CHAPTER 3

STATIC MODELING AND ANALYSIS OF WORKPIECE-FIXTURE SYSTEM

3.1 PROBLEM FORMULATION AND APPROACH

The accuracy of finished workpiece depends on the fixture contact elements in the workpiece-fixture system. The 98% of all system compliance is captured by modeling just workpiece-fixture contact tips (Siebenaler and Melkote 2006). 60% of the deformation and 90% of the damping in a fabricated structure can arise from various connections. Locators and clamping positions are more influenced in the workpiece deformation/deflection. Hence the positions of all locators and clamps must be in correct positions for giving supports and clamping in the workpiece-fixture system. In this research, the methods for predicting the normal force and contact pressure distribution between the workpiece-fixture contacts and the correct locating coordinates of all active and passive elements which is located in primary, secondary and tertiary surfaces of the workpiece-fixture system is found.

3.2 STATIC WORKPIECE-FIXTURE MODEL ANALYSIS

During clamping, the workpiece must be static equilibrium under the action of clamping loads and its weight. To calculate the elastic deformation of the workpiece at a fixture-workpiece contact due to clamping, the contact reaction force must be known. The principle of minimum total
complementary energy (Bau
chau 2004) is applied here to develop a model
that yields the reaction force at each fixture-workpiece contact. Because the
structural compliance of the workpiece is not considered, the total
complementary energy of the fixture-workpiece system is composed of two
parts that is energy from the fixture-workpiece contacts and energy from the
fixture elements. The complementary energy of fixture-workpiece contacts
can be computed from the contact force-displacement relationships, reported
in the contact mechanics literature. For a spherical-tipped fixture element
pressed against a curved workpiece surface, the relationship is given by
(Johnson 1987),

\[
\delta_z = \left[ \frac{9p^2}{16R(E')^2} \right]^{1/3}
\]

\[
\delta_j = \frac{Q_j}{8a}\left( \frac{2-\nu_w}{G_w} + \frac{2-\nu_f}{G_f} \right) \text{ for } j = x \text{ or } y
\]

Force-displacement relationships for other contact geometries are
also available (Johnson 1987). It is clear from Equation (3.1) that the
relationship between the contact force and the resulting deformation for a
spherical-tipped fixture element is nonlinear. Therefore, the complementary
energy from all fixture-workpiece contacts is written as (Haiyan Deng 2006),

\[
\prod_c = \sum_{i=1}^{(L+C)} \left( \int_0^{P_i} \delta_{zic} \, dp_i + \int_0^{Q_x} \delta_{xic} \, dQ_{ix} + \int_0^{Q_y} \delta_{yc} \, dQ_{iy} \right)
\]

(3.2)

where the subscript \(c\) refers to contact while \(i = 1, \ldots, (L+C)\).

The fixture elements are modelled as linear springs and the
complementary energy of the \((L+C)\) fixture elements is as follows,
\[ \Pi_i = \frac{1}{2} \left( \frac{P_i^2}{k_{ixf}} + \frac{Q_{ix}^2}{k_{ixf}} + \frac{Q_{iy}^2}{k_{iyf}} \right) \]  

(3.3)

where \( k_{ixf}, k_{iyf}, \) and \( k_{izf} \) are the structural stiffnesses of the \( i \)th fixture element in the \( x_i, y_i, \) and \( z_i \) directions, respectively.

The total complementary energy of the fixture-workpiece system is then given by,

\[ \Pi_t = \Pi_c + \Pi_f \]  

(3.4)

The contact reaction forces can be found by minimizing Equation (3.4) subject to a set of the following constraints.

The first constraint comes from the static equilibrium condition of the system and is given as follows,

\[ \sum F = 0, \]

\[ \sum M = 0, \]  

(3.5)

where \( F \) and \( M \) represent the resultant force and moment vectors at the center of gravity of the workpiece in the (xyz) frame, respectively.

The second constraint results from the assumption that constant clamping forces are used and is written as,

\[ P_j = F_{cj} \text{ for } j = (L+1), \ldots, (L+C) \]  

(3.6)

Assuming the Coulomb friction law applies at each fixture-workpiece contact, the third constraint is obtained as follows,
Since the fixture workpiece contact is strictly unilateral, the normal contact force \( P_i \) can only be compressive gives the fourth constraint as,

\[
P_i \geq 0
\]  

(3.8)

where it is assumed that normal forces directed into the workpiece is positive.

