \(P_a^x, P_a^y \in \tilde{X}\), \(\tilde{d}(P_a^x, P_a^y) < \tilde{\delta}\) implies \(\tilde{\rho}((\varphi, e)(P_a^x), (\varphi, e)(P_a^y)) < \tilde{\epsilon}\).

The following results will be utilized in this chapter.

**Theorem 4.1.1.** [11] Let \((\tilde{Y}, \tilde{d}_Y, A)\) be a soft metric subspace of a soft metric space \((\tilde{X}, \tilde{d}, A)\) and \(P_a^x \in (Y, A)\). Then for any soft open ball \(SS(B(P_a^x, \tilde{r}))\) in \(\tilde{X}\), \(SS(B(P_a^x, \tilde{r})) \cap (Y, A)\) is soft open ball in \((\tilde{Y}, \tilde{d}_Y, A)\), and also any soft open ball in \((\tilde{Y}, \tilde{d}_Y, A)\) is obtained as the intersection of a soft open ball in \(\tilde{X}\) with \((Y, A)\).

**Theorem 4.1.2.** [41] A soft mapping \((\varphi, e)\) is soft continuous at the soft point \(P_a^x \in \tilde{X}\) if and only if for every sequence of soft points \(\{P_a^{x_n}\}_n\) converging to the soft point \(P_a^x\) in the soft metric space \((\tilde{X}, \tilde{d}, A)\), the sequence \(\{(\varphi, e)(P_a^{x_n})\}_n\) in \((\tilde{Y}, \tilde{\rho}, B)\) converges to a soft point \((\varphi, e)(P_a^x)\) in \(\tilde{Y}\).

**Theorem 4.1.3.** [41] Every soft contraction mapping is soft continuous.

**Theorem 4.1.4.** [55] Let \((\tilde{X}, \tilde{d}, A)\) be a soft metric space. A soft point \(P_a^x \in \tilde{X}\) is a soft limit point of \((F, A)\) if and only if there is a soft sequence of soft points in \((F, A)\) converging to \(P_a^x\).
4.2 Soft continuous mappings and soft dense sets in soft metric spaces

In this section, we study soft continuity of a soft map between soft metric spaces in terms of soft dense subsets of the domain space. We begin by introducing the following definition of soft dense subset of $\widetilde{X}$. Throughout this chapter, $(Z, A)$ will denote an arbitrarily fixed soft dense subset of $\widetilde{X}$.

**Definition 4.2.1.** A soft set $(Z, A)$ of $\widetilde{X}$ will be called soft dense in $\widetilde{X}$ if $(Z, A) = \widetilde{X}$. In other words, for every $P_a^{x} \in \widetilde{X}$ and $\bar{c} > 0$ there exists soft point $P_b^{y} \in (Z, A)$ such that $\tilde{d}(P_b^{y}, P_a^{x}) \leq \bar{c}$.

Our first theorem follows directly from Theorem 4.1.4 above:

**Theorem 4.2.1.** A soft set $(Z, A)$ of $\widetilde{X}$ is said to be soft dense in $\widetilde{X}$ if and only if for every $P_a^{x} \in \widetilde{X}$ there exists a sequence $\{P_a^{x_n}\}_n$ of soft points in $(Z, A)$ converging to $P_a^{x}$.

Following theorem gives characterization of soft continuous maps in soft metric spaces in terms of soft dense subsets of domain space:

**Theorem 4.2.2.** Let $(\widetilde{X}, \tilde{d}, A)$ and $(\widetilde{Y}, \tilde{\rho}, B)$ be two soft metric spaces. A soft mapping $(\varphi, e) : (\widetilde{X}, \tilde{d}, A) \rightarrow (\widetilde{Y}, \tilde{\rho}, B)$ is soft continuous if and only if $(\varphi, e)|_{(Z, A)}$ is soft continuous at each point of $(Z, A)^c$ where $(Z, A)$ is soft dense in $\widetilde{X}$.

**Proof.** Let $P_a^{x} \in (Z, A)$ and suppose $(\varphi, e)$ is not soft continuous at $P_a^{x}$. Then $(\varphi, e)|_{(Z, A)}$ is soft continuous implies there exists an $\bar{c} > 0$ and a sequence $\{P_a^{x_n}\}_n$ in $(Z, A)^c$ such that $P_a^{x_n} \rightarrow P_a^{x}$ but $\tilde{d}((\varphi, e)(P_a^{x_n}), (\varphi, e)(P_a^{x})) \geq \bar{c}$ for every $n$. As $(Z, A)$ is soft dense in $\widetilde{X}$, for every $P_a^{y} \in \widetilde{X}$ and $\bar{c} > 0$ there exist $P_b^{y} \in (Z, A)$ such that $\tilde{d}(P_b^{y}, P_a^{x}) \leq \bar{c}$. Now since $(\varphi, e)$ is soft continuous at each point of $P_a^{x_n} \in (Z, A)^c$ and $(Z, A)$ is soft dense in $\widetilde{X}$ then for each $n$, there exists a soft point $P_b^{y_n} \in (Z, A)$ such that $\tilde{d}(P_a^{x_n}, P_b^{y_n}) \leq \frac{1}{n}$ and $\tilde{d}((\varphi, e)(P_a^{x_n}), (\varphi, e)(P_a^{y_n}))) \geq \frac{1}{n}$. By $P_a^{x_n} \rightarrow P_a^{x}$ and $\tilde{d}(P_a^{x_n}, P_b^{y_n}) \leq \frac{1}{n}$, we get $P_a^{y_n} \rightarrow P_a^{x}$. Since $(\varphi, e)|_{(Z, A)}$ is soft continuous at $P_a^{x}$, it follows that $(\varphi, e)(P_a^{y_n}) \rightarrow (\varphi, e)(P_a^{x})$ and so $\tilde{d}((\varphi, e)(P_a^{x_n}), (\varphi, e)(P_a^{y_n})) \leq \frac{1}{n}$ implies that $(\varphi, e)(P_a^{x_n}) \rightarrow (\varphi, e)(P_a^{x})$. This contradicts our assumption that $\tilde{d}((\varphi, e)(P_a^{x_n}), (\varphi, e)(P_a^{y_n})) \geq \bar{c}$ for every $n$. Hence $(\varphi, e)$ is soft continuous at $P_a^{x}$ and therefore, $(\varphi, e)$ is soft continuous. \hfill $\square$

The proof of the following theorem, on soft continuity of soft maps between soft metric space, follows from above theorem and easily proved fact
Theorem 4.2.3. Let \((\tilde{X}, \tilde{d}, A)\) and \((\tilde{Y}, \tilde{\rho}, B)\) be two soft metric spaces. Let \((\varphi, e) : (\tilde{X}, \tilde{d}, A) \to (\tilde{Y}, \tilde{\rho}, B)\) be a soft mapping and \(\tilde{X} = (F, A) \cup (G, A)\) where \((F, A)\) is soft open and \((G, A)\) is soft dense in \(\tilde{X}\). Then \((\varphi, e)\) is soft continuous if \((\varphi, e)|_{(F, A)}\) and \((\varphi, e)|_{(G, A)}\) are both soft continuous.

The following theorem is a map gluing theorem on soft continuity of a soft map between soft metric spaces:

Theorem 4.2.4. Let \((\tilde{X}, \tilde{d}, A)\) and \((\tilde{Y}, \tilde{\rho}, B)\) be two soft metric spaces. Let \((\varphi, e) : (\tilde{X}, \tilde{d}, A) \to (\tilde{Y}, \tilde{\rho}, B)\) be a soft mapping and \(\tilde{X} = (M, A) \cup (N, A)\) where \((M, A)\) \(\cap\) \((N, A)\) is soft dense in \(\tilde{X}\). Then \((\varphi, e)\) is soft continuous if \((\varphi, e)|_{(M, A)}\) and \((\varphi, e)|_{(N, A)}\) are both soft continuous.

