Chapter 5

Zip Rings Relative To Skew Monoid

This chapter deals with the some open problems on extensions of zip rings raised by Faith [16]. Here, we establish an equivalent relation between right zip ring and right zip skew monoid ring. This relation give a solution of the problem posed by Faith [16]. Further, we study this relation for right zip skew generalized power series ring by using $(S, \omega)$-Armendariz ring which provide a unified solution of the problem raised by Faith [16].

5.1 Introduction

A ring $\mathcal{R}$ is right (left) zip provided that if the right annihilator $r_{\mathcal{R}}(X)$ (resp., left annihilator $l_{\mathcal{R}}(X)$) of subset $X$ of $\mathcal{R}$ is zero, then there exists a finite subset $Y$ of $X$ such that $r_{\mathcal{R}}(Y) = 0$ (resp., $l_{\mathcal{R}}(Y) = 0$) Faith [15]. In [15], Faith raised the following questions: Does $\mathcal{R}$ being right zip imply $\mathcal{R}[x]$ a right zip?; Does $\mathcal{R}$ being a right zip imply $Mat_n(R)$ is right zip?; Does $\mathcal{R}$ being right zip ring imply $\mathcal{R}[\mathcal{G}]$ a right zip when $\mathcal{G}$ is a finite group? The concept of zip rings initiated by Zelmanowitz [63]. Zelmanowitz stated that any ring satisfying the descending chain condition on right annihilators is a right zip ring, but the converse does not hold. Beachy and Blair [5] studied rings that satisfy the condition that every faithful right ideal $\mathcal{I}$ of $\mathcal{R}$ is
co-faithful in the sense that $r_R(I_1)$ for a finite subset $I_1 \subseteq I$. Right zip rings have this property and conversely for commutative ring.

The extensions of zip rings were studied several authors. Beachy and Blair [5] showed that if $R$ is a commutative zip ring, then polynomial ring $R[x]$ over $R$ is a zip ring. Further, Faith [16] proved that if $R$ is a commutative zip ring and $S$ is a finite abelian group, then the group ring $R[G]$ of $G$ over $R$ is a zip ring. Afterwards, Cedo [9] studied that if $R$ is a commutative zip ring, then the $n \times n$ full matrix ring $Mat_n(R)$ over $R$ is zip; moreover, he settled negatively the questions posed by Faith [15]. Based on preceding results, Faith [16] again raised the following questions: When dose $R$ being a right zip ring imply $R[x]$ being right zip?; Characterize a ring $R$ such that $Mat_n(R)$ is right zip; When dose $R$ being a right zip ring imply $R[G]$ being right zip when $G$ is a finite group? Later, the extensions of noncommutative zip rings were studied by Hong et al. [24], Cortes [13] and Hashemi [20]. In this chapter, we generalize the results of noncommutative zip rings proved by Hong et al. [24], Cortes [13] and Hashemi [20] to skew monoid rings. Moreover, we give a unified solution of extension of right zip ring using $(S, \omega)$-Armendariz ring, where $S$ is a strictly ordered monoid and $\omega : S \rightarrow \text{End}(R)$ a monoid homomorphism of $R$.

This chapter is organized as follows. In section 5.2, we recall some known results of right zip rings which were proved by Hong et al. [24], Cortes [13] and Hashemi [20].

In section 5.3, we establish the relation between right (left) zip ring $R$ and skew monoid ring over $R$ and extend the results of preceding section.

In the final section, we study a relation between right (left) zip ring $R$ and skew generalized power series over $R$ using $(S, \omega)$-Armendariz ring and which give a unified approach to extensions of right zip rings.
5.2 Zip Rings

This section deals with basic definitions, examples and Properties of zip rings.

First, we give a definition of zip ring defined by Faith [15].

**Definition 5.2.1.** A ring $\mathcal{R}$ is called right zip if the annihilator $r_{\mathcal{R}}(X)$ of a subset $X$ of $\mathcal{R}$ is zero, $r_{\mathcal{R}}(Y) = 0$ for a finite subset $Y \subseteq \mathcal{R}$, similarly for left zip ring. A ring $\mathcal{R}$ is zip if it is right and left zip; equivalently, for the left ideal $\mathcal{L}$ of $\mathcal{R}$ with $r_{\mathcal{R}}(\mathcal{L}) = 0$, there exists a finitely generated left ideal $\mathcal{L}_1 \subseteq \mathcal{L}$ such that $r_{\mathcal{R}}(\mathcal{L}_1) = 0$. $\mathcal{R}$ is zip if it is right and left zip.

Recall that extensions of zip rings were studied by several authors. Beachy and Blair [5] showed that if $\mathcal{R}$ is a commutative zip ring, then polynomial ring $\mathcal{R}[x]$ over $\mathcal{R}$ is a zip ring. Further, Faith [16] proved that if $\mathcal{R}$ is a commutative zip ring and $\mathcal{S}$ is a finite abelian group, then the group ring $\mathcal{R}[\mathcal{G}]$ of $\mathcal{G}$ over $\mathcal{R}$ is a zip ring. Afterwards, Cedo [9] studied that if $\mathcal{R}$ is a commutative zip ring, then the $n \times n$ full matrix ring $\text{Mat}_n(\mathcal{R})$ over $\mathcal{R}$ is zip; moreover, he settled negatively the following questions posed by Faith [15]:

**Question 5.2.1.** Does $\mathcal{R}$ being a right zip ring imply $\mathcal{R}[x]$ being right zip?

**Question 5.2.2.** Does $\mathcal{R}$ being a right zip ring imply $\text{Mat}_n(\mathcal{R})$ being right zip?

**Question 5.2.3.** Does $\mathcal{R}$ being a right zip ring imply $\mathcal{R}[\mathcal{G}]$ being right zip when $\mathcal{G}$ is a finite group?