The last one is the non-yielding constraint on the contact stress and is given by,

\[
P_i - \sigma_y (\pi a_i^2) \leq 0
\]  

(3.9)

where \( \sigma_y \) is the yield strength of the workpiece material and \( a_i \) is the radius of the \( i^{th} \) contact region as noted earlier. Therefore, the static model is obtained by combining Equations (3.2) through (3.9) and is summarized as follows,

\[
\text{Minimize}
\]

\[
1_{ll}
\]

\[
P_i, Q_{ix}, Q_{iy}
\]

(3.10)

Subjected to:

\[
\sum F = 0, \sum M = 0
\]

\[
P_j = F c_j \quad \text{for } j = (L + 1), ..., (L + C)
\]

\[
\sqrt{(Q_{ix})^2 + (Q_{iy})^2} - \mu_i s P_i \leq 0 \quad \text{for } j = 1, ..., (L + C)
\]

\[
P_i \geq 0 \quad \text{for } j = 1, ..., (L + C)
\]

\[
P_i - \sigma_y (\pi a_i^2) \leq 0 \quad \text{for } j = 1, ..., (L + C)
\]
Solving Equation (3.10) yields the contact reaction forces, $P_i$, $Q_{ix}$, and $Q_{iy}$ with $i = 1$ to $(L+C)$, which are then substituted into Equation (3.1) to obtain the clamping force induced elastic deformation of the workpiece at each contact, $\delta_{ixc}$, $\delta_{yxc}$, and $\delta_{zxc}$.

3.3 TECHNIQUE USED FOR STATIC MODEL

3.3.1 Theoretical Analysis

Fixture contact elements in the workpiece-fixture system are important for the accuracy of the finished workpiece. The spherical-planar contact geometry obtained when a spherical tipped locator or clamp element makes contact with a planar workpiece surface as shown in Figure 3.1. The locators are assumed to have spherically tipped shapes, and that the area of contact is small compared with radius of fixture element. Considering a spherical locators contacting a flat surface of the workpiece, the deflection at the contact point is given by Faupel et al (1981).

$$y_{\text{fixed}} = \left[ 0.465 \frac{P^2}{R_f} \left( \frac{1}{E_w} + \frac{1}{E_f} \right)^2 \right]^{\frac{3}{2}} \tag{3.11}$$

Figure 3.1 Different contact forces acting on the planar surface of workpiece
The equations (3.12) to (3.15) are used to analyze the static conditions of the workpiece-fixture system. Equations (3.16) and (3.17) are used to find out the normal deformation by indentation and contact pressure distribution in the workpiece fixture contact surfaces (Johnson 1987).

Normal deformation $\delta = \left[ \frac{3p}{4E} \right]^\frac{2}{3} + \left[ \frac{1}{R} \right]^\frac{1}{3}$ (3.12)

Maximum contact pressure $p = \frac{1}{11} \left( \frac{6PE^2}{R^2} \right)^\frac{1}{3}$ (3.13)

Maximum shear stress $\tau = 0.31p$ (3.14)

Radius of equilibrium contact area $a = \frac{4PR}{\sqrt{11E}}$ (3.15)

Normal deformation by indentation on the surface

$\left( \sigma_w \right) = \left[ \frac{1}{E_w} \sqrt{\frac{\pi P}{4a}} \right] \left[ 2a^2 - r^2 \right] \quad 0 < r < a$ (3.16)

Contact pressure distribution $p(r) = p \left[ 1 - \frac{r^2}{a^2} \right]^\frac{1}{2} \quad 0 < r < a$ (3.17)

The contact modulus expresses the elastic properties of two bodies $i$ and $j$ as a series springs combination. The relative radius $R_r$ expresses a summation of curvatures (or inverse radii). Note that curvature is positive or negative as long as the relative radius is positive since it represents an equivalent sphere in contact with a plane. Quite different sets of contacting surfaces behave identically if they have identical contact module and relative radii.
There are two methods used to include the contact condition in the energy equation,

a) The Lagrange multiplier and
b) The Penalty function methods (Cook and Malkus 1989, Bathe 2004)

a) **Lagrange multiplier method**

Consider the variational formulation of a discrete structural model for a steady-state analysis,

\[ \Pi = \frac{1}{2} Q^T K Q - Q^T R \]  

(3.18)

where \( K \) is the element stiffness matrix, \( Q \) is the element nodal displacement vector and \( R \) is the vector of nodal force.