Proof. Let \(P^x_a \in \tilde{X}\). Without loss of generality, we may assume that \(P^x_a \not\in (M, A)\). Let \((\varphi, e)\) is not soft continuous at \(P^x_a\) then there exist \(\tilde{\epsilon} > 0\) and a sequence \(\{P^{x_n}_{a_n}\}_n\) in \(\tilde{X}\) such that \(P^{x_n}_{a_n} \to P^x_a\) but \(\tilde{\rho}((\varphi, e)(P^{x_n}_{a_n}),(\varphi, e)(P^x_a)) \geq \tilde{\epsilon}\) for every \(n\). Since, \((\varphi, e)|_{(M, A)}\) is soft continuous at \(P^x_a\), this sequence \(\{P^{x_n}_{a_n}\}_n\) can be taken to be in \((N, A)\). Now as \((\varphi, e)|_{(N, A)}\) is soft continuous at each \(P^{x_n}_{a_n}\) and \((M, A)\) \(\cap\) \((N, A)\) is soft dense in \(\tilde{X}\), there exists, for each \(n\), a soft point \(P^{x_n}_{b_n}\) \(\in (M, A) \cap (N, A)\) such that \(d(P^{x_n}_{a_n}, P^{x_n}_{b_n}) < \frac{1}{n}\) and \(\tilde{\rho}((\varphi, e)(P^{x_n}_{a_n}),(\varphi, e)(P^{x_n}_{b_n})) \geq \frac{1}{n}\). Therefore, \(P^{x_n}_{a_n} \to P^x_a\) and \(d(P^{x_n}_{a_n}, P^{x_n}_{b_n}) < \frac{1}{n}\) implies \(P^{x_n}_{b_n} \to P^x_a\). Again since \((\varphi, e)|_{(M, A)}\) is soft continuous at \(P^x_a\), \((\varphi, e)(P^{x_n}_{b_n}) \to (\varphi, e)(P^x_a)\) and therefore by, \(\tilde{\rho}((\varphi, e)(P^{x_n}_{a_n}),(\varphi, e)(P^{x_n}_{b_n})) \geq \frac{1}{n}\) we get \((\varphi, e)(P^{x_n}_{a_n}) \to (\varphi, e)(P^x_a)\), which contradicts our assumption \(\tilde{\rho}((\varphi, e)(P^{x_n}_{a_n}),(\varphi, e)(P^x_a)) \geq \tilde{\epsilon}\) for every \(n\). Hence \((\varphi, e)\) is soft continuous at \(P^x_a\). \(\square\)

Next, we give the following characterization of soft continuity of a soft map in terms of convergent sequence of soft points taken from a soft dense subset of domain of the soft map:

Theorem 4.2.5. Let \((\tilde{X}, \tilde{d}, A)\) and \((\tilde{Y}, \tilde{\rho}, B)\) be two soft metric spaces. For a soft mapping, \((\varphi, e) : (\tilde{X}, \tilde{d}, A) \to (\tilde{Y}, \tilde{\rho}, B)\), the following conditions are equivalent.

1. \((\varphi, e)\) is soft continuous.

2. \((\varphi, e)|_{(Z, A)} : (\tilde{Z}, \tilde{d}_Z, A) \to (\tilde{Y}, \tilde{\rho}, B)\) is soft continuous and \((\varphi, e)\) is soft continuous at each point of \((Z, A)^c\).

3. for any sequence of soft points \(\{P^{x_n}_{a_n}\}_n\) in \((Z, A)\) converging to a soft point \(P^x_a \in \tilde{X}\) implies \(\{(\varphi, e)(P^{x_n}_{a_n})\}_n\) converges to \((\varphi, e)(P^x_a)\).
Proof. (1) $\Leftrightarrow$ (2) by Theorem 4.2.2.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1) Suppose $\{P_{x_n}\}_n$ be a sequence of soft points in $\tilde{X}$ such that $P_{x_n}^n \to P_x^x$ where $P_{x_n}^x \in \tilde{X}$. Since $(Z,A)$ is soft dense in $\tilde{X}$, for each $n$, there is sequence $\{P_{d_n}^k\}$ of soft points in $(Z,A)$ converging to $P_{x_n}^x \in \tilde{X}$. Therefore by (3), $(\varphi,e)(P_{d_n}^k) \to (\varphi,e)(P_{x_n}^x)$.

Corollary 4.2.2. Let $\tilde{X}$ be soft cover of $(Z,A)$ such that $(\varphi,e)$ is soft continuous. Let $(\tilde{Z},\tilde{A})$ be the soft cover of $(Z,A)$ and $(\varphi,e)$ be a soft map from $(\tilde{Z},\tilde{A})$ to $(\tilde{X},\tilde{A})$. Then for each $n$, there exist positive integers $k_1(n)$ and $k_2(n)$ such that $\tilde{d}(P_{d_n}^{k_1(n)}, P_{x_n}^x) < \frac{1}{n}$ for all $k \geq k_1(n)$ and so $\tilde{d}(P_{d_n}^{k_1(n)}, P_{x_n}^x) \to 0$ as $n \to \infty$. Now as $P_{x_n}^x \to P_x^x$, we get $P_{d_n}^{k_1(n)} \to P_x^x$. Therefore, by (3) again, $(\varphi,e)(P_{d_n}^{k_1(n)}) \to (\varphi,e)(P_x^x)$ and so $(\varphi,e)(P_{d_n}^{k_1(n)}), (\varphi,e)(P_{x_n}^x) \to 0$ implies that $(\varphi,e)(P_{x_n}^x) \to (\varphi,e)(P_x^x)$. Hence $(\varphi,e)$ is soft continuous.

By using above theorem we get the following corollary:

Corollary 4.2.1. Let $(\varphi,e) : (\tilde{Z},\tilde{d}_z,A) \to (\tilde{Y},\tilde{\rho},B)$ be a soft continuous mapping where $(Z,A)$ is soft dense in $\tilde{X}$. Then there exist a largest soft subset $(W,A) \subseteq (W,A)$ such that $(\varphi,e)$ can be extended to a soft map $(f,e) : (\tilde{W},\tilde{d}_w,A) \to (\tilde{Y},\tilde{\rho},B)$ which is soft continuous.

Proof. Let $(Z',A) = (Z,A) \cup P_x^x$ and $(W,A) = \bigcup_{P_x^x \in S} P_x^x$ where $S$ be set of all soft points $P_x^x$ in $\tilde{X}$ for which there exist a soft continuous extension of $(\varphi,e)$, $(f,e) : (Z',\tilde{d}_z,A) \to (\tilde{Y},\tilde{\rho},B)$. Obviously, $(Z,A) \subseteq (W,A)$. Let $(f,e) : (\tilde{W},\tilde{d}_w,A) \to (\tilde{Y},\tilde{\rho},B)$ defined by $(f,e)(P_x^x) = (f,e)(P_x^x)$. By Theorem 4.2.5 (3), $(f,e)$ is soft continuous as $(Z,A)$ is soft dense in $\tilde{X}$. Now, let $(W_0,A)$ be another soft set such that $(Z,A) \subseteq (W_0,A)$ and $(f,e) : (\tilde{W}_0,\tilde{d}_w,A) \to (\tilde{Y},\tilde{\rho},B)$ be a soft continuous extension of $(\varphi,e)$ then $(\varphi,e)|_{(Z',A)} : (\tilde{Z},\tilde{d}_z,A) \to (\tilde{Y},\tilde{\rho},B)$ is soft continuous for all $P_x^x \subseteq (W_0,A)$ and so $(W_0,A) \subseteq (W,A)$ then $(W,A)$ is largest soft subset of $\tilde{X}$ where $(Z,A) \subseteq (W,A)$.

In the following corollary of above proved Theorem 4.2.2 we obtain the soft continuity of a soft map in terms of the soft continuity of its restrictions to members of a soft cover of the domain having soft dense intersection:

Corollary 4.2.2. Let $\{(F,A)_\alpha \mid \alpha \in \Lambda\}$ be soft cover of $\tilde{X}$, that is $\tilde{X} = \bigcup_{\alpha \in \Lambda} (F,A)_\alpha$ such that $(\tilde{F},\tilde{A})_\alpha$ is soft dense in $\tilde{X}$ then a map $(\varphi,e) : (\tilde{X},\tilde{d},A) \to (\tilde{Y},\tilde{\rho},B)$ is soft continuous if $(\varphi,e)|_{(F,A)_\alpha}$ is soft continuous.
**Proof.** By Theorem 4.2.5, it is sufficient to prove that if the sequence \( \{P_{a_n}^x\} \) of soft points in \( \tilde{\alpha}(F, A) \) converging to \( P_a^x \in \tilde{X} \) then \( (\varphi, e)(P_{a_n}^x) \to (\varphi, e)(P_a^x) \). As \( \{(F, A) \mid \alpha \in \Lambda\} \) be soft cover of \( \tilde{X} \), \( P_a^x \in (F, A) \) for some \( \alpha \), result follows from soft continuity of \( (\varphi, e)|_{(F, A)} \) at \( P_a^x \).