Based on preceding results, Faith [16] again raised the following questions:

**Question 5.2.4.** When does $\mathcal{R}$ being a right zip ring imply $\mathcal{R}[x]$ being right zip?

**Question 5.2.5.** Characterize a ring $\mathcal{R}$ such that $\text{Mat}_n(\mathcal{R})$ is right zip.

**Question 5.2.6.** When does $\mathcal{R}$ being a right zip ring imply $\mathcal{R}[\mathcal{G}]$ being right zip when $\mathcal{G}$ is a finite group?
In following Theorem, Hong et al. [24] proved Question 5.2.4 raised by Faith [16] using Armendariz ring.

**Theorem 5.2.1** ([24, Theorem 11]). Let $\mathcal{R}$ be an Armendariz ring. Then $\mathcal{R}$ is a right zip ring if and only if $\mathcal{R}[x]$ is a right zip ring.

Consideration of Armendariz ring in above Theorem is not superfluous, because the following example shows that their is no straight relation between Armendariz ring and zip ring.

**Example 5.2.1** ([24, Example 10(1)]). Reduced rings are Armendariz, but they need not be right (left) zip by the following. Let $F$ be a field and $\mathcal{R} = \langle \bigoplus_{i=1}^{\infty} F_i, 1_{\prod_{i=1}^{\infty} F_i} \rangle$ the $F$-subalgebra of $\prod_{i=1}^{\infty} F_i$ generated by $\bigoplus_{i=1}^{\infty} F_i$ and $1_{\prod_{i=1}^{\infty} F_i}$, where $F_i = F$ for all $I$. Then $\mathcal{R}$ is reduced and so it is Armendariz. Let $X = \bigoplus_{i=1}^{\infty} F_i$, then $r_\mathcal{R}(X) = l_\mathcal{R}(X) = 0$ but there can not be finite subset $X_0$ of $X$ such that $r_\mathcal{R}(X_0) = l_\mathcal{R}(X_0) = 0$; hence $\mathcal{R}$ is neither right nor left zip.

**Example 5.2.2** ([24, Example 10(2)]). Let $\mathbb{Z}_4$ be the ring of integers modulo 4. Notice that $\mathbb{Z}_4$ is Armendariz and zip. The trivial extension $T(\mathbb{Z}_4, \mathbb{Z}_4)$ is not Armendariz but it is zip by [24, Corollary 6].

Now, we quote some more results which were proved by Hong et al. [24].

**Proposition 5.2.1** ([24, Proposition 14]). Let $\mathcal{R}$ be a reduced ring and $S$ u.p.-product monoid. Then $\mathcal{R}$ is a right zip ring if and only if $\mathcal{R}[S]$ is a right zip ring.

**Theorem 5.2.2** ([24, Theorem 16]). Suppose $\mathcal{R}$ is a commutative ring and $S$ is a u.p.-monoid that contains infinite cyclic submonoid. Then $\mathcal{R}$ is a right zip ring if and only if $\mathcal{R}[S]$ is a right zip ring.

As a continuous study of extensions of zip rings, Cortes [13] studied skew polynomial extensions over zip rings by using skew version of Armendariz rings and generalized the results of Hong et al. [24]. The concept of skew version of Armendariz
ring was introduced by Hong et al. [23]. Here, we give the definitions related to skew version of Armendariz rings.

**Definition 5.2.2.** Suppose that $\sigma$ be an endomorphism of $R$. A ring $R$ is called skew Armendariz with the endomorphism $\sigma$ (simply, $\sigma$-skew Armendariz ring or satisfies $SA1'$ property) if for $f(x) = \sum_{i=0}^{n} a_i x^i, g(x) = \sum_{j=0}^{m} b_j x^j \in R[x;\sigma]$ satisfy $f(x)g(x) = 0$ which implies $a_i \sigma^i(b_j) = 0$ for every $0 \leq i \leq n$ and $0 \leq j \leq m$.

**Definition 5.2.3.** Suppose that $\sigma$ be an endomorphism of $R$. A ring $R$ is called skew Armendariz of power series type with the endomorphism $\sigma$ (simply, $\sigma$-skew Armendariz ring of power series type or satisfies $SA2'$ property) if for $f(x) = \sum_{i=0}^{\infty} a_i x^i, g(x) = \sum_{j=0}^{\infty} b_j x^j \in R[[x;\sigma]]$, satisfy $f(x)g(x) = 0$ which implies $a_i \sigma^i(b_j) = 0$ for every $i \geq 0$ and $j \geq 0$.

**Definition 5.2.4.** Suppose that $\sigma$ be an automorphism of $R$. A ring $R$ is called skew Armendariz of Laurent type with the endomorphism $\sigma$ (simply, $\sigma$-skew Armendariz ring of Laurent type or satisfies $SA3'$ property) if for $f(x) = q \sum_{i=s}^{\infty} a_i x^i, g(x) = n \sum_{j=t}^{\infty} b_j x^j \in R[[x,x^{-1};\sigma]]$ satisfy $f(x)g(x) = 0$ which implies $a_i \sigma^i(b_j) = 0$ for every $s \leq i \leq q$ and $t \leq j \leq n$.

**Definition 5.2.5.** Suppose that $\sigma$ be an automorphism of $R$. A ring $R$ is called skew Armendariz of Laurent power series type with the endomorphism $\sigma$ (simply, $\sigma$-skew Armendariz ring of Laurent power series type or satisfies $SA4'$ property) if for $f(x) = \sum_{i=s}^{\infty} a_i x^i, g(x) = \sum_{j=t}^{\infty} b_j x^j \in R[[x;\sigma]]$, satisfy $f(x)g(x) = 0$ which implies $a_i \sigma^i(b_j) = 0$ for every $i \geq s$ and $j \geq t$.