With the condition \( \frac{\partial \Pi}{\partial Q_i} = 0 \) for all \( i \)  

(3.19)

and assume that to impose the displacement at the degree of freedom \( Q_i \) with

\[ Q_i = Q_i^* \]  

(3.20)

In the Lagrange multiplier method the right hand side of Equation (3.18) to obtain

\[ \Pi = \frac{1}{2} Q^T K Q - Q^T R + \lambda(Q_i - Q_i^*) \]  

(3.21)

where \( \lambda \) is an additional variable, and \( Q_i^* \) is the contact force vector of the active contact node pairs.
Invoke $\partial [\Pi^*] = 0$, which gives

$$\partial Q^T K Q - \partial Q^T R + \lambda \partial Q_i + \lambda \partial (Q_i - Q^*_i) = 0 \quad (3.22)$$

Since $\partial Q$ and $\partial \lambda$ are arbitrary, we obtain

$$\begin{bmatrix} K & K_i^T \\ K_i & 0 \end{bmatrix} \begin{bmatrix} Q \\ \lambda \end{bmatrix} = \begin{bmatrix} R \\ Q^* \end{bmatrix} \quad (3.23)$$

where $K_i$ is a vector with all entries equal to zero except its entry, which is equal to 1. Hence the equilibrium equations without a constraint are amended with an additional equation that embodies the constraint condition. The Lagrange multiplier method has disadvantages, because the stiffness matrix in Equation (3.23) may contain a zero component in its diagonal. This will lead to computational stability problem.

**b) The Penalty function method**

In the penalty method the right-hand side of Equation (3.18) but without introducing an additional variable, it becomes,

$$\prod^{m*} = -\frac{1}{2} Q^T K Q - Q^T R + \alpha \left( Q_i - Q^*_i \right)^2 \quad (3.24)$$

In which $\alpha$ is a constant of relatively large magnitude, $\alpha \gg \max(k_i)$. The condition $\partial [\prod]^{m*} = 0$

Now yield

$$\partial Q^T K Q - \partial Q^T R + \alpha (Q_i - Q^*_i) \partial Q_i = 0 \quad (3.25)$$
\[(K + \alpha K_i K_i^T)Q = R + \alpha Q_i^T K_i \quad (3.26)\]

where \(\alpha K_i K_i^T\) is the penalty matrix.

Hence, using this technique, a large value is added to the \(i_{th}\) diagonal element of \(K\) and a corresponding force is added so that the required displacement \(Q_i\) is approximately equal to \(Q_i^*\). This is a general technique that has been used extensively to impose specified displacements or other variables. The method is effective because no additional equation is required. It is frequently used in the practical analysis because of its simple implementation.

To understand the above two methods a physical model of the contact of clamp/locator and workpiece is presented as shown in the Figure 3.2.

\[\lambda > 0; \ f_{ni} < 0; \ \lambda \cdot f_{ni} = 0\]

where \(\lambda\) = Distance from a contact point \(i\) and \(j\) in the normal direction of contact.
\( f_{ni} \) = Contact force acting on point \( i \) on workpiece.

To prevent interpenetration the separation distance \( \lambda \) for each contact pair must be greater or equal to zero.

If \( \lambda > 0 \), the contact force \( f_{ni} = 0 \), where \( \lambda = 0 \), the points are in contact and \( f_{ni} < 0 \). If \( \lambda < 0 \), penetration occurs.

In real physics, the actual contact area increases, and contact stiffness is enhanced when the load increase. Therefore, the contact deformation is non-linear as a function of the preload.

### 3.3.2 Clamping Force Calculation

The required clamping force is to be determined by the following analytical method. The clamping force is calculated by using the following machining parameter. Table 3.1 showing the machining details are used in this analysis.

