Chapter 3

ON SOFT CLOSED GRAPH AND SOFT CONTINUITY

3.1 Introduction

In 1999, Molodtsov [10] introduced the theory of soft sets as a new mathematical tool, which targets to solve many complicated problems in the fields of engineering, physics, computer science, economics and social sciences with uncertain data. This theory is applicable where there is no clearly defined mathematical model and is specially relevant to fuzzy system as every fuzzy set is a soft set. Topological structures of soft sets have been studied by some authors [66], [2], [59]. In 2011, Shabir and Naz [59] initiated the study of soft topological space. Consequently, they defined some basic notions of soft topological space such as soft open and closed sets, soft closure, soft separation axioms and established their elementary properties. Kharal and Ahmad [30], defined the notion of a soft mapping on soft classes and studied some properties of images and preimages of soft sets under soft mapping. Aygınoglu and Aygın [2], introduced soft continuity of soft mappings and soft product topology. Nazmul and Samanta [12], studied the new concept of soft points and neighbourhood properties in a soft topological space. In [50], Sahin and Kucuk introduced the concept of soft filter by using soft sets and investigated their related properties using convergence theory of soft filter. Demir and Özbakir [12], introduced some new notions such as cluster soft point, soft net and convergence of soft nets and defined soft hausdorff spaces by using soft points.

It is known that closed graph theorem (For a map $f : X \rightarrow Y$, where $X$ is arbitrary topological space and $Y$ is a compact Hausdorff space, the graph of $f$ is closed if and only if $f$ is continuous) is a basic result which characterizes continuous function in terms of their graphs. In soft theory, the concept of soft continuity of soft mappings have been introduced which is a generalization of continuity of a map. In this chapter, we obtain new and significant results on soft closed graphs and soft continuity to further understand this new and
important mathematical discipline.

In section 2, we first introduce soft graphs and soft closed graphs of soft mappings. We give characterization of soft closed graph of identity soft mapping [Theorem 3.2.1 below] and characterization of soft closure of a soft set in terms of convergence of soft net by using notion of soft points [Theorem 3.2.2 below]. Some other characterization of soft closed graph of soft mapping are obtained [Theorem 3.2.3 and Theorem 3.2.5 below].

In section 3, we show relationship between soft continuous mappings and their soft graph [Theorem 3.3.3 and Corollary 3.3.3 below]. We give characterization of soft continuity by using the concept of soft points [Theorem 3.3.2 below]. It is shown that soft equalizer $E_A$ of soft continuous mapping and a soft mapping with soft closed graph is soft closed [Theorem 3.3.4 below]. Also, we introduce soft filter generated by soft net and vice-versa and show their convergence results correspond to each other [Proposition 3.3.1 and Proposition 3.3.2 below].

In section 4, we give an example to show that if graph of a mapping $\varphi$ is closed then soft graph of soft mapping $(\varphi, e)$ defined in Definition 3.1.4 need not be soft closed [Example 3.4.1 below]. In Example 3.4.2 and Example 3.4.3, we show that a soft continuous mapping may not have soft closed graph and a soft mapping with soft closed graph need not be soft continuous, if the domain is not soft compact.

Remark 3.1.1. Some of the results of this chapter [Theorem 3.2.2 and Theorem 3.3.2 below] cannot be proved by using the concept of points in a soft set as used by many authors eg [24]. Instead, we see in this chapter that the concept of “soft points” is more useful and enables to prove better and stronger results.

Notation: Throughout, $X$ refers to an initial universe set, $P(X)$ the power set of $X$ and $A$ is a set of parameters. Moreover, $S(X, A)$ denotes the family of all soft sets over $X$ and $SP(X, A)$ denotes the family of all soft points over $X$ where a soft point will be denoted by $P^a_x$.

Conceptual Comments: In ordinary sets, a point $x$ in $X$ corresponds to a map $x : \{\ast\} \to X$ defined by $x(\ast) = x$. So, $X$ is homeomorphic to $Hom(\{x\}, X)$. Analogously more useful concept is obtained by considering $Hom(A, P(X))$ where a singleton $\{x\}$ is replaced by an arbitrary set $A$. Such a set is called a soft set where $A$ is to be taken as a parameterizing set.

We begin with the following preliminaries.
Zorlutuna, Min and Atmaca studied soft topological spaces in [66] and introduced the following important notions which we shall use in our results below. Let $X$ be an initial universe set, $P(X)$ the power set of $X$ and $A$ a set of parameters. A pair $(F, A)$ (or denoted by $F_A$) is called a soft set over $X$, where $F$ is a mapping given by, $F : A \rightarrow P(X)$. In other words, a soft set over $X$ is a parameterized family of subsets of the universe $X$. The family of all soft sets over $X$ is denoted by $S(X, A)$. The complement of a soft set $F_A$ is denoted by $F^c_A$ and is defined by $(F_A)^c = (F^c, A)$ where $F^c : A \rightarrow P(X)$ is a mapping given by $F^c(a) = X - F(a)$, for all $a \in A$. $F^c$ is said to be the soft complement function of $F$. $(F^c)^c$ is the same as $F$ and $((F_A)^c)^c = F_A$ i.e. subsets of $X$ corresponding to a given parameter $a$ for soft sets $F_A$ and $F^c_A$ are complements of each other. A soft set $F_A$ over $X$ is said to be a null soft set, denoted by $\Phi_A$, if for all $a \in A$, $F(a) = \phi$. A soft set $F_A$ over $X$ is said to be an absolute soft set, denoted by $\tilde{X}$, if for all $a \in A$, $F(a) = X$. For two soft sets $F_A$ and $G_A$ over a common universe $X$, $F_A$ is a soft subset of $G_A$ and $G_A$ is a soft superset of $F_A$ if $F(a) \subseteq G(a)$ for all $a \in A$ and is denoted by $F_A \subseteq G_A$. For two soft sets $F_A$ and $G_A$ over a common universe $X$, union of two soft sets of $F_A$ and $G_A$ is the soft set $H_A$, where $H(a) = F(a) \cup G(a)$ for all $a \in A$ and is denoted by $F_A \cup G_A = H_A$. For two soft sets $F_A$ and $G_A$ over a common universe $X$, intersection of two soft sets of $F_A$ and $G_A$ is the soft set $H_A$, where $H(a) = F(a) \cap G(a)$ for all $a \in A$ and is denoted by $F_A \cap G_A = H_A$.

Definition 3.1.1. [66]

1. A subcollection $\tau \subseteq S(X, A)$ is said to be a soft topology on $X$, if
   (i) $\Phi_A, \tilde{X}$ belong to $\tau$.
   (ii) the union of any number of soft sets in $\tau$ belongs to $\tau$.
   (iii) the intersection of any two soft sets in $\tau$ belongs to $\tau$.

   The triplet $(X, \tau, A)$ is called a soft topological space over $X$. The members of $\tau$ are called soft open sets. The soft complement of a soft open set is called a soft closed set in $(X, \tau, A)$. The family of all soft closed sets is denoted by $\tau^c$.

2. Let $(X, \tau, A)$ be a soft topological space over $X$ and $F_A$ a soft set over $X$. Then the soft closure of $F_A$, denoted by $\overline{F_A}$, is the intersection of all soft closed supersets of $F_A$ and $\overline{F_A}$ is the smallest soft closed set in $(X, \tau, A)$ which contains $F_A$.

We shall also make use of the fundamental notion of soft points introduced in [12]. A soft set $P_A$ over $X$ is said to be a soft point, denoted by $P^a_A$, if for the
element \(a \in A\), \(P(a) = \{x\}\) and \(P(a') = \phi\), for all \(a' \neq a\). The soft point \(P_a\) is said to be in the soft set \(F_A\), denoted by \(P_a \in F_A\), if \(x \in F(a)\). Two soft points \(P_{a_1}^{x_1}, P_{a_2}^{x_2}\) are said to be equal if \(a_1 = a_2\) and \(x_1 = x_2\). Thus, \(P_{a_1}^{x_1} \neq P_{a_2}^{x_2}\) \(\iff a_1 \neq a_2\) or \(x_1 \neq x_2\).