Cortes [13] extended the results of Hong et al. [24] and proved following results.

**Theorem 5.2.3 ([13, Theorem 2.8]).** Let $\sigma$ be an automorphism of $R$ and suppose $R$ satisfies $SA1'$. Then the following statements are equivalent.
1. $\mathcal{R}$ is right zip;

2. $\mathcal{R}[x, \sigma]$ is right zip;

3. $\mathcal{R}[x, x^{-1}, \sigma]$ is right zip.

**Theorem 5.2.4** ([13, Theorem 2.8]). Let $\sigma$ be an automorphism of $\mathcal{R}$ and suppose $\mathcal{R}$ satisfies $SA2'$. Then the following conditions are equivalent:

1. $\mathcal{R}$ is right zip;

2. $\mathcal{R}[[x, \sigma]]$ is right zip;

3. $\mathcal{R}[[x, x^{-1}, \sigma]]$ is right zip.

Now, we provide an Example related to zip ring which was discussed by Cortes [13].

**Example 5.2.3.** Let $\mathbb{F}$ be a field and $\sigma$ an automorphism of $\mathbb{F}$, and let

$$
\mathcal{R} = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \middle| a, b, c, d \in \mathbb{F} \right\}
$$

with usual addition and multiplication of matrix. Note that the monomorphism $\sigma$ is naturally extended to $\mathcal{R}$, and $\mathcal{R}$ has following one-sided ideals:

$$
\mathcal{I}_1 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} \middle| a \in \mathbb{F} \right\},
$$

$$
\mathcal{I}_2 = \left\{ \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| c \in \mathbb{F} \right\}.
$$
\( R \) and the zero ideal. We easily have \( r_R(I_2) \neq 0, l_R(I_2) \neq 0, r_R(I_1) \neq 0 \) and \( l_R(I_1) \neq 0 \). Now we clearly have that \( R \) is a zip ring and by [23, Proposition 17] \( R \) is \( \sigma \)-Armendariz. And \( R \) is also a \( \sigma \)-power Armendariz.

Hashemi [20] extended the results of Hong et al. [24].

**Theorem 5.2.5** ([20, Theorem 1.25]). Let \( R \) be a reversible ring and \( S \) a strictly totally ordered monoid. Then \( R \) is right zip if and only if \( R[S] \) is right zip.

### 5.3 Zip Rings Relative To Skew Monoid

In this section, we study the fundamental concept of skew monoid ring and \((S, \lambda)\)-Armendariz ring. Moreover, we investigate a relation between right zip property of a ring \( R \) and skew monoid ring over \( R \), and extend most of the results which were investigated by Hong et al. [24], Cortes [13] and Hashemi [20].

From [25, 44], let \( R \) be a ring and \( S \) a u.p.-monoid. Assume that there exists a monoid homomorphism \( \lambda : S \to Aut(R) \). We denote by \( \lambda^g(r) \) the image of \( r \in R \) under \( g \in S \). Then we can form a skew monoid ring \( R \star S \) (induced by the monoid homomorphism \( \lambda \)) by taking its elements to be finite formal combinations \( \sum_{i=1}^{n} a_i g_i \), with multiplication rule defined by \( gr = \lambda^g(r)g \).

To prove first Theorem of this section, we sought following definition.

**Definition 5.3.1.** Let \( \sigma \) be an automorphism of a ring \( R \). We define \( \sigma \) to be weakly rigid if \( ab = 0 \) implies \( a\sigma(b) = 0 \) or \( \sigma(a)b = 0 \) for any \( a, b \in R \).

A monoid homomorphism \( \sigma \) from a monoid \( S \) into the group of automorphism of \( R, x \to \lambda_x \) is called weakly rigid if \( \lambda_x \in Aut(R) \) is weakly rigid for every \( x \in S \).

We also need “McCoy Theorem” for skew monoid ring which was proved by Hong et al. [25]. Here, we quote only statements.
Theorem 5.3.1 ([25, Theorem 3]). Let $\mathcal{R}$ be a ring, $\mathcal{S}$ be a u.p.-monoid and $\mathcal{I}$ a right ideal of $\mathcal{R} * \mathcal{S}$. If $r_{\mathcal{R} * \mathcal{S}}(\mathcal{I}) \neq 0$, then $r_{\mathcal{R}}(\mathcal{I}) \neq 0$.

Hong et al. [24] gave an affirmative answer of the Faith question’s on noncommutative zip rings and proved that a ring $\mathcal{R}$ is right zip if and only if $\mathcal{R}[x]$ is right zip when $\mathcal{R}$ is Armendariz. Moreover, they showed that for a reduced ring $\mathcal{R}$ and u.p.-monoid $\mathcal{S}$, $\mathcal{R}$ is right zip if and only if $\mathcal{R}[\mathcal{S}]$ is right zip, and also investigated that $\mathcal{R}$ is zip if and only if $\mathcal{R}[\mathcal{S}]$ is zip when $\mathcal{R}$ is a commutative ring and $\mathcal{S}$ a u.p.-monoid. Further, cortes [13] studied skew polynomial extensions over zip rings by using skew version of Armendariz rings and generalized the results of Hong et al. [24]. Afterwards, Hashemi [20] extended the results of Hong et al. [24] and proved that a ring $\mathcal{R}$ is right zip if and only if $\mathcal{R}[\mathcal{S}]$ is right zip when $\mathcal{R}$ is reversible and $\mathcal{S}$ a strictly totally ordered monoid. In following Theorem, we study above results to skew monoid rings.

**Theorem 5.3.2.** Let $\mathcal{R}$ be a ring, $\mathcal{S}$ a u.p.-monoid and $\lambda : \mathcal{S} \to \text{Aut}(\mathcal{R})$ a monoid homomorphism. If $\lambda$ is weakly rigid, then skew monoid ring $\mathcal{R} * \mathcal{S}$ induced by $\lambda$ is a right zip ring if and only if $\mathcal{R}$ is a right zip ring.