#### Table 3.1 Machining details

<table>
<thead>
<tr>
<th>Sl. No.</th>
<th>Description</th>
<th>Details</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Diameter of end mill</td>
<td>6mm</td>
</tr>
<tr>
<td>2</td>
<td>Material of the cutter</td>
<td>HSS</td>
</tr>
<tr>
<td>3</td>
<td>Material of the workpiece</td>
<td>Aluminum</td>
</tr>
<tr>
<td>4</td>
<td>Cutting speed((V))</td>
<td>1.16 to 1.67m/sec</td>
</tr>
<tr>
<td>5</td>
<td>Spindle speed((S))</td>
<td>1500rpm</td>
</tr>
<tr>
<td>6</td>
<td>Feed</td>
<td>0.2mm/tooth</td>
</tr>
<tr>
<td>7</td>
<td>Horse power</td>
<td>1</td>
</tr>
</tbody>
</table>
The required clamping force based on cutting force is calculated. A simplified example appears below, with cutting force entirely horizontal is considered.

\[
\text{Cutting force } (F_c) = \frac{75 \times HP}{V} \quad (3.27)
\]

\[
\text{Clamping force } (P) = \frac{F_c}{\mu} \times \text{Safety factor} \quad (3.28)
\]

(Safety factor usually taken as 2)

where \( V \) is the cutting speed in m/sec and \( \mu \) is the static coefficient of friction.

![Figure 3.3  Cutting force calculation](image)

Using the above data, the cutting force \((F_c)\) and clamping force \((P)\) are found to be,

- Cutting force \((F_c)\) = 469N
- Clamping force \((P)\) = 1538N

The value of clamping force should always be greater than the cutting force in order to prevent the slippage of workpiece from the fixture elements. Thus the clamping force for case I problem has been taken in
between 1550N and 3550N. Figure 3.3 shows the cutting and clamping forces acting on the workpiece.

### 3.3.3 Contact Load Distribution

The analysis of locator and clamping surfaces of workpiece in fixture system, the contact load distribution is also one of the most important loads to analyses. In this contact load distribution analysis, first to calculate the clamping force by using machining parameter. Using the data in Table 3.1, the cutting speed \( V \) and cutting force \( F_c \) are found to be,

\[
\begin{align*}
\text{Cutting speed} \ (V) &= 1.57 \text{ m/s} \\
\text{Cutting force} \ (F_c) &= 478 \text{ N}
\end{align*}
\]

Since the cutting force \( F_c = 478 \text{N} \), the value of clamping force should be greater than or equal to the above value.

### 3.3.4 Finite Element Analysis (FEA)

ANSYS version 10 FEA package has been used for the finite element analysis with a single clamp model for a 3-2-1 locating scheme to reduce the computational time. All the fixture elements are spherically tipped in shape and assumed to follow the deformation pattern of Figure 3.4. The circular contact area between the fixture elements and the workpiece surface is small compared to the dimensions of the workpiece and fixture elements. The prismatic block and fixture element (locator/clamp) are assembled using solid models.
All components in the system were modelled as isotropic elastic bodies. The fixture tip, shown in Figure 3.4 was modelled as cylinders with spherical end. The prismatic block was modeled by Brick 45 element Solid92 and fixture tip was modeled by 10-node tetrahedral element Solid92 was used to mesh the solid bodies. Contact between the workpiece and fixture element was simulated using the quadratic surface-to-surface contact elements Target170 and Conta174.

### 3.4 SIMULATION STUDY

Single spherical tipped locator/clamp is used to support/clamp the workpiece having flat surfaces as shown in the Figure 3.1. The parameters like contact pressure distribution, maximum shear stress are calculated using ANSYS V10 and the results are compared with that of analytical calculations. The optimal value of Radius of curvature of locator/clamp (R) was calculated based on the following constraint. The maximum shear stress, $\tau_{\text{max}}$ should be less than the indentation limits $\frac{\sigma_y}{2}$ for different values of R.
Table 3.2 shows the properties of workpiece and fixture elements which are used in this simulation study.