The family of all soft points over \(X\) will be denoted by \(\text{SP}(X,A)\). The union of any collection of soft points can be considered as a soft set and every soft set can be expressed as union of all soft points belonging to it;

\[ F_A = \bigcup_{a \in F_a} P_a \]

**Definition 3.1.2.** Let \((X, \tau, A)\) be a soft topological space.

1. **[32]** A soft set \(F_A\) over \(X\) is called a soft neighbourhood of the soft point \(P_a\) if there exist a soft open set \(G_B\) such that \(P_a \in G_B \subseteq F_A\).

The soft neighbourhood system of soft point \(P_a\), denoted by \(\mathcal{N}_a(P_a)\), is the family of all its soft neighbourhoods.

2. **[32]** A soft point \(P_a^x\in F_A\) if and only if each soft neighbourhood of \(P_a^x\) intersects \(F_A\). We may say that a soft point \(P_a^x\) is in the soft closure of the soft set \(F_A\) if and only if it is softly near to \(F_A\).

**Definition 3.1.3.** **[24]** Let \((X, \tau_1, A)\) and \((Y, \tau_2, B)\) be two soft topological spaces. Then the space \((X \times Y, \tau_1 \times \tau_2, A \times B)\) is called soft product topological space over \(X \times Y\), if \(\tau_1 \times \tau_2\) is the collection of all soft unions of elements of \(\{F_A \times G_B \mid F_A \in \tau_1, G_B \in \tau_2\}\), where the cartesian product \(F_A \times G_B\) is defined by a soft set \(H_{A,B}\), i.e. \(H : A \times B \rightarrow P(X \times Y)\) and \(H(a, b) = F(a) \times G(b)\) for all \((a, b)\in A \times B\). In particular, cartesian product of soft points \(P_a^x\) in \(\text{SP}(X,A)\) and \(P_b^y\) in \(\text{SP}(Y,B)\) is a soft point \(P_{(a,b)}^x\) in \(X \times Y\) whose image at \((a, b)\) is \(P_a^x(a) \times P_b^y(b) = (x, y)\).

**Definition 3.1.4.** We now make note of the important concept of soft mappings between families \(S(X, A)\) and \(S(Y, B)\) of soft sets over \(X\) and \(Y\) respectively.

1. **[30]** If \(\varphi : X \rightarrow Y\) and \(e : A \rightarrow B\) are two mappings then for any soft set \(F_A\) in \(S(X, A)\) and \(b \in B\), \(\bigcup_{a \in e^{-1}(b)} \varphi(F(a)) \subset Y\) which is defined to be empty if \(e^{-1}(b) = \phi\). So for each \(F_A\), this gives us a soft set in \(S(Y, B)\) which is denoted by \((\varphi, e)(F_A)\). Therefore, this defines a map \((\varphi, e) : S(X, A) \rightarrow S(Y, B)\) so that \((\varphi, e)(F_A) : B \rightarrow P(Y)\) is defined by

\[
(\varphi, e)(F_A)(b) = \begin{cases} 
\bigcup_{a \in e^{-1}(b)} \varphi(F(a)), & \text{if } e^{-1}(b) \neq \phi \\
\phi, & \text{if } e^{-1}(b) = \phi
\end{cases}
\]

for all \(b \in B\). \((\varphi, e)(F_A)\) is called soft image of a soft set \(F_A\) and
\((\varphi, e) : S(X, A) \to S(Y, B)\) is called a soft mapping.

2. **Inverse image under soft mappings:** Let \((\varphi, e) : S(X, A) \to S(Y, B)\) be a soft mapping as above. If \(G_B\) is a soft set in \(S(Y, B)\). Then for all \(a\) in \(A\), \((\varphi, e)^{-1}(G_B)(a) = \varphi^{-1}(G(e(|a|)))\) is a subset of \(X\). This defines the soft set \((\varphi, e)^{-1}(G_B) : A \to P(X)\) in \(S(X, A)\). The soft set \((\varphi, e)^{-1}(G_B)\) is called *soft inverse image* of a soft set \(G_B\) under the soft mapping \((\varphi, e)\).

Concept of soft mapping defined above is illustrated by the following diagram.

**Definition 3.1.5.** \([24]\)

1. Let \(F_A\) be a soft set over \(X\) and \(Y\) be a nonempty subset of \(X\). Then the *sub-soft set* of \(F_A\) over \(Y\) denoted by \((^YF_A)\) is defined as, \(^YF(a) = Y \cap F(a)\), for each \(a \in A\).

2. Let \((X, \tau, A)\) be a soft topological space over \(X\) and \(Y\) is a nonempty subset of \(X\). Then \(\tau_Y = \{(^YF_A) | F_A \in \tau\}\) is said to be *soft relative topology* on \(Y\) and \((Y, \tau_Y, A)\) is called a *soft subspace* of \((X, \tau, A)\).

**Definition 3.1.6.** \([56]\) Let \((X, \tau, A)\) be a soft topological space.

1. A family \(C = \{(F_A)_i | i \in I\}\) of soft open sets in \((X, \tau, A)\) is called soft open cover of \(X\), if it satisfies \(\bigcup_{i \in I}(F_A)_i = \tilde{X}\). A finite subfamily of a soft open cover \(C\) of \(X\) is called a finite subcover of \(C\), if it is also a soft open cover of \(X\).
2. X is called soft compact if every soft open cover of X has a finite subcover.

Definition 3.1.7. A soft topological space \((X, \tau, A)\) is called soft Hausdorff space if for any two distinct soft points \(P_{a_1}^{x_1}, P_{a_2}^{x_2} \in SP(X, A)\) there exist soft open sets \(F_A\) and \(G_B\) such that \(P_{a_1}^{x_1} \not\in F_A\), \(P_{a_2}^{x_2} \not\in G_B\) and \(F_A \cap G_B = \emptyset\).

The following results will be utilized in this chapter.

Proposition 3.1.1. Let \(F_A, G_A \in S(X, A)\) and \(P_{a_1}^x \in SP(X, A)\) then we have,

1. \(P_{a_1}^x \not\in G_A\) if and only if \(P_{a_1}^x \notin G_A\).
2. \(P_{a_1}^x \not\in F_A \cup G_A\) if and only if \(P_{a_1}^x \not\in F_A\) or \(P_{a_1}^x \not\in G_A\).
3. \(P_{a_1}^x \not\in F_A \cap G_A\) if and only if \(P_{a_1}^x \not\in F_A\) and \(P_{a_1}^x \not\in G_A\).
4. \(F_A \not\subseteq G_A\) if and only if \(P_{a_1}^x \not\subseteq G_A\).

Proposition 3.1.2. Let \((X, \tau, A)\) be a soft topological space. A point \(P_{a_1}^x \not\subseteq F_A\) if and only if each soft neighbourhood of \(P_{a_1}^x\) intersects \(F_A\).

Proposition 3.1.3. Let \((X, \tau, A)\) be a soft compact topological space and \(F_A\) be a soft closed set on \(X\), then \(F_A\) is soft compact set on \(X\).

Proposition 3.1.4. Let \((\varphi, e) : S(X, A) \rightarrow S(Y, B)\) be a soft mapping and \(P_{a_1}^x \in SP(X, A)\). Then \((\varphi, e)(P^{x_1}) = P_{e(a)}^{\varphi(x)} \in SP(Y, B)\).

Proposition 3.1.5. Let \((\varphi, e) : S(X, A) \rightarrow S(Y, B)\) be a soft mapping and \(P_{b_1}^y \in SP(Y, B)\). Then \((\varphi, e)^{-1}(P_{b_1}^y) = P_{e^{-1}(b)}^{\varphi^{-1}(e)} \in SP(X, A)\).

Proposition 3.1.6. Let \((X, \tau, A)\) be a soft topological space over \(X\). Then the collection \(\tau_a = \{F(a) : F_A \in \tau\}\) for each \(a \in A\), defines a topology on \(X\).