## 4.3 Soft equicontinuity

In this section, we introduce the following definition of soft equicontinuity:

**Definition 4.3.1.** Let \( \mathcal{A} \) be a family of soft maps from \( \tilde{X} \) to \( \tilde{Y} \). Then the family \( \mathcal{A} \) will said to be

1. **soft equicontinuous** at a soft point \( P_a^x \in \tilde{X} \) if for \( \tilde{\epsilon} > 0 \) there exist \( \tilde{\delta} \) (depending on \( \tilde{\epsilon} \) and \( P_a^x \)) such that \( \tilde{d}(P_b^y, P_a^x) < \tilde{\delta} \) then \( \tilde{\rho}((\varphi, e)(P_b^y), (\varphi, e)(P_a^x)) < \tilde{\epsilon} \) for every \( (\varphi, e) \in \mathcal{A} \).

2. **soft pointwise equicontinuous** on a soft subset \((F, A)\) of \( \tilde{X} \) if it is soft equicontinuous at each soft point of \((F, A)\).

We illustrate this with the following example:

**Example 4.3.1.** Let \( \mathcal{F} \) be a set of all soft contraction mappings \((\varphi, e) : (\tilde{X}, \tilde{d}, A) \to (\tilde{Y}, \tilde{\rho}, B)\). Then \( \mathcal{F} \) is soft equicontinuous, since we can choose \( \tilde{\delta} = \tilde{\epsilon} \). To see this, we note that \( \tilde{d}(P_b^y, P_a^x) \leq \tilde{\delta} \) then \( \tilde{\rho}((\varphi, e)(P_b^y), (\varphi, e)(P_a^x)) \leq \tilde{\epsilon} \) for all \( P_b^y, P_a^x \in \tilde{X} \) and \((\varphi, e) \in \mathcal{F}\).

Next we introduce soft pointwise convergence of sequence of soft maps:

**Definition 4.3.2.** A sequence \( \{(\varphi, e)_n\} \) of soft functions from \( \tilde{X} \) to \( \tilde{Y} \) is soft pointwise converges to a soft function \((\varphi, e) : (\tilde{X}, \tilde{d}, A) \to (\tilde{Y}, \tilde{\rho}, B)\) if for all \( P_a^x \in \tilde{X} \) and \( \tilde{\epsilon} > 0 \) there exist \( N = N(\tilde{\epsilon}) \) such that \( \tilde{\rho}((\varphi, e)_n(P_a^x), (\varphi, e)(P_a^x)) \leq \tilde{\epsilon} \).

We illustrate this with the following example:

**Example 4.3.2.** Let \( A = \mathbb{R} \) be parameter set and \( X = \mathbb{R} \). Consider usual metric on these sets and define soft metric on \( \tilde{X} \) by \( \tilde{d}(P_b^y, P_a^x) = |\tilde{x} - \tilde{y}| + |\tilde{a} - \tilde{b}| \). Define \( (\varphi, e)_n : (\tilde{X}, \tilde{d}, A) \to (\tilde{X}, \tilde{d}, A) \) by \( (\varphi, e)_n(P_a^x) = (P_a^x) \) and \((\varphi, e) : (\tilde{X}, \tilde{d}, A) \to (\tilde{X}, \tilde{d}, A) \) by \( (\varphi, e)(P_a^x) = P_a^0 \). Therefore, \((\varphi, e)_n \to (\varphi, e)\).

Following theorem shows the soft continuity of soft pointwise limit of a sequence of soft maps when the family of that soft maps is soft equicontinuous.
**Theorem 4.3.1.** Let \( \{((\varphi,e))_{n} \}_{n} \) be a sequence of soft functions from \( \tilde{X} \) to \( \tilde{Y} \) with the property that \( ((\varphi,e))_{n}(P_{a}^{x}) \) converges to \( ((\varphi,e))(P_{a}^{x}) \) for \( P_{a}^{x} \in \tilde{X} \). Suppose further the family \( \{((\varphi,e))_{n}\}_{n=1}^{\infty} \) is soft equicontinuous. Then \( ((\varphi,e)) \) is soft continuous and the family \( \{((\varphi,e)),((\varphi,e))_{1},((\varphi,e))_{2},......\} \) is also soft equicontinuous.

*Proof.* Fix any \( \tilde{\epsilon} > 0 \) and \( P_{a}^{x} \in \tilde{X} \), there exist \( \tilde{\delta}(\text{depending on } \tilde{\epsilon} \text{ and } P_{a}^{x}) \) such that whenever \( \tilde{d}(P_{b}^{y},P_{a}^{x}) < \tilde{\delta} \) then \( \tilde{\rho}((\varphi,e))_{n}(P_{b}^{y}),((\varphi,e))_{n}(P_{a}^{x})) < \frac{\tilde{\epsilon}}{3} \) for every \( P_{b}^{y} \in \tilde{X} \) and \( n \in \mathbb{N} \). Further as \( ((\varphi,e))_{n}(P_{a}^{x}) \rightarrow ((\varphi,e))(P_{a}^{x}) \) for every \( P_{a}^{x} \in \tilde{X} \), there exist \( N \) such that \( \tilde{\rho}((\varphi,e))_{n}(P_{a}^{x}),((\varphi,e))(P_{a}^{x})) < \frac{\tilde{\epsilon}}{3} \) and \( \tilde{\rho}((\varphi,e))_{n}(P_{b}^{y}),((\varphi,e))(P_{b}^{y})) < \frac{\tilde{\epsilon}}{3} \) for all \( n \geq N \). Fix a value \( n \), then for all \( P_{b}^{y} \in \tilde{X} \) with \( \tilde{d}(P_{b}^{y},P_{a}^{x}) < \tilde{\delta} \) we have \( \tilde{\rho}((\varphi,e))(P_{b}^{y}),((\varphi,e))_{n}(P_{a}^{x})) \leq \tilde{\rho}((\varphi,e))_{n}(P_{a}^{x}),((\varphi,e))(P_{a}^{x})) + \tilde{\rho}((\varphi,e))_{n}(P_{b}^{y}),((\varphi,e))(P_{b}^{y})) < \tilde{\epsilon} \). □

For our next results, we need the following definitions:

**Definition 4.3.3.** A sequence \( \{(\varphi,e))_{n}\}_{n} \) of soft functions from \( \tilde{X} \) to \( \tilde{Y} \) is said to be cauchy sequence if for every \( P_{a}^{x} \in \tilde{X} \) and \( \tilde{\epsilon} > 0 \) there exist \( N \) such that for all \( n,m \geq N \), \( \tilde{\rho}((\varphi,e))_{n}(P_{a}^{x}),((\varphi,e))_{m}(P_{a}^{x})) < \tilde{\epsilon} \).

**Definition 4.3.4.** A soft metric space \( (\tilde{X},\tilde{d},A) \) is soft complete if every cauchy sequence in \( \tilde{X} \) converges to some soft point of \( \tilde{X} \).

In the following theorem, soft pointwise convergence of a sequence of soft maps which are soft pointwise equicontinuous are discussed, when codomain space is soft complete.

**Theorem 4.3.2.** Let \( \{(\varphi,e))_{n}\}_{n=1}^{\infty} \) be a sequence of soft maps from \( \tilde{X} \) to \( \tilde{Y} \) such that family \( \{(\varphi,e))_{n}(P_{a}^{x})\}_{n} \) is soft pointwise equicontinuous on \((Z,A)^{c}\) and \((\tilde{Y},\tilde{\rho},B)\) is soft complete. If \( \{(\varphi,e))_{n}(P_{a}^{x})\}_{n} \) converges for all \( P_{a}^{x} \in (Z,A) \) then \( \{(\varphi,e))_{n}(P_{a}^{x})\}_{n} \) converges for all \( P_{a}^{x} \in (Z,A) \).