**Table 3.2 Properties of workpiece and fixture elements**

<table>
<thead>
<tr>
<th>Description</th>
<th>Workpiece</th>
<th>Locator/Clamp</th>
</tr>
</thead>
<tbody>
<tr>
<td>Materials</td>
<td>Aluminum AISI 7178</td>
<td>Steel</td>
</tr>
<tr>
<td>Elasticity modulus (N/mm²)</td>
<td>75x10³</td>
<td>201x10³</td>
</tr>
<tr>
<td>Density(kg/mm³)</td>
<td>2.7664 x10⁻⁵</td>
<td>7.848x10⁻⁵</td>
</tr>
<tr>
<td>Yield strength(N/mm²)</td>
<td>120</td>
<td>-</td>
</tr>
<tr>
<td>Poisson ratio</td>
<td>0.33</td>
<td>0.3</td>
</tr>
<tr>
<td>Geometry (mm)</td>
<td>150x120x70</td>
<td>25mm</td>
</tr>
</tbody>
</table>

The optimal value of $R$ is found to be 225mm for which the max shear stress is $\tau_{\text{max}}=57.66$N/mm² from the analytical results which are shown in Table A 1.1 in ‘Appendix 1’. The graph (Figure 3.5) shows the effect of variation of radius of locator/clamping elements on the contact pressure.

![Figure 3.5 Radius Vs Maximum contact pressure](image-url)
From the graph it is inferred that the contact pressure is inversely proportional to contact radius. Correspondingly the Figures 3.6 to 3.9 shows the normal deformation, normal deformation by indentation, maximum shear stress and contact pressure distribution are inversely proportional to contact radius. Figure 3.10 showing half width contact area are directly proportional to contact radius.

![Graph 1](image1.png)

**Figure 3.6 Radius Vs Normal deformations**

![Graph 2](image2.png)

**Figure 3.7 Radius Vs Normal deformation by indentation**
**Figure 3.8** Radius Vs Maximum shear stress

**Figure 3.9** Radius Vs Contact pressure distributions

**Figure 3.10** Radius Vs Half width contact area
Table 3.3 Analytical result Vs FEM result (R=225mm)

<table>
<thead>
<tr>
<th>Description</th>
<th>Analytical result</th>
<th>FEM Result</th>
<th>% of Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal deformation(δ), mm</td>
<td>0.0057</td>
<td>0.0043</td>
<td>24.56</td>
</tr>
<tr>
<td>Maximum contact pressure(p), N/mm²</td>
<td>186</td>
<td>132</td>
<td>29.03</td>
</tr>
<tr>
<td>Maximum shear stress(τ), N/mm²</td>
<td>57.66</td>
<td>52.67</td>
<td>8.65</td>
</tr>
<tr>
<td>Normal deformation by indentation(δ_i), mm</td>
<td>0.0035</td>
<td>0.0031</td>
<td>11.43</td>
</tr>
<tr>
<td>Contact pressure distribution(p(n)), N/mm²</td>
<td>161.081</td>
<td>110.718</td>
<td>31.36</td>
</tr>
</tbody>
</table>

The Table 3.3 shows the comparison of analytical result and FEM result for the optimal value of radius of curvature.

3.5 SUMMARY OF THE RESULTS

Using the general formulae for machining, the value of cutting speed (V) and cutting force (F_c), are calculated. From those values, it is found that the clamping force to keep the workpiece in position without damaging during machining is adequate with sufficient contact force. The contact pressure, normal deformation, normal deformation by indentation, maximum shear stress and contact pressure distribution in a frictional workpiece-fixture system are calculated by FEA in ANSYS Version10 Software and the same are compared with that of analytical values. The percentage of error is listed in Table 3.3. It is found that predicted value from FEA shows agreement within 30% of the analytical values. Without using cost consuming process 70-80% results obtained by using FEA are good, and so that it can be considered. This shows that the generated FEA model is a better representation of the real world problem. The graph (Figure 3.5) shows the effect of variation of radius of locator/clamping elements on the contact point. From the graph it is inferred that the contact pressure is inversely proportional to contact radius. The above graphs are useful to calculate the optimal value of the contact pressure to keep the workpiece in position without damaging the workpiece during machining.