### 3.2 Soft closed graph and its characterizations

It is well known that the graph of a function \(f : X \rightarrow Y\) is the collection of all ordered pair \((x, f(x))\) which is subset of \(X \times Y\). In this section, we analogously define soft graph of a soft mapping using the notion of soft points. Firstly, it must be a soft set in \(X \times Y\) with parameterizing set \(A \times B\). For defining this, let \(x \in X\) and \(P_{a_1}^x \in SP(X, A)\). Now, by definition of soft mapping \((\varphi, e)\),

\[
(\varphi, e)(P_{a_1}^x)(b) = \begin{cases} 
\bigcup_{a' \in e^{-1}(b)} \varphi(P_{a'}^x), & \text{if } e^{-1}(b) \neq \emptyset, \\
\phi, & \text{if } e^{-1}(b) = \emptyset.
\end{cases}
\]

\[
= \begin{cases} 
\varphi(x), & \text{if } a \in e^{-1}(b) \\
\phi, & \text{if } a \notin e^{-1}(b)
\end{cases}
\]

where \((\varphi, e)(P_{a_1}^x)(b) \subseteq Y\).
Therefore, \[ \bigcup_x \bigcup_a \varphi(x)(P^\varphi_a)(b) = \left\{ \begin{array}{ll} \bigcup_x \varphi(x) & \text{if } b = e(a) \\ \phi & \text{if } b \neq e(a) \end{array} \right. \]

\[ \mathcal{G}(\varphi) \quad \text{if } b = e(a) \]
\[ \phi, \quad \text{if } b \neq e(a) \]

which is a subset of \( X \times Y \).

As every soft set is soft union of its soft points, we get the following definition of soft graph whose image of parameter \((a, b)\) will be \[ \bigcup_x \bigcup_a \varphi(x)(P^\varphi_a)(b) \]

\[ = \left\{ \begin{array}{ll} \mathcal{G}(\varphi) & \text{if } b = e(a) \\ \phi & \text{if } b \neq e(a) \end{array} \right. \]

Motivated by these considerations we make the following

**Definition 3.2.1.** Let \( S(X, A) \) and \( S(Y, B) \) be the families of all soft sets over \( X \) and \( Y \) respectively and \((\varphi, e): S(X, A) \to S(Y, B)\) be a soft mapping. Then the **soft graph of** \((\varphi, e)\) is a soft set \( \mathcal{G}(\varphi, e)_{A \times B} \), where \( \mathcal{G}(\varphi, e): A \times B \to \mathcal{P}(X \times Y) \) is defined by

\[ \mathcal{G}(\varphi, e)(a, b) = \left\{ \begin{array}{ll} \mathcal{G}(\varphi) & \text{if } b = e(a) \\ \phi & \text{if } b \neq e(a) \end{array} \right. \]

Here \( \mathcal{G}(\varphi) \) is usual graph of the function \( \varphi \).

**Definition 3.2.2.** Let \((X, \tau_1, A)\) and \((Y, \tau_2, B)\) be two soft topological spaces. If \( \mathcal{G}(\varphi, e)_{A \times B} \) is soft closed in the soft product topological space \((X \times Y, \tau_1 \times \tau_2, A \times B)\), then we say that \((\varphi, e)\) is a soft mapping with **soft closed graph**.

**Remark 3.2.1.** We show in Example 3.4.1 below that although soft closed graph of \((\varphi, e)\) will give us a closed graph of \( \varphi \) but the converse is not true. It is straight forward to see that for soft topological spaces \((X, \tau_1, A)\) and \((Y, \tau_2, B)\), if soft mapping \((\varphi, e): S(X, A) \to S(Y, B)\) has a soft closed graph then the graph of \( \varphi: X \to Y \) is closed.

We begin by giving characterization of soft closed graph of identity soft mapping

**Theorem 3.2.1.** Let \((X, \tau, A)\) be a soft topological space. Let \( 1_X : X \to X \) and \( 1_A : A \to A \) be identity mappings then the soft mapping \((1_X, 1_A): S(X, A) \to S(X, A)\) has a soft closed graph if and only if \((X, \tau, A)\) is soft hausdorff space.

**Proof.** Let \((X, \tau, A)\) be soft hausdorff space. Soft graph of \((1_X, 1_A)\) is a soft set \( \mathcal{G}(1_X, 1_A): A \times A \to \mathcal{P}(X \times X) \) defined by

\[ \mathcal{G}(1_X, 1_A)(a, b) = \left\{ \begin{array}{ll} \{(x, x)|x \in X\}, & \text{if } a = b \\ \phi, & \text{if } a \neq b \end{array} \right. \]

Assume that \( P^{(x,y)}_{(a,b)} \subseteq \mathcal{G}(1_X, 1_A)_{A \times A} \) then \( P^x_a \neq P^y_b \). Since \((X, \tau, A)\) is soft hausdorff space, there exist soft open sets \( F_A \) and \( G_B \) such that \( P^x_a \subseteq F_A \) and \( P^y_b \subseteq G_A \) and \((F_A) \cap (G_A) = \Phi\). Therefore, \( P^{(x,y)}_{(a,b)} \subseteq F_A \times G_A \subseteq \mathcal{G}(1_X, 1_A)_{A \times A} \)

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where $F_A \times G_A$ is soft open set in $X \times X$. This implies $G(1_X, 1_A)^c$ is soft open and hence, $G(1_X, 1_A)_{A \times A}$ is soft closed in $X \times X$.

Conversely, Let $P_a^x, P_b^y \in SP(X, A)$ be any two distinct soft points. Then, $P_{(a,b)}^{(x,y)} \not\in G(1_X, 1_A)^c_{A \times A}$ which is soft open in $X \times X$ implies there exist soft open sets $M_A$ and $N_B$ in $X$ such that $P_{(a,b)}^{(x,y)} \not\in M_A \times N_A \subseteq G(1_X, 1_A)_{A \times A}$. Therefore, $P_a^x \not\in M_A, P_b^y \not\in N_A$ and $(M_A) \cap (N_A) = \emptyset$, proving the converse.

Next, we need the characterization of soft closure in terms of convergence of soft nets to study soft graphs further. For this, we need the following

**Definition 3.2.3.** [12] Let $X$ be a set and $(D, \preceq)$ be a directed set [29] where a directed set is a set with a relation $\preceq$ (by b $\succeq$ a, we mean a $\preceq$ b) which is reflexive, transitive and upwards directive (where a set $D$ is directed if for each $m, n \in D$, there is some $p \in D$ such that $p \succeq m, p \succeq n$). The function $T : D \rightarrow SP(X, A)$ defined by $T(\alpha) = P_{a_{\alpha}}^{x_{\alpha}}$ is called a soft net in $X$. For $\alpha \in D$, $T(\alpha)$ is denoted by $T_\alpha$ and hence a soft net is denoted by $\{T_\alpha = P_{a_\alpha}^{x_\alpha} \mid \alpha \in D\}$.

**Example 3.2.1.** [12] Since $(N_\tau(P_a^x), \preceq^*)$ is a directed set (where the relation $\preceq^*$ is defined by $F_A \preceq^* G_B$ if and only if $G_B \subseteq F_A$, the function $T : N_\tau(P_a^x) \rightarrow SP(X, A)$ is a soft net $\{T_{F_A} \mid F_A \in N_\tau(P_a^x)\}$ where $T_{F_A} = P_a^x \tilde{\in} F_A$.

**Remark 3.2.2.** Let $(X, \tau_1, A)$ and $(Y, \tau_2, B)$ be two soft topological spaces and $(\varphi, e) : S(X, A) \rightarrow S(Y, B)$ be a soft mapping. Let $\{T_\alpha = P_{a_\alpha}^{x_\alpha} \mid \alpha \in D\}$ be a soft net in $(X, \tau_1, A)$ then $\{(\varphi, e)(T_\alpha) = (\varphi, e)(P_{a_\alpha}^{x_\alpha}) = P_{e(a_\alpha)}^{\varphi(x_\alpha)} \mid \alpha \in D\}$ is a soft net in $(Y, \tau_2, B)$ and $\{P_{(a_\alpha, e(a_\alpha))}^{\varphi(x_\alpha)} \mid \alpha \in D\}$ is a soft net in $(X \times Y, \tau_1 \times \tau_2, A \times B)$.