In particular, this theorem holds if \( \{(\varphi,e))_{n}\}_{n=1}^{\infty} \) is soft equicontinuous family of soft functions.

*Proof.* Let \( P_{a}^{x} \in (Z,A)^{c} \) and \( \tilde{\epsilon} > 0 \). Since \( \{(\varphi,e))_{n}\}_{n} \) is soft pointwise equicontinuous at \( P_{a}^{x} \) then there exist \( \tilde{\delta} > 0 \) such that for every \( n \), \( \tilde{\rho}((\varphi,e))_{n}(P_{a}^{x}),((\varphi,e))_{n}(P_{b}^{d})) < \tilde{\epsilon} \) whenever \( \tilde{d}(P_{a}^{x},P_{b}^{d}) < \tilde{\delta} \) for all \( P_{b}^{d} \in (Z,A) \). Fix one such \( P_{b}^{d} \) in \((Z,A) \).

Then since \( \{(\varphi,e))_{n}(P_{b}^{d})\}_{n} \) is convergent in \( \tilde{Y} \), there exist a positive integer \( N_{0} = N_{0}(\tilde{\epsilon}) \) such that for all \( m,n \geq N_{0} \), \( \tilde{\rho}((\varphi,e))_{n}(P_{b}^{d}),((\varphi,e))_{m}(P_{b}^{d})) < \tilde{\epsilon} \). It follows that \( \tilde{\rho}((\varphi,e))_{n}(P_{a}^{x}),((\varphi,e))_{m}(P_{a}^{x})) \leq \tilde{\rho}((\varphi,e))_{n}(P_{a}^{x}),((\varphi,e))_{n}(P_{b}^{d})) + \tilde{\rho}((\varphi,e))_{n}(P_{b}^{d}),((\varphi,e))_{m}(P_{b}^{d})) + \tilde{\rho}((\varphi,e))_{m}(P_{b}^{d}),((\varphi,e))_{m}(P_{a}^{x})) < \tilde{\epsilon} + \tilde{\epsilon} + \tilde{\epsilon} = \tilde{\epsilon} \) for all \( m,n \geq N_{0} \). Thus, \( \{(\varphi,e))_{n}(P_{a}^{x})\}_{n} \) is a cauchy sequence in \( \tilde{X} \). Hence it converges, since \((\tilde{Y},\tilde{\rho},B)\) is soft complete, □
Finally, we give characterization of soft pointwise equicontinuity of soft map in terms of convergence sequence of soft points in soft dense subset:

**Theorem 4.3.3.** For a family $\mathcal{A}$ of soft maps from $\tilde{X}$ to $\tilde{Y}$, the following conditions are equivalent

1. $\mathcal{A}$ is soft pointwise equicontinuous on $\tilde{X}$.

2. The family $\mathcal{A}|_{(Z,A)} = \{((\varphi,e)|_{(Z,A)}|((\varphi,e) \in \mathcal{A})\}$ is soft pointwise equicontinuous on $(Z,A)^c$.

3. For any sequence of soft points $\{P_{an}^x\}_n$ in $(Z,A)$ converging to a soft point $P_a^x \in \tilde{X}$ implies that for every $\tilde{\epsilon} > 0$, there exists a positive integer $n_0 = n_0(\text{depending on } \tilde{\epsilon} \text{ and } P_a^x)$ such that for every $(\varphi,e) \in \mathcal{A}$ and for all $n \geq n_0$, $\tilde{\rho}((\varphi,e)(P_{an}^x), (\varphi,e)(P_a^x)) < \tilde{\epsilon}$.

4. For any sequence of soft points $\{P_{xn}^x\}_n$ in $\tilde{X}$ converging to a soft point $P_a^x \in \tilde{X}$ implies that for every $\tilde{\epsilon} > 0$, there exists a positive integer $n_0 = n_0(\text{depending on } \tilde{\epsilon} \text{ and } P_a^x)$ such that for every $(\varphi,e) \in \mathcal{A}$ and for all $n \geq n_0$, $\tilde{\rho}((\varphi,e)(P_{xn}^x), (\varphi,e)(P_a^x)) < \tilde{\epsilon}$.

**Proof.** (1) $\Rightarrow$ (4): Let $\tilde{\epsilon} > 0$ and $P_a^x \in \tilde{X}$, then since $\mathcal{A}$ is soft pointwise equicontinuous there exist $\tilde{\delta} = \tilde{\delta}(\tilde{\epsilon}, P_a^x) > 0$ such that for every $P_b^y \in \tilde{X}$, $\tilde{\rho}((\varphi,e)(P_b^y), (\varphi,e)(P_a^x)) < \tilde{\epsilon}$ whenever $\tilde{d}(P_b^y, P_a^x) < \tilde{\delta}$ for every $(\varphi,e) \in \mathcal{A}$.

Now, since $P_{an}^x \rightarrow P_a^x$, there exist positive integer $N_0 = N_0(\tilde{\delta}, P_a^x) = N_0(\tilde{\epsilon}, P_a^x)$ such that for every $n \geq N_0$, $\tilde{d}(P_{xn}^x, P_a^x) < \tilde{\delta}$. It implies that $\tilde{\rho}((\varphi,e)(P_{xn}^x), (\varphi,e)(P_a^x)) < \tilde{\epsilon}$ for every $(\varphi,e) \in \mathcal{A}$.

(4) $\Rightarrow$ (1): Assume that (1) does not hold. Then there exist $P_a^x \in \tilde{X}$ and $\tilde{\epsilon} > 0$ such that for every $\tilde{\delta} > 0$, there exist $(f,e) \in \mathcal{A}$ and $P_b^y \in \tilde{X}$ depending on $\tilde{\delta}$ such that $\tilde{d}(P_b^y, P_a^x) < \tilde{\delta}$ but $\tilde{\rho}((f,e)(P_b^y), (f,e)(P_a^x)) \geq \tilde{\epsilon}$. In particular, by taking $\tilde{\delta} = \frac{\tilde{\epsilon}}{n}$, then there exists $(f,e)_n \in \mathcal{A}$ and $P_{an}^x$ in $\tilde{X}$ for every $n$ satisfying $\tilde{d}(P_{xn}^x, P_a^x) \leq \frac{\tilde{\epsilon}}{n}$ which contradicts (4).

(1) $\Rightarrow$ (2) is obvious. (2) $\Rightarrow$ (3) follows from (1) $\Rightarrow$ (4).

(3) $\Rightarrow$ (4): Let $P_{an}^x \rightarrow P_a^x$ and $\tilde{\epsilon} > 0$. Since $(Z,A)$ is soft dense in $\tilde{X}$, for every $n$, there exists a sequence of soft points $\{P_{an}^{dn_k}\}_n$ in $(Z,A)$ such that $P_{an}^{dn_k} \rightarrow P_{an}^x$. Therefore, there exist $k_1(n)$ such that $\tilde{d}(P_{an}^{dn_k}, P_{an}^x) < \frac{\tilde{\epsilon}}{n}$ for all $k \geq k_1(n)$. Then by (3), it follows that for every $n$, there exist $k_2(n)$ such that for every $(\varphi,e) \in \mathcal{A}$, $\tilde{\rho}((\varphi,e)(P_{an}^{dn_k}), (\varphi,e)(P_{an}^x)) < \tilde{\epsilon}$. In particular for
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\[ k' = \max\{k_1(n), k_2(n)\} \] and for every \((\varphi, e)\) in \(A\), \(\tilde{\rho}((\varphi, e)(P_{a_{k'}}^n), (\varphi, e)(P_{a_k}^n)) \leq \frac{1}{n} \rightarrow \bar{0} \text{ as } n \rightarrow \infty. \] 