**Proof.** Suppose $\mathcal{R}$ is a right zip ring and $U \subseteq \mathcal{R} * \mathcal{S}$ with $r_{\mathcal{R} * \mathcal{S}}(U) = 0$. Let $V$ be the set of coefficients of elements of $U$, $V \subseteq \mathcal{R}$. If any element $p \in r_{\mathcal{R}}(V)$, then $Vp = 0$ implies $vp = 0 \ \forall \ v \in V$. Now, take any element $\beta = v_1g_1 + v_2g_2 + \ldots + v_ng_n \in U$, where $v_i \in V$ and $g_i \in \mathcal{S}$ for each $i$. So

$$
\beta p = (v_1g_1 + \ldots + v_ng_n)p \\
= v_1g_1p + \ldots + v_ng_np \\
= v_1\lambda^{g_1}(p)g_1 + \ldots + v_n\lambda^{g_n}(p)g_n = 0
$$

since $\lambda$ is weakly rigid. It follows that $p \in r_{\mathcal{R} * \mathcal{S}}(U) = 0$, $p = 0$ thus $p \in r_{\mathcal{R}}(V) = 0$. By assumption, $\mathcal{R}$ is right zip and $V \subseteq \mathcal{R}$ then there exists a non empty finite subset $V_0 = \{a_1, a_2, \ldots a_m\}$ of $V$ such that $r_{\mathcal{R}}(V_0) = 0$. For any $a_i \in V_0$, there exists $\alpha_i \in U$ such
that some coefficients of \( \alpha_i \) is \( a_i \). Consider \( \alpha_i = b_{i1}g_{i1} + \ldots + b_{it_i}g_{it_i} \), where \( b_{i1}, \ldots b_{it_i} \) are nonzero elements in \( R \) and \( g_{i1}, \ldots g_{it_i} \) for each \( i \). Since \( S \) be an u.p.-monoid, there exists two nonempty subsets \( A = \{ h_1, h_2, \ldots, h_m \} \) and \( B = \{ g_{i1}, \ldots, g_{it_i} \} \) such that \( h_1g_{i1}, \ldots, h_1g_{it_i}, h_2g_{i2}, \ldots, h_2g_{it_i}, \ldots, h_mg_{im}, \ldots, h_mg_{mt_m} \) are distinct to each other. Let \( \alpha = h_1\alpha_1 + \ldots + h_m\alpha_m \). If there is a nonzero \( \gamma \in R^*S \) such that \( \alpha_i\gamma = 0 \) for each \( i \), then \( \alpha\gamma = 0 \). So by Theorem 5.3.1, there exists a nonzero element \( r \in R \) such that \( \alpha r = 0 \) which implies that \( a_i r = 0 \) for each \( i \). Therefore \( r \in r_{R}(V_0) = 0, r = 0 \) which is contradiction. Hence \( R^*S \) is a right zip ring.

Conversely, assume that \( R^*S \) is a right zip ring and any subset \( V \subseteq R \) with \( r_{R}(V) = 0 \). Let any element \( \alpha = a_1h_1 + \ldots + a_nh_n \in r_{R^*S}(V) \), where \( a_i \in R \) and \( h_i \in S \) for each \( i \). Then \( V\alpha = 0 \) that means \( v\alpha = 0 \) for all \( v \in V \). Thus \( va_i \) is \( 0 \) for all \( i = 1, 2, \ldots, n \) which implies \( a_i \in r_{R}(V) = 0 \) so \( a_i = 0 \) for all \( i = 1, 2, \ldots, n \). Thus \( \alpha = 0 \) it means \( r_{R^*S}(V) = 0 \). Since \( R^*S \) is right zip rings, then there exists a finite subset \( V_0 \subseteq V \) such that \( r_{R^*S}(V_0) = 0 \). Therefore \( r_{R}(V_0) = r_{R^*S}(V_0) \cap R = 0 \) implies \( r_{R}(V_0) = 0 \). Hence \( R \) is right zip ring.

Remark 5.3.1. Above Theorem gives the answer of the question raised by Faith [16].

In the following definition, we introduce the concept of Armendariz ring to skew monoid ring with the help of construction of skew monoid ring.

Definition 5.3.2. Let \( R \) be a ring, \( S \) a u.p.-monoid and \( \lambda : S \rightarrow Aut(R) \) a monoid homomorphism. A ring \( R \) is called \( (S, \lambda) \)-Armendariz if whenever \( \alpha = \sum_{i=1}^{n} a_ig_i, \beta = \sum_{j=1}^{m} b_jh_j \in R^*S \) such that \( \alpha\beta = 0 \), then \( a_i\lambda^{g_i}(b_j) = 0 \) for all \( i, j \).

To prove Theorem 5.3.3, we need to show following Lemma.

Lemma 5.3.1. Let \( R \) be ring, \( S \) a u.p.-monoid, \( \lambda : S \rightarrow Aut(R) \) a monoid homomorphism and \( A = R^*S \). Then the following conditions are equivalent

1. \( R \) is \( (S, \lambda) \)-Armendariz;
2. For any nonempty subset $Y \subseteq \mathcal{A}$, $r_{\mathcal{A}}(Y) = r_{\mathcal{R}}(C_{Y})\mathcal{A}$, where $C_{Y} = \bigcup_{\alpha \in Y} C_{\alpha}$ such that $C_{\alpha} = \left\{ \lambda^{-f_{i}}(a_{i}) \mid 1 \leq i \leq p, \alpha = \sum_{i=1}^{p} a_{i}f_{i} \in Y \right\}$.