**Definition 3.2.4.** [12] Let $(X, \tau, A)$ be a soft topological space, $\{T_\alpha = P_{a_\alpha}^{x_\alpha} \mid \alpha \in D\}$ be a soft net in $X$ and $F_A \in S(X, A)$

1. The soft net $\{P_{a_\alpha}^{x_\alpha} \mid \alpha \in D\}$ is called in $F_A$ if $P_{a_\alpha}^{x_\alpha} \tilde{\in} F_A$, for all $\alpha \in D$.

2. The soft net $\{P_{a_\alpha}^{x_\alpha} \mid \alpha \in D\}$ is called eventually in $F_A$ if there exist some $\alpha_0 \in D$ such that $P_{a_\alpha}^{x_\alpha} \tilde{\in} F_A$ for all $\alpha \succeq \alpha_0$.

**Definition 3.2.5.** [12] A soft net $\{P_{a_\alpha}^{x_\alpha} \mid \alpha \in D\}$ in a soft topological space $(X, \tau, A)$ is said to converge to $P_a^x$, if it is eventually in every soft neighbourhood of $P_a^x$.

**Proposition 3.2.1.** If $P_{a_\alpha}^{x_\alpha} \rightarrow P_a^x$ where $P_a^x \in SP(X, A)$ and $P_{e(a_\alpha)}^{\varphi(x_\alpha)} \rightarrow P_b^y$ where $P_b^y \in SP(X, A)$ if and only if $P_{(a_\alpha, e(a_\alpha))}^{\varphi(x_\alpha)} \rightarrow P_{(a, b)}^{(x, y)}$ where $P_{(a, b)}^{(x, y)} \in SP(X \times Y, A \times B)$.
Proof. Let $P_{x_a}^x \to P_a^x$ and $P_{e(a_a)}^x \to P_{y}^y$. Assume $G_{A \times B} \in \mathcal{N}_{\tau}(P_{(a,b)}^{(x,y)})$ then there exist soft open sets $F_A$ and $H_B$ in $X$ and $Y$ respectively such that $P_{(a,b)}^{(x,y)} \in F_A \times H_B \subseteq G_{A \times B}$ which implies $P_{a}^x \in F_A$ and $P_{y}^y \in H_B$. As $P_{x_a}^x \to P_a^x$ and $P_{e(a_a)}^x \to P_{y}^y$ there exist $\alpha' \in D$ and $\alpha'' \in D$ such that $P_{a}^x \in F_A$ for all $\alpha \geq \alpha'$ and $P_{e(a_a)}^x \in H_B$ for all $\alpha \geq \alpha''$. Then for $\alpha', \alpha'' \in D$ there exists $\alpha^0 \in D$ such that $\alpha^0 \geq \alpha'$ and $\alpha^0 \geq \alpha''$. This implies $P_{(a_a,e(a_a))}^{(x,a)} \subseteq F_A \times H_B \subseteq G_{A \times B}$ for all $\alpha \geq \alpha^0$ and hence $P_{(a_a,e(a_a))}^{(x,a)} \to P_{(a,b)}^{(x,y)}$.

Conversely, Let $F_A$ and $G_B$ be soft open sets in $X$ and $Y$ respectively such that $P_{a}^x \subseteq F_A$ and $P_{y}^y \subseteq G_B$ which implies $F_A \times G_B$ be soft open sets in $X \times Y$ such that $P_{(a,b)}^{(x,y)} \subseteq F_A \times G_B$. As $P_{(a_a,e(a_a))}^{(x,a)} \to P_{(a,b)}^{(x,y)}$, there exist an index $\alpha' \in D$ such that $P_{a}^x \subseteq F_A$ for all $\alpha \geq \alpha'$. This implies $P_{a}^x \subseteq F_A$ and $P_{e(a_a)}^x \subseteq G_B$ for all $\alpha \geq \alpha'$ and therefore, $P_{a}^x \to P_{a}^x$ and $P_{e(a_a)}^x \to P_{y}^y$.

The following theorem gives characterization of soft closure of a soft set in terms of convergence of its soft net by using soft points. It is important to note that sufficient part of the following theorem is not true when we use concept of points belongs to a soft set, rather than soft points and therefore, by using the notion of soft points we get better results and are able to characterize soft closure below.

**Theorem 3.2.2.** Let $(X, \tau, A)$ be a soft topological space. Let $G_A \in S(X, A)$ be a soft set and $P_a^x \in SP(X, A)$. Then $P_{a}^x \subseteq \overline{G_A}$ if and only if there exist a soft net $\{P_{a}^x|\alpha \in D\}$ in $G_A$ i.e. $P_{a}^x \subseteq G_A$ for all $\alpha \in D$ such that $P_{a}^x \to P_a^x$.

**Proof.** Let $(D, \leq)$ be a directed set and $\{P_{a}^x|\alpha \in D\}$ be a soft net in $G_A$ such that $P_{a}^x \to P_a^x$. Let $F_A \in \mathcal{N}_{\tau}(P_{a}^x)$. Now as $P_{a}^x \to P_a^x$, there exists an index $\alpha' \in D$ such that $P_{a}^x \subseteq F_A$ for all $\alpha \geq \alpha'$. Thus, $P_{a_a'}^x \subseteq F_A \cap G_A$ i.e. $x_{\alpha'} \subseteq F_A(a_{\alpha'}) \cap G_A(a_{\alpha'})$. In order to satisfy $x_{\alpha'} \subseteq F_A(a_{\alpha'}) \cap G_A(a_{\alpha'})$ and therefore we have $F_A \cap G_A \neq \emptyset$. Hence, $P_a^x \subseteq \overline{G_A}$.

Conversely, let $P_{a}^x \subseteq \overline{G_A}$. This implies every soft neighborhood of $P_{a}^x$ intersects $G_A$, that is $H_A \cap G_A \neq \emptyset$ for every $H_A \in \mathcal{N}_{\tau}(P_{a}^x)$. Then there exists an $a \in A$ such that $H_A(a) \cap G_A(a) \neq \emptyset$. Let $y_{H_A} \in H_A(a) \cap G_A(a)$ then $P_{a}^y \subseteq H_A \cap G_A$. Now $(\mathcal{N}_{\tau}(P_{a}^x), \subseteq^*)$ is a directed set (where the relation $\subseteq^*$ is defined by $G_B \subseteq^* F_A$ if and only if $F_A \subseteq G_B$). Define a soft net, $T : \mathcal{N}_{\tau}(P_{a}^x) \to SP(X, A)$ as $T_{H} = P_{a}^{H_{A}} \subseteq H_A \cap G_A$. Now $P_{a}^{H_{A}} \subseteq G_A$ for every $H_A \in \mathcal{N}_{\tau}(P_{a}^x)$. This implies $P_{a}^{H_{A}}$ is a soft net in $G_A$. For proving $P_{a}^{H_{A}} \to P_a^x$, assume $F_A \in \mathcal{N}_{\tau}(P_{a}^x)$ and $F_A \subseteq^* H_A$ such that $H_A \subseteq F_A$. Therefore, $T_{H} = P_{a}^{H_{A}} \subseteq H_A \subseteq F_A$ which implies $P_{a}^{H_{A}} \subseteq F_A$, for every $F_A \subseteq^* H_A$. Hence $P_{a}^{H_{A}} \to P_a^x$. □
We are now able to give another characterization of soft closed graph of a soft mapping as given below:

**Theorem 3.2.3.** Let \((X, \tau_1, A)\) and \((Y, \tau_2, B)\) be two soft topological spaces. A soft mapping \((\varphi, e) : S(X, A) \to S(Y, B)\) has a soft closed graph if and only if whenever \(P^x_{a_\alpha} \to P^x_a\) in \(X\) and \((\varphi, e)(P^x_{a_\alpha}) = P^y_{e(a_\alpha)} \to P^y_b\) in \(Y\) then \(P^y_b = (\varphi, e)(P^x_a) = P^x(\varphi(e))\).