Since \(\tilde{P}_{a_{k'}}^n \rightarrow P_{a_k}^{P_{a_{k'}}} \rightarrow P_{a_k}^x\), implies \(P_{a_{k'}}^n \rightarrow P_{a_k}^{P_{a_{k'}}}\). Therefore by (3) again, there exist \(n_0(\varepsilon, P_{a_k}^x)\) such that for every \((\varphi, e)\) in \(A\) and for all \(n \geq n_0 \tilde{\rho}((\varphi, e)(P_{a_{k'}}^n), (\varphi, e)(P_{a_k}^x)) < \frac{\varepsilon}{2}\). Hence for all \(n \geq n_0\) and for all \((\varphi, e)\) in \(A\), \(\tilde{\rho}((\varphi, e)(P_{a_{k'}}^n), (\varphi, e)(P_{a_k}^x)) \leq \tilde{\rho}((\varphi, e)(P_{a_{k'}}^n), (\varphi, e)(P_{a_{k'}})), (\varphi, e)(P_{a_k}^x)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \]

From Theorem 4.2.2 and 4.3.3 we have the following:

**Theorem 4.3.4.** Let \(\{(\varphi, e)_n\}_n\) be a sequence of soft functions from \(\tilde{X}\) to \(\tilde{Y}\) with the property that \((\varphi, e)_n(P_{a_k}^x)\) converges to \((\varphi, e)(P_{a_k}^x)\) for \(P_{a_k}^x \in \tilde{X}\). If the family \(\{(\varphi, e)_n|_{(Z,A)}\}_n\) is soft pointwise equicontinuous on \((Z,A)\) then \((\varphi, e)\) is soft continuous if and only if \((\varphi, e)\) is soft continuous at each soft point in \((Z,A)^c\).

### 4.4 Soft uniform continuity

In this section, we obtain analogs of the results in Section 2 for soft uniform continuity. We begin with the following characterization of soft uniformly continuous maps in terms of the sequences of soft points:

**Theorem 4.4.1.** Let \((\tilde{X}, \tilde{d}, A)\) and \((\tilde{Y}, \tilde{\rho}, B)\) be two soft metric spaces. A soft map \((\varphi, e)\) is soft uniformly continuous if and only if for any sequences \(\{P_{a_{k'}}^x\}_n\) and \(\{P_{b_{k'}}^n\}_n\) of soft points in \(\tilde{X}, \tilde{d}(P_{a_{k'}}^x, P_{b_{k'}}^n) \rightarrow \bar{0}\) implies \(\tilde{\rho}((\varphi, e)(P_{a_{k'}}^x), (\varphi, e)(P_{b_{k'}}^n)) \rightarrow \bar{0}\).

**Proof.** Suppose \(\tilde{d}(P_{a_{k'}}^x, P_{b_{k'}}^n) \rightarrow \bar{0}\) implies \(\tilde{\rho}((\varphi, e)(P_{a_{k'}}^x), (\varphi, e)(P_{b_{k'}}^n)) \rightarrow \bar{0}\) for all sequences \(\{P_{a_{k'}}^x\}_n\) and \(\{P_{b_{k'}}^n\}_n\) of soft points in \(\tilde{X}\). Let \((\varphi, e)\) is not soft uniformly continuous, then there exists an \(\bar{\varepsilon} > \bar{0} \) such that for every positive integer \(n\), there exist soft points \(P_{a_{k'}}^x\) and \(P_{b_{k'}}^n\) in \(\tilde{X}\) satisfying \(\tilde{d}(P_{a_{k'}}^x, P_{b_{k'}}^n) \leq \frac{1}{n}\) and \(\tilde{\rho}((\varphi, e)(P_{a_{k'}}^x), (\varphi, e)(P_{b_{k'}}^n)) \geq \bar{\varepsilon}\). Therefore, there exist sequences \(\{P_{a_{k'}}^x\}_n\) and \(\{P_{b_{k'}}^n\}_n\) of soft points in \(\tilde{X}\) such that \(\tilde{d}(P_{a_{k'}}^x, P_{b_{k'}}^n) \rightarrow \bar{0}\) but \(\tilde{\rho}((\varphi, e)(P_{a_{k'}}^x), (\varphi, e)(P_{b_{k'}}^n))\) does not converge to \(\bar{0}\), which contradicts our assumption. Hence \((\varphi, e)\) is soft uniformly continuous.

Conversely, Let \((\varphi, e)\) is soft uniformly continuous, \(\bar{\varepsilon} \leq \bar{0}\) and \(\{P_{a_{k'}}^x\}_n\) and \(\{P_{b_{k'}}^n\}_n\) be sequences of soft points such that \(\tilde{d}(P_{a_{k'}}^x, P_{b_{k'}}^n) \rightarrow \bar{0}\). Since \((\varphi, e)\) is soft uniformly continuous, there exist \(\bar{\delta} = \delta(\bar{\varepsilon}) \leq \bar{0}\) such that for any pair of points \(P_{a_k}^x, P_{b_k}^n\) in \(\tilde{X}\), \(\tilde{d}(P_{a_k}^x, P_{b_k}^n) \leq \bar{\delta}\) implies \(\tilde{\rho}((\varphi, e)(P_{a_k}^x), (\varphi, e)(P_{b_k}^n)) \leq \bar{\varepsilon}\). Also,
we have \( \tilde{d}(P_{a_n}^x, P_{b_n}^y) \to 0 \) implies that there exists a positive integer \( n_0 \) such that \( \tilde{d}(P_{a_n}^x, P_{b_n}^y) \to \tilde{\delta} \) for all \( n \geq n_0 \) and so, \( \tilde{\rho}((\varphi, e)(P_{a_n}^x), (\varphi, e)(P_{b_n}^y)) < \tilde{\epsilon} \) for all \( n \geq n_0 \). Therefore, \( \tilde{\rho}((\varphi, e)(P_{a_n}^x), (\varphi, e)(P_{b_n}^y)) \to 0. \) □

Following theorem is a map gluing theorem of soft uniform continuity of a soft map in terms of its restriction to soft sequentially compact soft subsets of \( \tilde{X} \):

**Theorem 4.4.2.** Let \( (\varphi, e) : (\tilde{X}, \tilde{d}, A) \to (\tilde{Y}, \tilde{\rho}, B) \) be a soft mapping and \( \tilde{X} = (F, A) \sqcup (G, A) \) where \( (F, A) \) is arbitrary and \( (G, A) \) is soft sequentially compact. Then \( (\varphi, e) \) is soft uniformly continuous if \( (\varphi, e)|_{(F,A)} \) is soft uniformly continuous and \( (\varphi, e) \) is soft continuous at each point of \( (G, A) \).

**Proof.** Let \( (\varphi, e) \) is not soft uniformly continuous. Since \( (\varphi, e)|_{(F,A)} \) and \( (\varphi, e)|_{(G,A)} \) are both soft uniformly continuous, then by above theorem there exists an \( \tilde{\epsilon} > 0 \) such that for any positive integer \( n \), there exist soft points \( P_{a_n}^x \) and \( P_{b_n}^y \) in \( (F, A) \) and \( (G, A) \) respectively satisfying \( \tilde{d}(P_{a_n}^x, P_{b_n}^y) < \frac{1}{n} \) and \( \tilde{\rho}((\varphi, e)(P_{a_n}^x), (\varphi, e)(P_{b_n}^y)) \geq \tilde{\epsilon} \). Then as \( (G, A) \) is soft sequential compact, \( \{P_{b_n}^y\}_n \) has a subsequence \( \{P_{b_{nk}}^y\} \) converging to a soft point \( P_b^y \) in \( (G, A) \), i.e. \( \tilde{d}(P_{b_{nk}}^y, P_b^y) \to 0. \)

It follows that corresponding subsequence \( \{P_{a_{nk}}^x\} \) of \( \{P_{a_n}^x\}_n \) also converges to \( P_b^y \) since \( \tilde{d}(P_{a_{nk}}^x, P_{b_{nk}}^y) < \frac{1}{n} \) and \( \tilde{d}(P_{b_{nk}}^y, P_b^y) \to 0. \) Now, by soft continuity of \( (\varphi, e) \) at \( P_b^y \) \( \tilde{\rho}((\varphi, e)(P_{a_{nk}}^x) \to (\varphi, e)(P_{b_{nk}}^y) \to (\varphi, e)(P_b^y)) \), contradicting our earlier assertion that \( (\varphi, e)(P_{a_n}^x), (\varphi, e)(P_{b_n}^y)) \geq \tilde{\epsilon} \) for every \( n \). Hence \( (\varphi, e) \) is soft uniformly continuous. □

Following theorem is another map gluing theorem on soft uniformly continuous maps which characterizes soft uniform continuity in terms of soft dense subsets of \( \tilde{X} \):

**Theorem 4.4.3.** Let \( (\varphi, e) : (\tilde{X}, \tilde{d}, A) \to (\tilde{Y}, \tilde{\rho}, B) \) be a soft map. Then \( (\varphi, e) \) is soft uniformly continuous if and only if \( (\varphi, e)|_{(Z,A)} \) is soft uniformly continuous and \( (\varphi, e) \) is soft continuous at each point of \( (Z,A) \) for some soft dense subset \( (Z,A) \) of \( \tilde{X} \).