Proof. (1)⇒(2) Suppose $\alpha = \sum_{i=1}^{p} a_{i}f_{i} \in Y \subseteq \mathcal{A}$. Now set $V = C_{Y} = \bigcup_{\alpha \in Y} C_{\alpha}$ such that $C_{\alpha} = \{ \lambda^{-f_{i}}(a_{i}) : 1 \leq i \leq p \}$. We show that $r_{\mathcal{A}}(\alpha) = r_{\mathcal{A}}(C_{\alpha})$. Assume any $\beta = \sum_{j=1}^{q} b_{j}g_{j} \in r_{\mathcal{A}}(\alpha)$. Then $\alpha\beta = 0$ which implies $a_{i}\lambda^{f_{i}}(b_{j}) = 0$ for all $i,j$, since $\mathcal{R}$ is $(S,\lambda)$-Armendariz. Thus $\lambda^{-f_{i}}(a_{i})b_{j} = 0$. It follows that $\beta \in r_{\mathcal{A}}(C_{\alpha})$. Therefore $r_{\mathcal{A}}(\alpha) \subseteq r_{\mathcal{A}}(C_{\alpha})$. Suppose $\gamma = \sum b_{i}c_{i}h_{i} \in r_{\mathcal{A}}(C_{\alpha})$. Then $C_{\alpha}\gamma = 0$ which implies $\lambda^{-f_{i}}(a_{i})c_{k} = 0$ for all $i$ and $k$. Thus $\alpha\gamma = 0$ which implies $\gamma \in r_{\mathcal{A}}(\alpha)$. Therefore $r_{\mathcal{A}}(C_{\alpha}) \subseteq r_{\mathcal{A}}(\alpha)$ thus $r_{\mathcal{A}}(\alpha) = r_{\mathcal{A}}(C_{\alpha})$. Now, we have $r_{\mathcal{A}}(Y) = r_{\mathcal{A}} \left( \bigcup_{\alpha \in Y} C_{\alpha} \right)$. Thus $r_{\mathcal{A}}(Y) = \bigcap_{\alpha \in Y} r_{\mathcal{A}}(C_{\alpha})$. By the above proof $r_{\mathcal{A}}(Y) = \bigcap_{\alpha \in Y} r_{\mathcal{A}}(C_{\alpha})$. It follows that $r_{\mathcal{A}}(Y) = r_{\mathcal{R}}(C_{Y})\mathcal{A}$.

(2)⇒(1) Let $\alpha = \sum_{i=1}^{m} a_{i}f_{i}$, $\beta = \sum_{j=1}^{n} b_{j}g_{j}$ are in $\mathcal{A}$ such that $\alpha\beta = 0$. Then $\beta \in r_{\mathcal{A}}(\alpha) = r_{\mathcal{R}}(C_{\alpha})\mathcal{A}$. Thus $\lambda^{-f_{i}}(a_{i})\beta = 0$ which implies $a_{i}\lambda^{f_{i}}(b_{j}) = 0$ for all $i$ and $j$. Therefore $\mathcal{R}$ is $(S,\lambda)$-Armendariz.

Here, we prove Theorem 5.3.3, by using $(S,\lambda)$-Armendariz ring and remove the condition that $\sigma$ is weakly rigid.

**Theorem 5.3.3.** Let $\mathcal{S}$ be a u.p.-monoid, $\lambda : \mathcal{S} \to \text{Aut}(\mathcal{R})$ a monoid homomorphism and $\mathcal{A} = \mathcal{R} * \mathcal{S}$. If $\mathcal{R}$ is $(S,\lambda)$-Armendariz, then the following conditions are equivalent:

1. $\mathcal{R}$ is right zip;
2. $\mathcal{A}$ is right zip.

Proof. (1)⇒(2) Suppose $\mathcal{R}$ is right zip and $Y$ a nonempty subset of $\mathcal{A}$ such that $r_{\mathcal{A}}(Y) = 0$. Let $V$ be a set of all coefficients of $Y$ defined by $V = C_{Y} = \bigcup_{\alpha \in Y} C_{\alpha}$ such that $C_{\alpha} = \{ \lambda^{-f_{i}}(a_{i}) : 1 \leq i \leq n \}$, where $\alpha = \sum_{i=1}^{n} a_{i}f_{i} \in Y$. Then $r_{\mathcal{R}}(V) = 0$. Since $\mathcal{R}$ is...
right zip, there exists a nonempty subset \( V_1 = \{ \lambda^{-f_{i_1}}(a_{i_1}), \lambda^{-f_{i_2}}(a_{i_2}), \ldots, \lambda^{-f_{i_n}}(a_{i_n}) \} \) such that \( r_R(V_1) = 0 \). For each \( \lambda^{-f_{i_j}}(a_{i_j}) \in V_1 \), there exists \( f_{i_j} \in Y \) such that some of coefficients of \( f_{i_j} \) are \( a_{i_j} \). Consider \( Y_0 \) be a minimal subset of \( Y \). Then \( V_1 \subseteq V_0 \). Thus \( r_R(V_0) = 0 \) which implies \( r_R(V_0) = 0 \). By Lemma 5.3.1, \( r_A(Y_0) = r_R(C_{Y_0})A = 0 \). Hence \( A \) is right zip.

(2)\(\Rightarrow\)(1) Suppose \( A \) is right zip and \( V \) a nonempty subset of \( R \) such that \( r_R(V) = 0 \). Then \( r_A(V) = 0 \). Since \( A \) is right zip, there exists a nonempty subset \( V_1 \) of \( V \) such that \( r_A(V_1) = 0 \). Thus \( r_R(V_1) = r_A(V_1) \cap R = 0 \). Hence \( R \) is right zip.

Following Example shows that the consideration of “\((S, \lambda)\)-Armendariz ring” in Theorem 5.3.3 is not superfluous.