**Proof.** Suppose \((\varphi, e)\) has a soft closed graph and \(P^x_{a_\alpha} \to P^x_a\) in \(X\) and \((\varphi, e)(P^x_{a_\alpha}) = P^y_{e(a_\alpha)} \to P^y_b\) in \(Y\). Then by Proposition 3.2.1, \(P^{(x,y)}_{(a_\alpha, e(a_\alpha))} \to P^{(x,y)}_{(a, b)}\) where \(P^{(x,y)}_{(a_\alpha, e(a_\alpha))} \subseteq \mathbb{G}(\varphi, e)_{A \times B}\), soft graph of \((\varphi, e)\). It implies \(P^{(x,y)}_{(a_\alpha, e(a_\alpha))}\) is a soft net in \(\mathbb{G}(\varphi, e)_{A \times B}\) converging to \(P^{(x,y)}_{(a, b)}\). Therefore, by Theorem 3.2.2, \(P^{(x,y)}_{(a, b)} \subseteq \mathbb{G}(\varphi, e)_{A \times B}\). Since \(\mathbb{G}(\varphi, e)_{A \times B}\) is soft closed, then \(P^{(x,y)}_{(a, b)} \subseteq \mathbb{G}(\varphi, e)_{A \times B}\). Hence \(b = e(a)\) and \(y = \varphi(x)\). Therefore, \(P^y_b = P^x(\varphi(e)) = (\varphi, e)(P^x_a)\).

Conversely, suppose \(P^x_{a_\alpha} \to P^x_a\) and \((\varphi, e)(P^x_{a_\alpha}) = P^y_{e(a_\alpha)} \to P^y_b\) implies \(P^y_b = (\varphi, e)(P^x_a) = P^x(\varphi(e))\) and let \(P^{(x,y)}_{(a_\alpha, e(a_\alpha))} \subseteq \mathbb{G}(\varphi, e)_{A \times B}\). Then by Theorem 3.2.2, there exists a soft net \(P^{(x,y)}_{(a_\alpha, e(a_\alpha))}\) in \(\mathbb{G}(\varphi, e)_{A \times B}\) converging to \(P^{(x,y)}_{(a, b)}\) which implies \(P^x_{a_\alpha} \to P^x_a\) and \((\varphi, e)(P^x_{a_\alpha}) = P^y_{e(a_\alpha)} \to P^y_b\). Thus by assumption \(P^y_b = P^x(\varphi(e)) = (\varphi, e)(P^x_a)\) implies \(b = e(a)\) and \(y = \varphi(x)\). Therefore, \(P^{(x,y)}_{(a, b)} \subseteq \mathbb{G}(\varphi, e)_{A \times B}\) and hence, \(\mathbb{G}(\varphi, e)_{A \times B}\) is soft closed. \(\square\)

**Remark 3.2.3.** We note that the soft graph of the restriction of soft mapping with soft closed graph is also soft closed where restriction of soft mapping is defined by

**Definition 3.2.6.** Let \((\varphi, e) : S(X, A) \to S(Y, B)\) be a soft mapping and \(Z \subseteq X\). Then the restriction of \((\varphi, e)\) to \(S(Z, A)\) is a soft mapping \((\varphi, e)|_{S(Z, A)} : S(Z, A) \to S(Y, B)\) defined by the mappings \(e : A \to B\) and \(\varphi|_Z : Z \to Y\) where \(\varphi|_Z\) is the restriction of \(\varphi\) to \(Z\).

**Theorem 3.2.4.** Let the soft mapping \((\varphi, e) : S(X, A) \to S(Y, B)\) have a soft closed graph and \(M \subset X\), \(N \subset Y\) such that \(\varphi(M) \subset N\) then \((\varphi, e)|_{S(M, A)} : S(M, A) \to S(Y, B)\) also has a soft closed graph.

**Proof.** Let \(\{P^x_{a_\alpha} | \alpha \in D\}\) be a soft net in \(M\) such that \(P^x_{a_\alpha} \to P^x_a\) where \(P^x_a \in SP(M, A)\) and \(\{(\varphi, e)(P^x_{a_\alpha}) = P^y_{e(a_\alpha)} | \alpha \in D\}\) be a soft net in \(N\) such that \(P^y_{e(a_\alpha)} \to P^y_b\) where \(P^y_b \in SP(N, B)\). Now \(\{P^x_{a_\alpha} | \alpha \in D\}\) is a soft net in \(X\) and \(\{P^x_{e(a_\alpha)} | \alpha \in D\}\) is a soft net in \(Y\). Then by Theorem 3.2.3, \(P^y_b = (\varphi, e)(P^x_a) = P^x(\varphi(e)) = (\varphi, e)|_{S(M, A)}(P^x_a)\) since \((\varphi, e)\) has a soft closed graph. \(\square\)
Finally, in this section we give another characterization of soft closed graph using soft open sets.

**Theorem 3.2.5.** The soft mapping $(\varphi, e) : S(X, A) \rightarrow S(Y, B)$ has a soft closed graph if and only if for every $P^a_\varphi \in SP(X, A)$ and $P^y_\varphi \in SP(Y, B)$ where $P^y_\varphi \neq (\varphi, e)(P^a_\varphi) = P^x_\varphi$ there exist soft open sets $F_A$ in $X$ where $P^a_\varphi \subseteq F_A$ and $G_B$ in $Y$ where $P^y_\varphi \subseteq G_B$ such that $(\varphi, e)(F_A) \cap G_B = \Phi$.

**Proof.** Suppose the given condition holds. For proving that the soft graph $G(\varphi, e)_{A \times B}$ is soft closed, we prove $G(\varphi, e)_{A \times B} = G(\varphi, e)_{A \times B}$. Let us assume $P^{(x,y)}_{(a,b)} \in G(\varphi, e)_{A \times B}$ which implies either $b \neq e(a)$ or $y \neq \varphi(x)$. Then $P^y_\varphi \neq (\varphi, e)(P^a_\varphi) = P^{x(a)}_\varphi$. Therefore by hypothesis, there exist soft open sets $F_A$ in $X$ where $P^a_\varphi \subseteq F_A$ and $G_B$ in $Y$ where $P^y_\varphi \subseteq G_B$ such that $(\varphi, e)(F_A) \cap G_B = \Phi$, which implies $(F_A \times G_B) \cap G(\varphi, e)_{A \times B} = \Phi$, where $F_A \times G_B$ is soft open set such that $P^{(x,y)}_{(a,b)} \subseteq F_A \times G_B$. Hence $P^{(x,y)}_{(a,b)} \subseteq G(\varphi, e)_{A \times B}$ and so $G(\varphi, e)_{A \times B}$ is soft closed.

Conversely, let $G(\varphi, e)_{A \times B}$ be soft closed and $P^x_\varphi \in SP(X, A)$. Assume $P^y_\varphi \in SP(Y, B)$ where $P^y_\varphi \neq P^{x(a)}_\varphi$, Therefore either $b \neq e(a)$ or $y \neq \varphi(x)$, which implies $P^{(x,y)}_{(a,b)} \notin G(\varphi, e)_{A \times B}$. As $G(\varphi, e)_{A \times B}$ is soft closed, $P^{(x,y)}_{(a,b)} \notin G(\varphi, e)_{A \times B}$. Then there exist soft open sets $F_A$ and $G_B$ in $X$ and $Y$ respectively where $P^{(x,y)}_{(a,b)} \subseteq F_A \times G_B$ such that $(F_A \times G_B) \cap G(\varphi, e)_{A \times B} = \Phi$, which implies $(\varphi, e)(F_A) \cap G_B = \Phi$. □

### 3.3 Soft closed graph and soft continuity

In this section, we show relationship between soft continuous mappings and their soft graphs. Also, we introduce definition of soft filter generated by soft net and vice-versa. We prove equivalence of convergence theory of soft filter and soft net. Therefore, results proved in [54, 93] for soft filter will also hold for soft nets.

**Definition 3.3.1.** [54] Let $(X, \tau_1, A)$ and $(Y, \tau_2, B)$ be two soft topological spaces. Let $(\varphi, e) : S(X, A) \rightarrow S(Y, B)$ be a soft mapping and $P^x_\varphi \in SP(X, A)$

1. $(\varphi, e)$ is soft continuous at $P^x_\varphi$ if for each $G_B \in \mathcal{N}^e_{\tau}(P^{x(a)}_\varphi)$ there exist $F_A \in \mathcal{N}^e_{\tau}(P^x_\varphi)$ such that $(\varphi, e)(F_A) \subseteq G_B$.