**Proof.** Let \( \tilde{\epsilon} > 0 \). Since \( (\varphi, e)|_{(Z,A)} \) is soft uniformly continuous, there exists \( \tilde{\delta} > 0 \) such that for any pair of soft points \( P_a^x \) and \( P_b^y \) in \( (Z, A) \), \( \tilde{\rho}((\varphi, e)(P_a^x), (\varphi, e)(P_b^y)) \leq \tilde{\epsilon} \). Let \( \tilde{\rho}((\varphi, e)(P_a^x), (\varphi, e)(P_b^y)) > \tilde{\epsilon} \). Since \( (Z, A) \) is dense in \( \tilde{X} \), there exist \( P_c^x \) and \( P_f^y \) in \( (Z, A) \) such that \( \tilde{d}(P_c^x, P_f^y) < \frac{\epsilon}{3} \) and \( \tilde{d}(P_a^x, P_c^x) < \frac{\epsilon}{3} \) and \( \tilde{d}(P_b^y, P_f^y) < \frac{\epsilon}{3} \). Now as \( (\varphi, e) \) is soft continuous at each point of \( (Z,A) \), \( \tilde{\rho}((\varphi, e)(P_a^x), (\varphi, e)(P_c^x)) < \frac{\epsilon}{3} \) and \( \tilde{\rho}((\varphi, e)(P_b^y), (\varphi, e)(P_f^y)) < \frac{\epsilon}{3} \). Thus for all \( P_a^x, P_b^y \in \tilde{X} \), whenever \( \tilde{d}(P_a^x, P_b^y) < \frac{\epsilon}{3} \) we have, \( \tilde{\rho}((\varphi, e)(P_a^x), (\varphi, e)(P_b^y)) < \tilde{\epsilon} \).
such that \( \tilde{\rho}(\varphi,e)(P^x_n), (\varphi,e)_n(P^y_n) + \tilde{\rho}(\varphi,e)(P^y_n), (\varphi,e)(P^y_n) \) \( \leq \tilde{\epsilon} + \hat{\epsilon} + \tilde{\epsilon} = \epsilon \). Hence \((\varphi,e)\) is soft uniformly continuous.

By using Theorem 4.2.4, we get the following corollary of above theorem

**Corollary 4.4.1.** \((\varphi,e) : (X,\tilde{A},A) \to (Y,\tilde{\rho},B)\) be a soft map, \( \tilde{X} = (F,A) \sqcup (G,A) \) where \((F,A) \sqcap (G,A)\) is soft dense in \( \tilde{X} \). Then \((\varphi,e)\) is soft uniformly continuous if \((\varphi,e)|_{(F,A)}(\tilde{X},\tilde{\rho})\) is soft uniformly continuous and \((\varphi,e)|_{(G,A)}\) and \((\varphi,e)|_{(F,A)}\) are both soft continuous.

**Proof.** By Theorem 4.2.4 if \((\varphi,e)|_{(F,A)}\) and \((\varphi,e)|_{(G,A)}\) are both soft continuous then \((\varphi,e)\) is soft continuous. Then by above theorem, \((\varphi,e)\) is soft uniformly continuous.

The following theorem is another characterization of soft uniform continuity of a soft map

**Theorem 4.4.4.** A soft map \((\varphi,e)\) is soft uniformly continuous if and only if for any sequence of soft points \( \{P^a_n\}_n \) and \( \{P^x_n\}_n \) in \((Z,A)\) and \( \tilde{X} \) respectively, \( \tilde{d}(P^a_n, P^x_n) \to \tilde{0} \) implies \( \tilde{d}(\varphi,e)(P^a_n), (\varphi,e)(P^x_n) \to \tilde{0} \).

**Proof.** Let \( \tilde{d}(P^a_n, P^x_n) \to \tilde{0} \) implies \( \tilde{d}(\varphi,e)(P^a_n), (\varphi,e)(P^x_n) \to \tilde{0} \) for all sequence of soft points \( \{P^a_n\}_n \) and \( \{P^x_n\}_n \) in \((Z,A)\) and \( \tilde{X} \) respectively. Now assume \( \{P^a_n\}_n \) and \( \{P^x_n\}_n \) be sequence of soft points in \( \tilde{X} \) such that \( \tilde{d}(P^a_n, P^x_n) \to \tilde{0} \). Now, as \((Z,A)\) is soft dense in \( \tilde{X} \), for each \( n \), there exist soft points \( P^a_n \) and \( P^x_n \) in \((Z,A)\) such that \( \tilde{d}(P^a_n, P^x_n) \to \tilde{0} \). Then by assumption \( \tilde{d}(\varphi,e)(P^a_n), (\varphi,e)(P^x_n) \to \tilde{0} \). It follows that if \( \tilde{d}(P^a_n, P^x_n) \to \tilde{0} \) then \( \tilde{d}(\varphi,e)(P^a_n), (\varphi,e)(P^x_n) \to \tilde{0} \). Combining Theorem 4.4.3 and Theorem 4.4.4 above, we get the following characterizations of soft uniform continuity of soft maps which is analog of Theorem 4.2.5 for soft uniformly continuous maps

**Theorem 4.4.5.** For any soft map, \((\varphi,e) : (X,\tilde{A},A) \to (Y,\tilde{\rho},B)\), the following are equivalent

1. \((\varphi,e)\) is soft uniformly continuous.

2. for any sequence of soft points \( \{P^a_n\}_n \) and \( \{P^x_n\}_n \) in \((Z,A)\) and \( \tilde{X} \) respectively, \( \tilde{d}(P^a_n, P^x_n) \to \tilde{0} \) implies \( \tilde{d}(\varphi,e)(P^a_n), (\varphi,e)(P^x_n) \to \tilde{0} \).
3. \((\varphi, e)|_{(Z, A)} : (\tilde{Z}, \tilde{d}_Z, A) \to (\tilde{Y}, \tilde{\rho}, B)\) is soft uniformly continuous and 
\((\varphi, e)\) is soft continuous at each point of \((Z, A)^c\).

From above theorem, we get the following corollary

**Corollary 4.4.2.** Let \((\varphi, e) : (\tilde{Z}, \tilde{d}_Z, A) \to (\tilde{Y}, \tilde{\rho}, B)\) be a soft continuous mapping where \((Z, A)\) is soft dense in \(\tilde{X}\). Then there exist a largest soft subset \((W, A)\) of \(\tilde{X}\), \((Z, A)\tilde{\subseteq}(W, A)\) such that \((\varphi, e)\) can be extended to a soft map \((f, e) : (\tilde{W}, \tilde{d}_w, A) \to (\tilde{Y}, \tilde{\rho}, B)\) which is soft uniformly continuous.

**Proof.** Let \((Z', A) = (Z, A) \bigcup P^x_a\) and \((W, A) = \bigcup_{P^x_a \in \mathcal{S}} P^x_a\) where \(\mathcal{S}\) be set of all soft points \(P^x_a\) in \(\tilde{X}\) for which there exists a soft continuous extension of \((\varphi, e)\), \((f_x, e) : (\tilde{Z}', \tilde{d}_Z', A) \to (\tilde{Y}, \tilde{\rho}, B)\). Then by Corollary 4.2.1, \((W, A)\) is the largest soft subset of \(\tilde{X}\), \((Z, A)\tilde{\subseteq}(W, A)\) such that \((\varphi, e)\) can be extended to a soft map \((f, e) : (\tilde{W}, \tilde{d}_w, A) \to (\tilde{Y}, \tilde{\rho}, B)\) defined by \((f, e)(P^x_a) = (f_x, e)(P^x_a)\) and \((f, e)\) is soft continuous. Now by Theorem 4.4.5, we get \((f, e)\) is soft uniformly continuous.