**Example 5.3.1.** There exists a right zip ring which is not \((S, \lambda)\)-Armendariz.

*Proof.* Let \( \mathbb{Z}_8 \) be the ring of integers of modulo 8. Then \( \mathbb{Z}_8 \) is a right zip ring. Thus [30, Corollary 6], the trivial extension \( R = T(\mathbb{Z}_8, \mathbb{Z}_8) \) is also a right zip. Suppose \( \lambda : S \to Aut(R) \) defined by \( \lambda^g(a, b) = (b, a) \), for any \( g \in S \) and \( (a, b) \in T(\mathbb{Z}_8, \mathbb{Z}_8) \), where \( S \) be a u.p.-monoid it is easy to check that \( \lambda \) is a monoid homomorphism. Assume \( e, g \in S \) with \( e \neq g \) and \( \alpha = (0, 4)e + (4, 1)g, \beta = (0, 4)e + (1, 4)g \in R \ast S \). Then \( \alpha \beta = 0 \), while \( (4, 1)\lambda^g(0, 4) \neq 0 \). Hence \( R \) is not \((S, \lambda)\)-Armendariz. \( \square \)

In following Theorem, we generalize Theorem 5.3.2 and Theorem 5.3.3 by using McCoy Theorem [25, Theorem 3].

**Theorem 5.3.4.** Let \( S \) be a u.p.-monoid \( \lambda : S \to Aut(R) \) a monoid homomorphism and \( A = R \ast S \). Then \( R \) is right zip if and only if \( A \) is right zip.

*Proof.* Suppose \( R \) is right zip and \( Y \) a nonempty subset of \( A \) such that \( r_A(Y) = 0 \). Let \( V \) be the set of all coefficients of elements of \( Y \) and defined by \( V = C_Y = \bigcup_{\alpha \in Y} C_\alpha \) such
that \( C_\alpha = \{ \lambda^{-g_i}(a_i) : 1 \leq i \leq n \} \), where \( \alpha = \sum_{i=1}^{n} a_i g_i \in Y \). Then \( r_{\mathcal{R}}(V) = 0 \). Since \( \mathcal{R} \) is right zip, there exists a nonempty subset \( V_1 = \{ \lambda^{-g_11}(a_{i_1}), \lambda^{-g_{i_2}}(a_{i_2}), \ldots, \lambda^{-g_{in}}(a_{i_n}) \} \) such that \( r_{\mathcal{R}}(V_1) = 0 \). For each \( \lambda^{-g_{ij}}(a_{i_j}) \in V_1 \), there exists \( \beta_{ij} \in Y \) such that some of coefficients of \( \beta_{ij} \) are \( a_{i_j} \) for each \( 1 \leq j \leq n \). Let \( \beta_k = b_{k_1}g_{k_1} + b_{k_2}g_{k_2} + \ldots + b_{kt_k}g_{kt_k} \), where \( i_j = k \) and \( 1 \leq k, j \leq n \), and \( b_{k_1}, \ldots, b_{kt_k} \) are nonzero elements in \( \mathcal{R} \). Since \( S \) be an u.p.-monoid, there exists two nonempty subsets \( A = \{ h_1, h_2, \ldots, h_n \} \) and \( B = \{ g_{k_1}, \ldots, g_{kt_k} \} \) such that

\[
\begin{align*}
h_1g_{11}, \ldots, h_1g_{1t_1}, h_2g_{21}, \ldots, h_2g_{2t_2}, \ldots, h_ng_{n1}, \ldots, h_ng_{nt_n},
\end{align*}
\]

are distinct to each other. Let \( \beta = h_1\beta_1 + \ldots + h_n\beta_n \). Now, assume that there is nonzero \( \gamma = \sum_{s=1}^{m} c_s f_s \in \mathcal{A} \) such that \( \beta_k\gamma = 0 \) for each \( k = 1, 2, \ldots, n \). Then \( h_k\beta_k\gamma = 0 \) for each \( k = 1, 2, \ldots, n \) which implies that \( \beta\gamma = 0 \). So by Theorem 5.3.1, there exists a nonzero \( p \in \mathcal{R} \) such that \( \beta p = 0 \) which implies \( \lambda^{-g_k}(a_k)p = 0 \) for each \( 1 \leq k \leq n \).

It is follows that \( p \in r_{\mathcal{R}}(V_1) = 0 \), which is contradiction. Therefore \( \mathcal{A} \) is right zip.

Conversely, if \( \mathcal{A} \) is right zip, then \( \mathcal{R} \) is right zip (proof is similar to the argument of [20, Theorem 1.25]).

Now, we state two important Corollaries.

**Corollary 5.3.1.** Let \( \mathcal{R} \) be a ring, \( (\mathcal{S}, \leq) \) a strictly totally ordered monoid, \( \lambda : \mathcal{S} \to \text{End}(\mathcal{R}) \) a monoid homomorphism and \( \mathcal{A} = \mathcal{R} * \mathcal{S} \). Then \( \mathcal{R} \) is right zip if and only if \( \mathcal{A} \) is right zip.

Recall that a ring \( \mathcal{R} \) is called right duo if all right ideals are two sided ideals. Left duo rings are defined similarly, and a ring is called duo if it is both right and left duo. Clearly, duo rings are semicommutative.

**Corollary 5.3.2.** Let \( \mathcal{R} \) be a duo ring and \( \mathcal{S} \) a strictly totally ordered monoid. Then \( \mathcal{R} \) is right zip if and only if \( \mathcal{R}[\mathcal{S}] \) is right zip.
5.4 Zip Skew Generalized Power Series Rings

In this section, we study the concept of skew generalized power series rings, \((S, \omega)\)-Armendariz rings and \(S\)-compatible rings which were introduced [47, 49]. Moreover, we investigate a relation between right (left) zip property of a ring \(R\) and skew generalized power series over \(R\), which is an extension of main results of the section 5.2 and section 5.3.