2. $(\varphi, e)$ is soft continuous if $(\varphi, e)$ is soft continuous at each soft point in $SP(X, A)$.

We shall make use of the following.
Theorem 3.3.1. [62] Let \((X, \tau_1, A)\) and \((Y, \tau_2, B)\) be two soft topological spaces. A soft mapping \((\varphi, e) : S(X, A) \rightarrow S(Y, B)\) is soft continuous if and only if \((\varphi, e)^{-1}(F_B)\) is soft closed in \(X\) for all soft closed subsets \(F_B\) of \(Y\).

As in the characterization of soft closure in Theorem [3.2.2] use of soft points below enable us to give characterization of soft continuity which is not possible by using points of the soft set

Theorem 3.3.2. Let \((X, \tau_1, A)\) and \((Y, \tau_2, B)\) be two soft topological spaces. A soft mapping \((\varphi, e) : S(X, A) \rightarrow S(Y, B)\) is soft continuous at \(P^x_a \in SP(X, A)\) if and only if \(P^x_{a\alpha} \rightarrow P^x_a\) implies \(P^{\varphi(x)_{\alpha}}_{e(a\alpha)} \rightarrow P^{\varphi(x)}_{e(a)}\).

Proof. Let \((\varphi, e)\) is soft continuous at \(P^x_a\) and \(G_B \in \mathcal{N}_r(P^{\varphi(x)}_{e(a)})\). Then, there exist a soft neighbourhood \(F_A\) of \(P^x_a\) such that \((\varphi, e)(F_A) \subseteq G_B\). If \(P^x_{a\alpha} \rightarrow P^x_a\), then soft net \(\{P^x_{a\alpha} \mid \alpha \in D\}\) is eventually in \(F_A\) and therefore, \(\{P^{\varphi(x)_{\alpha}}_{e(a\alpha)} \mid \alpha \in D\}\) is eventually in \(G_B\). Hence, \(P^{\varphi(x)_{\alpha}}_{e(a\alpha)} \rightarrow P^{\varphi(x)}_{e(a)}\).

Conversely, Suppose \((\varphi, e)\) is not soft continuous at \(P^x_a\), then there is a soft neighbourhood \(G_B\) of \(P^{\varphi(x)}_{e(a)}\) such that \((\varphi, e)(H_A) \nsubseteq G_B\) for every \(H_A \in \mathcal{N}_r(P^x_a)\). Therefore, for every \(H_A\) there is some \(a_{H_A}\) such that \((\varphi, e)(H_A)(a_{H_A}) \nsubseteq G_B(a_{H_A})\). Define a soft net, \(T : \mathcal{N}_r(P^x_a) \rightarrow SP(X, A)\) as \(T_{H_A} = P^{x_{H_A}}_{a_{H_A}}\) where \(P^{x_{H_A}}_{a_{H_A}} \in H_A\) for which \(P^{\varphi(x)_{H_A}}_{e(a_{H_A})} \nsubseteq G_B\). Now, \(P^{x_{H_A}}_{a_{H_A}} \rightarrow P^x_a\) but \(P^{\varphi(x)_{H_A}}_{e(a_{H_A})}\) does not converge to \(P^{\varphi(x)}_{e(a)}\), which contradicts our assumption. ☐

For our next results, we make use of the following

Lemma 3.3.1. [62] A soft topological space \((X, \tau, A)\) is a soft hausdorff space if and only if every soft net in \((X, \tau, A)\) converges to atmost one soft point.

Following theorem shows that if co-domain of a soft continuous mapping is soft hausdorff then that soft mapping has soft closed graph

Theorem 3.3.3. Let \((\varphi, e) : S(X, A) \rightarrow S(Y, B)\) be a soft continuous mapping where \((Y, \tau, B)\) is soft hausdorff space then \(G(\varphi, e)_{AXB}\) is soft closed.

Proof. Suppose \(P^{(x,y)}_{(a,b)} \in G(\varphi, e)_{AXB}\). Then by Theorem 3.3.2, there exist a soft net \(P^{(x_{a\alpha},\varphi(x)_{a\alpha})}_{(a\alpha,e(a\alpha))}\) in \(G(\varphi, e)_{AXB}\) such that \(P^{(x_{a\alpha},\varphi(x)_{a\alpha})}_{(a\alpha,e(a\alpha))} \rightarrow P^{(x,y)}_{(a,b)}\), which implies \(P^{x_{a\alpha}}_{a\alpha} \rightarrow P^x_a\) and \(P^{\varphi(x)_{a\alpha}}_{e(a\alpha)} \rightarrow P^y_b\). Since \((\varphi, e)\) be a soft continuous implies \(P^{\varphi(x)}_{e(a)} \rightarrow P^y_b\) by Theorem 3.3.2. Now, as \(P^{\varphi(x)}_{e(a)} \rightarrow P^y_b\) and \((Y, \tau, B)\) be a soft hausdorff space then by Lemma 3.3.1 \(P^y_b = P^{\varphi(x)}_{e(a)}\) which implies \(b = e(a)\) and \(y = \varphi(x)\) and therefore, \(P^{(x,y)}_{(a,b)} \in G(\varphi, e)_{AXB}\). Hence, \(G(\varphi, e)_{AXB}\) is soft closed. ☐
The following theorem shows that the soft equalizer $E_A$ of soft continuous mapping and a soft mapping with soft closed graph is soft closed.

**Theorem 3.3.4.** Let $(\varphi, e), (f, e') : S(X, A) \to S(Y, B)$ be two soft mappings such that one of them has soft closed graph and the other is soft continuous and $S = \{P^x_a \in SP(X, A) \mid (\varphi, e)(P^x_a) = (f, e')(P^x_a)\}$ be a subset of $SP(X, A)$ then the soft set $E_A = \bigcup_{P^x_a \in S} P^x_a$ is soft closed in $X$.

**Proof.** Assume $(\varphi, e)$ is soft continuous mapping and $(f, e')$ has soft closed graph. Let $P^x_a \in E_A$ then by Theorem 3.2.2, there exist a soft net $P^x_{a_\alpha}$ in $E_A$ such that $P^x_{a_\alpha} \to P^x_a$. Since $(\varphi, e)$ is soft continuous, $(\varphi, e)(P^x_{a_\alpha}) \to (\varphi, e)(P^x_a)$ by Theorem 3.3.2. Also, $P^x_{a_\alpha} \in E_A$ which implies $(\varphi, e)(P^x_{a_\alpha}) = (f, e')(P^x_{a_\alpha})$. Now, $(f, e')$ has a soft closed graph then $P^x_{a_\alpha} \to P^x_a$ and $(f, e')(P^x_{a_\alpha}) \to (\varphi, e)(P^x_a)$ implies $(\varphi, e)(P^x_a) = (f, e')(P^x_a)$. Therefore, $P^x_a \in E_A$ and hence, $E_A$ is soft closed in $X$. \qed

From Theorems 3.3.4 and 3.3.3 above we have the following.

**Corollary 3.3.1.** Let $(\varphi, e), (f, e') : S(X, A) \to S(Y, B)$ be two soft continuous mappings and $(Y, \tau_2, B)$ is soft hausdorff space then $F_A$ defined in above theorem is soft closed in $X$.

In the remainder of this chapter, we discuss soft filters, prove their equivalence relation with soft nets and use soft nets to obtain more results on soft graphs.

**Definition 3.3.2.** [56] A soft filter on $X$ is a non-empty subset $F \subseteq S(X, A)$ such that

1. $\Phi \notin F$,
2. If $F_A, G_B \in F$, then $F_A \cap G_B \in F$,
3. If $F_A \in F$ and $F_A \subseteq G_B$, then $G_B \in F$.

The definition implies that the intersection of a finite number of a soft filter is non-empty and the union of any number of members of a soft filter belongs to soft filter. Also (3) follows that $X_A \in F$.

**Definition 3.3.3.** [56] Let $F_1$ and $F_2$ be two soft filters on $X$. Then, $F_2$ is finer than $F_1$ (or $F_1$ is coarser than $F_2$) if $F_1 \subseteq F_2$.