In the following corollary of above Theorem 4.4.5, we get analog of Corollary 4.2.2 for soft uniform continuity

**Corollary 4.4.3.** Let \(\{(F, A)_\alpha \mid \alpha \in \Lambda\}\) be soft cover of \(\tilde{X}\), that is \(\tilde{X} = \bigcup_{\alpha \in \Lambda} (F, A)_\alpha\) such that \(\tilde{\cap}(F, A)_\alpha\) is soft dense in \(\tilde{X}\) then a map \((\varphi, e) : (\tilde{X}, \tilde{d}, A) \to (\tilde{Y}, \tilde{\rho}, B)\) is soft uniformly continuous if \((\varphi, e)|_{(F, A)_\alpha}\) is soft uniformly continuous for some \(\alpha\) and soft continuous for every \(\alpha\).

**Proof.** Since \((\varphi, e)|_{(F, A)_\alpha}\) is soft uniformly continuous for some \(\alpha\) and since by Corollary 4.2.2, the soft map \((\varphi, e)\) is soft continuous. Then by above Theorem 4.4.5, \((\varphi, e)\) is soft uniformly continuous.

### 4.5 Soft uniform equicontinuity

In this section, we begin by introducing soft uniformly equicontinuous family of soft maps from \(\tilde{X}\) to \(\tilde{Y}\)

**Definition 4.5.1.** A family \(\mathcal{A}\) be a family of soft maps from \(\tilde{X}\) to \(\tilde{Y}\) is said to be soft uniformly equicontinuous on a soft subset \((F, A)\) of \(\tilde{X}\) if for all \(\tilde{\epsilon} > \tilde{0}\) there exist \(\tilde{\delta}(\text{depending on } \tilde{\epsilon})\) such that for every \((\varphi, e)\) in \(\mathcal{A}\) and \(P^x_a, P^y_b \in (F, A), \tilde{\rho}((\varphi, e)(P^y_b), (\varphi, e)(P^x_a)) < \tilde{\epsilon}\) whenever \(\tilde{d}(P^y_b, P^x_a) < \tilde{\delta}\).

Next, we introduce soft uniform convergence of sequence of soft maps
Definition 4.5.2. A sequence \( \{(\varphi, e)_n\}_n \) of soft maps from \( \tilde{X} \) to \( \tilde{Y} \) is soft uniformly converges to a soft map \( (\varphi, e) : (\tilde{X}, \tilde{d}, A) \to (\tilde{Y}, \tilde{\rho}, B) \) if for all \( \varepsilon > 0 \) there exists \( N = N(\varepsilon, \tilde{\rho}, \tilde{d}) \) such that \( \tilde{\rho}((\varphi, e)_n(P^x_a), (\varphi, e)(P^x_a)) \leq \varepsilon \) for every \( n \geq N \) and for every \( P^x_a \in \tilde{X} \).

Following theorem shows that limit of soft uniformly convergent sequence of soft continuous maps is soft continuous.

Theorem 4.5.1. Let \((\tilde{X}, \tilde{d}, A)\) and \((\tilde{Y}, \tilde{\rho}, B)\) be two soft metric spaces and assume that \( \{(\varphi, e)_n\}_n \) is a sequence of soft continuous maps, where \((\varphi, e)_n : (\tilde{X}, \tilde{d}, A) \to (\tilde{Y}, \tilde{\rho}, B) \) is soft uniformly converges to a soft map \((\varphi, e)\). Then \((\varphi, e)\) is soft continuous.

Proof. Let \( P^x_a \in \tilde{X} \). Given an \( \varepsilon > 0 \), we find \( \delta > 0 \) such that \( \tilde{\rho}((\varphi, e)(P^x_a), (\varphi, e)(P^x_a)) \leq \varepsilon \) whenever \( \tilde{d}(P^y_b, P^x_a) < \delta \) for every \( P^y_b \in \tilde{X} \). Since \((\varphi, e)_n\) soft uniformly converges to \((\varphi, e)\), there is natural number \( N \) such that when \( n \geq N \), \( \tilde{\rho}((\varphi, e)_n(P^y_b), (\varphi, e)(P^x_a)) < \varepsilon/3 \) for all \( P^y_b \in \tilde{X} \). Also as \((\varphi, e)_n\) is soft continuous at \( P^x_a \), there is a \( \delta > 0 \) such that \( \tilde{\rho}((\varphi, e)_n(P^y_b), (\varphi, e)_n(P^x_a)) < \varepsilon/3 \) whenever \( \tilde{d}(P^y_b, P^x_a) < \delta \). It follows that \( \tilde{\rho}((\varphi, e)_n(P^y_b), (\varphi, e)(P^x_a)) \leq \tilde{\rho}((\varphi, e)_n(P^y_b), (\varphi, e)_n(P^x_a)) + \tilde{\rho}((\varphi, e)_n(P^y_b), (\varphi, e)(P^x_a)) \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \). Hence, \((\varphi, e)\) is soft continuous at \( P^x_a \).

In the following theorem, we show that a soft equicontinuous family of soft maps with soft sequentially compact domain space is soft uniformly equicontinuous.

Theorem 4.5.2. A soft equicontinuous family of soft maps from soft sequentially compact metric space to any soft metric space is soft uniformly equicontinuous.

Proof. Suppose \((\tilde{X}, \tilde{d}, A)\) be a soft sequential compact metric space and \( F \) is a family of soft maps \((\varphi, e) : (\tilde{X}, \tilde{d}, A) \to (\tilde{Y}, \tilde{\rho}, B)\) that is not soft uniformly equicontinuous then there is \( \varepsilon > 0 \), such that for every \( n \in \mathbb{N} \), there are soft points \( P^x_{a_n}, P^y_{b_n} \in \tilde{X} \) and a soft map \((\varphi, e)_n \in F\) with \( \tilde{d}(P^x_{a_n}, P^y_{b_n}) < \frac{1}{n} \) and \( \tilde{\rho}((\varphi, e)_n(P^x_{a_n}), (\varphi, e)_n(P^y_{b_n})) > \varepsilon \). Since \((\tilde{X}, \tilde{d}, A)\) is a soft sequential compact, the sequence \( \{P^x_{a_n}\}_n \) of soft points has a subsequence \( \{P^x_{a_{n_k}}\}_k \) converging to a soft point \( P^x_a \in \tilde{X} \). Also by \( \tilde{d}(P^x_{a_n}, P^y_{b_n}) < \frac{1}{n} \), we get corresponding subsequence \( \{P^y_{b_{n_k}}\}_k \) of \( \{P^y_{b_n}\}_n \) also converges to \( P^y_a \). Hence, for all \( \varepsilon > 0 \), there are soft points \( P^x_{a_{n_k}}, P^y_{b_{n_k}} \) such that \( \tilde{d}(P^x_{a_{n_k}}, P^x_a) < \delta \) and \( \tilde{d}(P^y_{b_{n_k}}, P^y_a) < \delta \). Then we have either \( \tilde{\rho}((\varphi, e)_n(P^x_{a_{n_k}}), (\varphi, e)_n(P^x_a)) > \varepsilon \) or \( \tilde{\rho}((\varphi, e)_n(P^y_{b_{n_k}}), (\varphi, e)_n(P^y_a)) \). Hence \( F \) is not soft equicontinuous at \( P^x_a \), which is a contradiction. Hence \( F \) is soft uniformly equicontinuous.
Finally, we get analogous result of Theorem 4.3.3 for soft uniform equicontinuity.

**Theorem 4.5.3.** For a family $\mathcal{A}$ of soft maps from $\tilde{X}$ to $\tilde{Y}$, the following conditions are equivalent

1. $\mathcal{A}$ is soft uniformly equicontinuous on $\tilde{X}$.

2. for any sequence of soft points $\{P_{x_n}^a\}_n$, $\{P_{b_n}^y\}_n$ of soft points in $\tilde{X}$, $\tilde{d}(P_{x_n}^a, P_{b_n}^y) \to 0$ implies that for every $\bar{\epsilon} > 0$, there exists a positive integer $N$ (depending on $\bar{\epsilon}$) such that for all $n \geq N$ and for every $(\varphi, e)$ in $\mathcal{A}$, $\tilde{p}((\varphi, e)(P_{x_n}^a), (\varphi, e)(P_{b_n}^y)) \geq \bar{\epsilon}$.