For the construction of skew generalized power series ring, we need some definitions. Let \((S, \leq)\) be a partially ordered set. Then \((S, \leq)\) is called artinian if every strictly decreasing sequence of elements of \(S\) is finite, and \((S, \leq)\) is called narrow if every subset of pairwise order-incomparable elements of \(S\) is finite. Thus, \((S, \leq)\) is artinian and narrow if and only if every nonempty subset of \(S\) has at least one but only a finite number of minimal elements.

An ordered monoid is a pair \((S, \leq)\) consisting of a monoid \(S\) and an order \(\leq\) on \(S\) such that for all \(a, b, c \in S\), \(a \leq b\) implies \(ca \leq cb\) and \(ac \leq bc\). An ordered monoid \((S, \leq)\) is said to be strictly ordered if for all \(a, b, c \in S\), \(a < b\) implies \(ca < cb\) and \(ac < bc\).

Let \(R\) be a ring, \((S, \leq)\) a strictly ordered monoid and \(\omega : S \to \text{End}(R)\) a monoid homomorphism. For \(s \in S\), let \(\omega_S\) denote the image of \(s\) under \(\omega\), that is, \(\omega_s = \omega(s)\). Let \(A\) be the set of all functions \(\alpha : S \to R\) such that the \(\text{supp}(\alpha) = \{s \in S : \alpha(s) \neq 0\}\) is artinian and narrow. Then for any \(s \in S\) and \(\alpha, \beta \in A\) the set

\[
X_s(\alpha, \beta) = \{(x, y) \in \text{supp}(\alpha) \times \text{supp}(\beta) : s = xy\}
\]

is finite. Thus one can define the product \(\alpha \beta : G \to R\) of \(\alpha, \beta \in A\) as follows

\[
(\alpha \beta)(s) = \sum_{(x, y) \in X_s(\alpha, \beta)} \alpha(x) \cdot \omega_x(\beta(y))
\]

with point-wise addition and multiplication as defined above, \(A\) becomes a ring, called the ring of skew generalized power series with coefficients in \(R\) and exponents in \(S\) (see
denoted by \( R[[S, \omega, \leq]] \) (or by \( R[[S, \omega]] \)). The skew generalized power series ring \( R[[S, \omega]] \) is a compact generalization of (skew) polynomial rings, (skew) Laurent polynomial rings, (skew) power series rings, (skew) group rings, (skew) monoid rings, Mal’cev-Neumann Laurent series rings and generalized power series rings. The symbol \( 1 \) denote the identity element of multiplicative monoid \( S \), ring \( R \) and \( R[[S, \omega]] \).

To each \( r \in R \) and \( s \in S \), we associate elements \( c_r, e_s \in R[[S, \omega]] \) defined by

\[
c_r(x) = \begin{cases} 
  r & \text{if } x = 1 \\
  0 & \text{if } x \in S \setminus \{1\}
\end{cases}, \quad 

\[
e_s(x) = \begin{cases} 
  1 & \text{if } x = s \\
  0 & \text{if } x \in S \setminus \{s\}
\end{cases}
\]

It is clear that \( r \mapsto c_r \) is a ring embedding of \( R \) into \( R[[S, \omega]] \) and \( s \mapsto e_s \) is a monoid embedding of \( S \) into a multiplicative monoid of ring \( R[[S, \omega]] \), and \( e_s c_r = c_{\omega_s(r)} e_s \).

Moreover, for each nonempty subset \( X \) of \( R \) we put \( X[[S, \omega]] = \{ \alpha \in R[[S, \omega]] : \alpha(s) \in X \cup \{0\} \text{ for every } s \in S \} \) denotes a subset of \( R[[S, \omega]] \), and for each nonempty subset \( Y \) of \( R[[S, \omega]] \) we put \( C_Y = \{ \beta(t) : \beta \in Y, t \in S \} \) denotes a subset of \( R \).

With the help of definition of skew generalized power series ring \( R[[S, \omega]] \), Marks et al. [47, 49] introduced the concept of Armendariz property to skew generalized power series and given a unified approach to all the classes of Armendariz ring.

**Definition 5.4.1.** Let \( R \) be a ring, \((S, \leq)\) a strictly ordered monoid and \( \omega : S \to \text{End}(R) \) a monoid homomorphism. We say that \( R \) is \((S, \omega)\)-Armendariz if whenever \( \alpha \beta = 0 \) for \( \alpha, \beta \in R[[S, \omega]] \), then \( \alpha(\beta).\omega_s(\beta(t)) = 0 \) for all \( s, t \in S \). If \( S = \{1\} \) then every ring is \((S, \omega)\)-Armendariz.

We recall the definition of compatible endomorphism from [47, Definition 2.3].

**Definition 5.4.2.** An endomorphism \( \sigma \) of a ring \( R \) is compatible if for all \( a, b \in R \),

\[
ab = 0 \iff a\sigma(b) = 0.
\]

**Definition 5.4.3.** Let \( R \) be a ring, \((S, \leq)\) a strictly ordered monoid and \( \omega : S \to \text{End}(R) \) a monoid homomorphism. We say that \( R \) is \( S \)-compatible if \( \omega_s \) is compatible
for every $s \in S$.

To prove the main result of this section, we need Lemma 5.4.1 and Lemma 5.4.2 introduced by Marks et al. [47]. We here quote only the statements.