**Definition 3.3.4.** [56] Let $F$ be a soft filter on $X$. Then a subfamily $C$ of $F$ is called a soft filterbase for $F$ if for any $F_A \in F$ there exist $G_B \in C$ such that $G_B \subseteq F_A$. 

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Definition 3.3.5. A soft filter $\mathcal{F}$ on a soft topological space $(X, \tau, A)$ is said to converge $P^x_a \in SP(X, A)$, if $\mathcal{N}_x(P^x_a) \subseteq \mathcal{F}$ and we write $\mathcal{F} \rightarrow P^x_a$.

For the application of results on soft filters we define the following.

Definition 3.3.6. Soft filter generated by soft net: Let $(D, \subseteq)$ be a directed set and $\{P^x_{\alpha} \mid \alpha \in D\}$ be a soft net in $X$ and $B = \{(G_{\alpha})_A = \bigcup \{P^x_{\alpha} \mid \alpha \in D\}$.

Definition 3.3.7. Soft net generated by soft filter: Let $\mathcal{F}$ be a soft filter on $X$ and let $D_\mathcal{F} = \{(P^x_a, F_A) \mid P^x_a \subseteq F_A \in \mathcal{F}\}$ then $(D, \subseteq)$ be a directed set where the relation $\subseteq$ is defined by $(P^x_a, G_A) \subseteq (P^x_a, F_A)$ if and only if $F_A \subseteq G_A$. Then, the mapping $T : D_\mathcal{F} \rightarrow SP(X, A)$ defined by $T(P^x_a, F_A) = P^x_a$ is soft net generated by soft filter $\mathcal{F}$.

We have established how to generate soft net from soft filter and soft filter from soft net. In order to get some results, we must also show their convergence results correspond to each other.

Proposition 3.3.1. Let $(X, \tau, A)$ is a soft topological space.

1. A soft net $P^x_{\alpha} \rightarrow P^x_a$ if and only if its associated soft filter $\mathcal{F} \rightarrow P^x_a$.

2. A soft filter $\mathcal{F} \rightarrow P^x_a$ if and only if its associated soft net $P^x_{\alpha} \rightarrow P^x_a$.

Proof. 1. Let $\{P^x_{\alpha} \mid \alpha \in D\}$ be a soft net such that $P^x_{\alpha} \rightarrow P^x_a$ and $F_A \in \mathcal{N}_x(P^x_a)$ then there exist $\alpha_0 \in D$ such that $P^x_{\alpha_0} \in F_A$ for all $\alpha \geq \alpha_0$ which implies $\bigcup \{P^x_{\alpha} \mid \alpha \geq \alpha_0\} \subseteq F_A$. Now $(G_{\alpha_0})_A = \bigcup \{P^x_{\alpha} \mid \alpha \geq \alpha_0\} \in B$ where $B$ is a soft filterbase for $\mathcal{F}$. Hence $F_A \in \mathcal{F}$ and therefore, $\mathcal{F} \rightarrow P^x_a$.

Conversely, Assume $F_A$ be soft neighbourhood of $P^x_a$ and $F_A \in \mathcal{F}$ then there exist $G_A \in B$ such that $G_A \subseteq F_A$ where $G_A = (G_{\alpha_0})_A = \bigcup \{P^x_{\alpha} \mid \alpha \geq \alpha_0\}$ which implies $P^x_{\alpha} \subseteq F_A$ for every $\alpha \geq \alpha_0$. Hence $P^x_{\alpha} \rightarrow P^x_a$.

2. Let $\mathcal{F} \rightarrow P^x_a$ and $F_A$ be soft neighbourhood of $P^x_a$. Then $F_A \in \mathcal{F}$ implies $(P^x_a, F_A) \in D_\mathcal{F}$. Let $T : D_\mathcal{F} \rightarrow SP(X, A)$ be a soft net then for $(P^y_b, G_A) \geq (P^x_a, F_A)$, $T(P^y_b, G_A) = P^y_b \subseteq G_A \subseteq F_A$ implies $T(P^y_b, G_A) \subseteq F_A$.

Conversely, Let $T \rightarrow P^x_a$ and $F_A$ be soft neighbourhood of $P^x_a$ then there exist $(P^y_b, G_A) \subseteq F_A$ where $G_A \in \mathcal{F}$ such that for every $(P^y_b, H_A) \geq P^y_b \subseteq G_A \subseteq F_A$.
Proof.
1. Let \( \alpha \in A \). From Definition 3.3.8, \( (P_{b_0}^y, G_A), T(P_{b}^y, H_A) \in F_A \). In particular, for every \( P_{b_0}^y \in G_A \) and \( (P_{b_0}^y, G_A) \simeq (P_{b_0}^y, G_A) \). We get \( T(P_{b}^y, G_A) \in F_A \) implies \( P_{b}^y \in F_A \) and which further implies \( G_A \subseteq F_A \). Hence, \( F_A \in \mathcal{F} \).

We now discuss soft cluster point of nets in terms of soft points

Definition 3.3.8. A soft point \( P_a^x \) is soft cluster point of soft net \( \{P_{a_\alpha}^x \mid \alpha \in D\} \) if for every \( F_A \in \mathcal{N}_r(P_a^x) \) and \( \alpha_0 \in D \) there exist \( \alpha \simeq \alpha_0 \) such that \( P_{a_\alpha}^x \in F_A \).

Definition 3.3.9. \([43] \) Let \( \mathcal{F} \) be a soft filter on a soft topological space \((X, \tau, A)\), a soft point \( P_a^x \) is soft cluster point of \( \mathcal{F} \), if \( P_a^x \in \overline{G_A} \) for every \( G_A \in \mathcal{F} \).

We show now that the known concept of soft cluster points of soft filters is equivalent to our concept of soft cluster points of soft nets

Proposition 3.3.2. Let \((X, \tau, A)\) is a soft topological space.

1. \( P_a^x \) is a soft cluster point of soft net \( \{P_{a_\alpha}^x \mid \alpha \in D\} \) if and only if \( P_a^x \) is a soft cluster point of the associated soft filter \( \mathcal{F} \).

2. \( P_a^x \) is a soft cluster point of soft filter \( \mathcal{F} \) if and only if \( P_a^x \) is a soft cluster point of the associated soft net \( \{P_{a_\alpha}^x \mid \alpha \in D\} \).

Proof.
1. Let \( F_A \) be soft neighbourhood of \( P_a^x \). Now as \( P_a^x \) is a soft cluster point of soft net \( P_{a_\alpha}^x \) then for all \( \alpha_0 \in D \) there exist \( \alpha \simeq \alpha_0 \) such that \( P_{a_\alpha}^x \in F_A \). But by definition \( (G_{a_\alpha})_A = \bigcup_{\alpha \simeq \alpha_0} P_{a_\alpha}^x \in \mathcal{B} \) implies \( (G_{a_\alpha})_A \cap F_A \neq \Phi \). This implies \( P_a^x \) is a soft cluster point of the associated soft filter base \( \mathcal{B} \) and hence, is a soft cluster point of associated filter \( \mathcal{F} \).

Conversely, Let \( P_a^x \) is a soft cluster point of the associated soft filter \( \mathcal{F} \) and \( F_A \in \mathcal{N}_r(P_a^x) \) then for all \( G_A \in \mathcal{F} \) implies \( G_A \cap F_A \neq \Phi \). In particular, for all \( (G_{a_\alpha})_A \in \mathcal{B} \) we have, \( (G_{a_\alpha})_A \cap F_A \neq \Phi \). Now \( \mathcal{B} \) consists of members of type \( (G_{a_\alpha})_A = \bigcup_{\alpha \simeq \alpha_0} P_{a_\alpha}^x \). Hence, for all \( \alpha_0 \in D \) there exists \( \alpha \simeq \alpha_0 \) we get \( P_{a_\alpha}^x \in F_A \).