3. for any sequence of soft points $\{P_{c_n}^d\}_n$, $\{P_{b_n}^y\}_n$ in $(Z, A)$ and $\tilde{X}$ respectively, $\tilde{d}(P_{c_n}^d, P_{b_n}^y) \to 0$ implies that for every $\bar{\epsilon} > 0$, there exists a positive integer $N$ (depending on $\bar{\epsilon}$) such that for all $n \geq N$ and for every $(\varphi, e)$ in $\mathcal{A}$, $\tilde{p}((\varphi, e)(P_{c_n}^d), (\varphi, e)(P_{b_n}^y)) \to 0$.

4. the family $\mathcal{A}|_{(Z,A)} = \{(\varphi, e)|_{(Z,A)} | (\varphi, e) \in \mathcal{A}\}$ is soft uniformly equicontinuous on $(Z, A)$ and every $(\varphi, e)$ in $\mathcal{A}$ is soft continuous at each soft point of $(Z, A)^c$.

**Proof.** (1) $\Rightarrow$ (2): If (1) holds then for given $\bar{\epsilon} > 0$, there exists $\tilde{\delta} = \tilde{\delta}(\bar{\epsilon}) > 0$ such that for any $(\varphi, e)$ in $\mathcal{A}$, $\tilde{p}((\varphi, e)(P_{x_n}^a), (\varphi, e)(P_{b_n}^y)) \geq \bar{\epsilon}$ whenever $\tilde{d}(P_{x_n}^a, P_{b_n}^y) < \tilde{\delta}$. Then, if $\tilde{d}(P_{x_n}^a, P_{b_n}^y) \to 0$ there exists a positive integer $N=N(\tilde{\delta})=N(\bar{\epsilon})$ such that for every $n \geq N$, $\tilde{d}(P_{x_n}^a, P_{b_n}^y) < \tilde{\delta}$. Therefore, for every $(\varphi, e)$ in $\mathcal{A}$ and $n \geq N$, $\tilde{p}((\varphi, e)(P_{x_n}^a), (\varphi, e)(P_{b_n}^y)) \geq \bar{\epsilon}$ and hence (2) holds.

(2) $\Rightarrow$ (1): If $\mathcal{A}$ is not soft uniformly equicontinuous on $\tilde{X}$, then there exist $\bar{\delta} > 0$ such that for every positive integer $n$, there exists $(\varphi, e)_n$ in $\mathcal{A}$ and soft points $P_{x_n}^a$ and $P_{b_n}^y$ in $\tilde{X}$ such that $\tilde{d}(P_{x_n}^a, P_{b_n}^y) < \frac{1}{n}$ and so, $\tilde{d}(P_{x_n}^a, P_{b_n}^y) \to 0$ but $\tilde{p}((\varphi, e)_n(P_{x_n}^a), (\varphi, e)_n(P_{b_n}^y)) < \bar{\epsilon}$, which is a contradiction.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (4): by proof of (2) $\Rightarrow$ (1), we see that if (3) holds then $\mathcal{A}|_{(Z,A)}$ is soft uniformly equicontinuous on $(Z, A)$. Now let $P_{a}^x \in (Z, A)^c$ and $(\varphi, e)$ in $\mathcal{A}$. Take $P_{b_n}^y = P_{a}^x$ for every $n$ in (3), then we get a sequence $\{P_{c_n}^d\}_n$ of soft points in $(Z, A)$ converging to $P_{a}^x$ which implies $(\varphi, e)(P_{c_n}^d) \to (\varphi, e)(P_{a}^x)$. Hence $(\varphi, e)$ is soft continuous by Theorem 4.2.5.
(4) ⇒ (1): Let \( \tilde{\epsilon} \geq 0 \). By soft uniform equicontinuity of \( \mathcal{A}|_{(Z,A)} \), there exists \( \tilde{\delta} = \tilde{\delta}(\tilde{\epsilon}) > 0 \) such that for every \((\varphi,e)\) in \( \mathcal{A} \), \( \tilde{\rho}((\varphi,e)(P^d_e), (\varphi,e)(P^d_f)) \prec \frac{\tilde{\epsilon}}{3} \), for any pair of soft points \( P^d_e \) and \( P^d_f \) in \((Z,A)\) satisfying \( \tilde{d}(P^d_e, P^d_f) \prec \tilde{\delta} \). Let \((\varphi,e)\) in \( \mathcal{A} \) and \( P^d_a, P^d_b \in \tilde{X} \) with \( \tilde{d}(P^d_a, P^d_b) \prec \frac{\tilde{\delta}}{3} \). Now as \((Z,A)\) is dense in \( \tilde{X} \), there exist \( P^d_a \) and \( P^d_f \) in \((Z,A)\) such that \( \tilde{d}(P^d_a, P^d_f) \prec \frac{\tilde{\delta}}{3} \) and \( \tilde{d}(P^d_b, P^d_f) \prec \frac{\tilde{\delta}}{3} \). Since \((\varphi,e)\) is soft continuous at each point of \((Z,A)^c \) implies that \( \tilde{\rho}((\varphi,e)(P^d_x), (\varphi,e)(P^d_y)) \prec \frac{\tilde{\epsilon}}{3} \) and \( \tilde{\rho}((\varphi,e)(P^d_y), (\varphi,e)(P^d_f)) \prec \frac{\tilde{\epsilon}}{3} \). Therefore, we get \( \tilde{d}(P^d_a, P^d_f) \prec \frac{\tilde{\delta}}{3} \) and hence, \( \tilde{\rho}((\varphi,e)(P^d_a), (\varphi,e)(P^d_f)) \prec \frac{\tilde{\epsilon}}{3} + \frac{\tilde{\epsilon}}{3} + \frac{\tilde{\epsilon}}{3} = \frac{\tilde{\epsilon}}{3} \). Therefore, \( \tilde{\rho}((\varphi,e)(P^d_x), (\varphi,e)(P^d_y)) + \tilde{\rho}((\varphi,e)(P^d_y), (\varphi,e)(P^d_f)) + \tilde{\rho}((\varphi,e)(P^d_f)) \prec \frac{\tilde{\epsilon}}{3} + \frac{\tilde{\epsilon}}{3} + \frac{\tilde{\epsilon}}{3} = \tilde{\epsilon} \). □

### 4.6 Examples

Following example shows that converse of Theorem 4.3.1 need not be true.

**Example 4.6.1.** Consider soft metric defined in Example 4.3.2. Define \((f,e)_n : (\tilde{X}, \tilde{d}, A) \to (\tilde{Y}, \tilde{d}, A)\) by \((f,e)_n(P^d_a) = P^{nx}_a\) and \((f,e) : (\tilde{X}, \tilde{d}, A) \to (\tilde{Y}, \tilde{d}, A)\) by \((f,e)(P^d_a) = P^0_a\). Here, \((f,e)\) is soft continuous but \( (f,e)_n(P^d_a) \) does not converge to \((f,e)(P^d_a)\).

The following example shows that assumption of soft completeness cannot be dropped from Theorem 4.3.2.

**Example 4.6.2.** Let \( X = \mathbb{R} \) and \( Y = \mathbb{R} - \{0\} \) and \( A = \mathbb{R} \) be parameter set. Define \((\varphi,e)_n : (\tilde{X}, \tilde{d}, A) \to (\tilde{Y}, \tilde{d}, A)\) by \((\varphi,e)_n(P^d_a) = P^y_{a,n}\) where \( P^y_{a,n}(a) = x + \frac{1}{n} \) and \( P^y_{a,n}(a') = \phi \) for all \( a' \in A - \{a\} \). Now \((Z,A) = \mathbb{R} - \{0\}\) is soft dense in \( \tilde{X} \). For \( P^0_a \in (Z,A) \) then sequence \( \{(\varphi,e)_n(P^0_a)\}_n \) converges. But for \( P^0_a \in \tilde{X} \) then limit of \( \{(\varphi,e)_n(P^0_a)\}_n \) does not exist in \( \tilde{Y} \).