**Lemma 5.4.1.** Let $\mathcal{R}$ be a ring, $(S, \leq)$ a strictly ordered monoid, $\omega : S \to \text{End}(\mathcal{R})$ a monoid homomorphism, and $\mathcal{A} = \mathcal{R}[[S, \omega]]$. Then the following conditions are equivalent:

1. $\mathcal{R}$ is $S$-compatible;

2. for any $a \in \mathcal{R}$ and any nonempty subset $Y \subseteq \mathcal{A}$,

\[ a \in \text{ann}_r^\mathcal{R}(C_Y) \Leftrightarrow c_a \in \text{ann}_r^\mathcal{A}(Y). \]

**Proof.** See [47, Lemma 3.1].

**Lemma 5.4.2.** Let $\mathcal{R}$ be a ring, $(S, \leq)$ a strictly ordered monoid, $\omega : S \to \text{End}(\mathcal{R})$ a monoid homomorphism, and $\mathcal{A} = \mathcal{R}[[S, \omega]]$. If $\mathcal{R}$ is $S$-compatible, then for any nonempty subset $X \subseteq \mathcal{R}$, $\text{ann}_r^\mathcal{R}(X[[S, \omega]]) = \text{ann}_r^\mathcal{A}(X[[S, \omega]])$.

**Proof.** See [47, Lemma 3.2].

In next Theorem, we establish a relation between right zip ring and skew generalized power series over right zip ring which is an extension of the results mentioned in section 5.2 and section 5.3.

**Theorem 5.4.1.** Let $\mathcal{R}$ be a ring, $(S, \leq)$ a strictly ordered monoid, and $\omega : \mathcal{R} \to \text{End}(\mathcal{R})$ a monoid homomorphism. If $\mathcal{R}$ is $(S, \omega)$-Armendariz ring and $S$-compatible. Then $\mathcal{R}[[S, \omega]]$ is right zip if and only if $\mathcal{R}$ is right zip.

**Proof.** Suppose that $\mathcal{R}[[S, \omega]]$ is right zip. We show that $\mathcal{R}$ is right zip. For this consider $Y \subseteq \mathcal{R}$ with $r_\mathcal{R}(Y) = 0$. Since $Y \subseteq \mathcal{R}$, so we put $Y[[S, \omega]] = \{ \alpha : \alpha(s) \in \mathcal{R} \text{ and } s \in S \} \subseteq \mathcal{R}[[S, \omega]]$ for all $s, t \in S$. Let any $\beta \in r_\mathcal{R}[[S, \omega]](Y[[S, \omega]])$. Then $\alpha \beta = 0$
which implies $\alpha(s)\beta(t) = 0$ since $\mathcal{R}$ is $\mathcal{S}$-compatible and $(\mathcal{S}, \omega)$-Armendariz. Thus $\beta(t) \in r_{\mathcal{R}}(\alpha(s)) = 0$ for all $\alpha(s) \in \mathcal{R}$ which implies $\beta = 0$ for all $\beta \in \mathcal{R}[[\mathcal{S}, \omega]]$ therefore $r_{\mathcal{R}[[\mathcal{S}, \omega]]}(Y[[\mathcal{S}, \omega]]) = 0$. Since $\mathcal{R}[[\mathcal{S}, \omega]]$ is right zip, there exists a subset $V \subseteq Y[[\mathcal{S}, \omega]]$ such that $r_{\mathcal{R}[[\mathcal{S}, \omega]]}(V) = 0$. Then we put $C_V = \{\gamma(u) : u \in S$ and $\gamma \in V\}$ is a subset of $Y$. By Lemma 5.4.1, for any $a \in r_{\mathcal{R}}(C_V) \iff c_a \in r_{\mathcal{R}[[\mathcal{S}, \omega]]}(V)$ since $\mathcal{R}$ is $\mathcal{S}$-compatible. Thus we have $r_{\mathcal{R}}(C_V) = 0$. Hence $\mathcal{R}$ is right zip.

Conversely, suppose $\mathcal{R}$ is right zip ring and a subset $U \subseteq \mathcal{R}[[\mathcal{S}, \omega]]$ with $r_{\mathcal{R}[[\mathcal{S}, \omega]]}(U) = 0$. We put $C_U = \{\beta(t) : \beta \in U$ and $t \in \mathcal{S}\}$ which is nonempty subset of $\mathcal{R}$. By Lemma 5.4.1, for any $p \in r_{\mathcal{R}}(C_U) \iff c_p \in r_{\mathcal{R}[[\mathcal{S}, \omega]]}(U)$, since $\mathcal{R}$ is $\mathcal{S}$-compatible. Thus $r_{\mathcal{R}}(C_U) = 0$. Since $\mathcal{R}$ is right zip ring, there exists a nonempty subset $X \subseteq C_U$ such that $g_{\mathcal{R}}(X) = 0$. So we put $X[[\mathcal{S}, \omega]] = \{\alpha \in \mathcal{R}[[\mathcal{S}, \omega]] : \alpha(s) \in X \cup \{0\} \text{ and } s \in \mathcal{S}\}$. Thus by Lemma 5.4.2, $r_{\mathcal{R}[[\mathcal{S}, \omega]]}(X[[\mathcal{S}, \omega]]) = r_{\mathcal{R}}(X)[[\mathcal{S}, \omega]] = 0$, since $\mathcal{R}$ is $\mathcal{S}$-compatible. Therefore $\mathcal{R}[[\mathcal{S}, \omega]]$ is right zip. \hfill $\square$

Here, we get an important Corollary.

**Corollary 5.4.1.** Let $\mathcal{R}$ be a ring, $(\mathcal{S}, \leq)$ a strictly totally ordered monoid, and $\omega : \mathcal{S} \to \text{End}(\mathcal{R})$ a monoid homomorphism. If $\mathcal{R}$ is $(\mathcal{S}, \omega)$-Armendariz ring and $\mathcal{S}$-compatible. Then $\mathcal{R}[[\mathcal{S}, \omega]]$ is right zip if and only if $\mathcal{R}$ is right zip.