2. Let \( P_a^x \) is a soft cluster point of soft filter \( \mathcal{F} \). Assume \( G_A \in \mathcal{N}_r(P_a^x) \) and \( (P_{b}^y, H_A) \in D_F \) be any element where \( H_A \in \mathcal{F} \) then by assumption we get \( G_A \cap H_A \neq \Phi \) then there exist \( a_0 \) such that \( G(a_0) \cap H(a_0) \neq \phi \). Let \( y_0 \in G(a_0) \cap H(a_0) \). Therefore, \( P_{a_0}^y \in G_A \cap H_A \). Then \( (P_{a_0}^y, H_A) \in D_F \).
and \((P_{a_0}^y, H_A) \succeq (P_b^y, H_A)\), we have \(T(P_{a_0}^y, H_A) = P_{a_0}^y \tilde{\in} G_A\).

Conversely, let \(G_A \in \mathcal{N}(P_a^y)\) and \(F_A \in \mathcal{F}\). Then, \((P_b^y, F_A) \in \mathcal{D}(\mathcal{F})\) and therefore, there exists \((P_{b'}^y, H_A) \succeq (P_b^y, F_A)\) such that \(T(P_{b'}^y, H_A) = P_{b'}^y \tilde{\in} H_A \cap G_A\). Now \(H_A \subseteq F_A\) implies \(P_{b'}^y \tilde{\in} F_A \cap G_A\). Therefore, there exist \(y' \in F(b') \cap G(b')\) and hence \(F_A \cap G_A \neq \emptyset\).

\[\square\]

**Definition 3.3.10.** Let \(X\) be a set and \((D, \preceq)\) and \((M, \preceq)\) be two directed sets. Let \(T : D \to \text{SP}(X, A)\) be a soft net and \(\eta : M \to D\) be a map such that \(\eta\) is monotonic and co-final where as usual co-final means that for every \(\alpha_0 \in D\) there exist \(\mu_0 \in M\) such that \(\eta(\mu) \succeq \alpha_0\) whenever \(\mu \succeq \mu_0\).

\[
\begin{array}{c}
D \xrightarrow{T} \text{SP}(X, A) \\
\eta \downarrow \quad T \circ \eta \\
M
\end{array}
\]

then we say that \(T \circ \eta : M \to \text{SP}(X, A)\) be a soft subnet of the soft net \(T\). We write \(\eta(\mu) = \alpha_\mu\) and \((T \circ \eta)(\mu) = T(\eta(\mu)) = P_{a_\mu}^{\alpha_\mu}\).

**Lemma 3.3.2.** \([64]\) The following are equivalent for a soft topological space \((X, \tau, A)\)

1. A soft topological space \((X, \tau, A)\) is soft compact.
2. Every soft filter on \(X\) has a soft cluster point.
3. Every maximal soft filter on \(X\) converges to a soft point.

Now by study of soft nets generated by soft filters and vice versa discussed above, we see in the following corollary that above theorem is also true in terms of soft nets

**Corollary 3.3.2.** Let \((X, \tau, A)\) be a soft topological space then following are equivalent

1. A soft topological space \((X, \tau, A)\) is soft compact.
2. Every soft net on \(X\) has a soft cluster point.
3. Every universal soft net on \(X\) converges to a soft point.

**Theorem 3.3.5.** Let \((\varphi, e) : S(X, A) \to S(Y, B)\) be a soft mapping and \((\varphi, e)\) has a soft closed graph then \((\varphi, e)^{-1}(F_B)\) is soft closed in \(X\) for all soft compact subsets \(F_B\) of \(Y\).
Proof. If possible, let there exist a soft compact subset \( F_B \) of \( Y \) such that \((\varphi, e)^{-1}(F_B)\) is not soft closed in \( X \). Therefore, there exist \( S P^x_a \) such that \( P^x_a \notin (\varphi, e)^{-1}(F_B) \) then by Theorem 3.3.2, there exist a soft net \( P^x_{a_n} \) in \((\varphi, e)^{-1}(F_B)\) such that \( P^x_{a_n} \to P^x_a \). This implies \((\varphi, e)(P^x_{a_n}) = P^x(e(a_n))\) is a soft net in \( F_B \). Now, \( F_B \) is soft compact then by Corollary 3.3.2, \((\varphi, e)(P^x_{a_n})\) has a soft subnet \((\varphi, e)(P^x_{a_{n_p}})\) which converges to \( P^y_b \notin F_B \). This implies \( P^x_{a_{n_p}} \) is a soft subnet of \( P^x_{a_n} \) and therefore, \( P^x_{a_{n_p}} \to P^x_a \). Since, \((\varphi, e)(P^x_{a_{n_p}}) \to P^y_b \) and \((\varphi, e)\) has a soft closed graph, \( F^y_b = (\varphi, e)(P^x_a) \) by Theorem 3.3.3. This implies \( P^x_a \notin (\varphi, e)^{-1}(P^y_b) \) which contradicts our supposition. □

Finally from Theorem 3.3.1 and Theorem 3.3.5 above we are able to achieve the converse conclusion of Theorem 3.3.3

**Corollary 3.3.3.** Let \((\varphi, e) : S(X, A) \to S(Y, B)\) be a soft mapping with soft closed graph and \((Y, \tau_2, B)\) is soft compact space then \((\varphi, e)\) is soft continuous.

### 3.4 Examples

Following example validates that our Remark 3.2.1 is not true in general.

**Example 3.4.1.** Let \( X = \{x, y, z\}, A = \{0, 1\} \) and \( \tau = \{\Phi, X, (F_1, A), (F_2, A), (F_3, A), (F_4, A), (F_5, A), (F_6, A)\} \) where,

\[
F_1(0) = \{x\}, F_1(1) = \{z\};
\]

\[
F_2(0) = \{y\}, F_2(1) = \{x\};
\]

\[
F_3(0) = \{z\}, F_3(1) = \{y\};
\]

\[
F_4(0) = \{x, y\}, F_4(1) = \{x, z\};
\]

\[
F_5(0) = \{x, z\}, F_5(1) = \{y, z\};
\]

\[
F_6(0) = \{y, z\}, F_6(1) = \{x, y\}.
\]

Let \( \varphi = 1_X \) and \( e = 1_A \) and for each \( a \in A \), \((X, \tau_a)\) is a Hausdorff space then, \( \Delta(\varphi) = \{(x, x) : x \in X\} \) is closed subset of \( X \times X \). Now \((\varphi, e) : S(X, A) \to S(X, A) \) and soft graph of \((\varphi, e)\), \( G(\varphi, e) : A \times A \to P(X \times X)\) is defined by,

\[
G(\varphi, e)(a, b) = \begin{cases} \Delta(\varphi), & \text{if } a = b \\ \phi, & \text{if } a \neq b \end{cases}
\]

is not soft closed in \( X \times X \).

A soft continuous mapping may not have soft closed graph, as shown in following example. Therefore, the condition of the space \((Y, \tau_2, B)\) to be soft hausdorff cannot be dropped for Theorem 3.3.3.
Example 3.4.2. Let \((X, \tau, A)\) be the soft topological space of Example 3.4.1 which is not Hausdorff. Let \((\varphi, e) : S(X, A) \to S(X, A)\) be soft mapping where \(\varphi = 1_X\) and \(e = 1_A\). Then \((\varphi, e)\) is soft continuous but soft graph defined in Example 3.4.1 is not soft closed.

Following example shows that a soft mapping with soft closed graph need not be soft continuous, if the domain is not soft compact (c.f. Corollary 3.3.3).

Example 3.4.3. Let \(X = \mathbb{R}\), the set of real numbers. Let \(A = \{a\}\) and \(\tau = \{\emptyset, X\} \cup \{(F_U)_A\}\) where, \((F_U)_A = \{(a, U)\}\), where \(U\) is usual open set in \(\mathbb{R}\). Then \((X, \tau, A)\) is a soft topological space. Define \(\varphi : \mathbb{R} \to \mathbb{R}\) by, \(\varphi(x) = \begin{cases} 1/x, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}\) and \(e : A \to A\) by \(e(a) = a\). Then \((\varphi, e) : S(X, A) \to S(X, A)\) be soft mapping and soft graph of \((\varphi, e)\), \(G(\varphi, e) : A \times A \to \mathcal{P}(X \times X)\) defined by, 
\[
G(\varphi, e)(a, a) = \{(x, 1/x)|x \in \mathbb{R}\} \cup \{0, 0\},
\]
is soft closed in \(X \times X\). But \((\varphi, e)\) is not soft continuous at \(P^0_a\).