Chapter 2

Preliminaries

The present thesis deals with the study of generalized measures on quantum structures, measurability, approximations, and quantum dynamical systems, leading to the theory of entropy of quantum spaces of orthomodular lattices. In this chapter we give some prerequisites for the subsequent study. In Section 1, basics of posets, lattices, Boolean algebras and orthomodular lattices are collected. Section 2 gives a brief exposition on effect algebras, orthoalgebras and other quantum structures; interrelationship among these structures is also given. The last section provides an introduction to some basic notions in the existing literature of measure theory on quantum structures.

2.1 Orthomodular lattices

2.1.1 [14, 53, 107] A set $P$ together with a relation $\leq$, which is reflexive, antisymmetric and transitive is called a partially ordered set (hereafter abridged to a poset); the relation $\leq$ is called a partial order relation (or a partial ordering) on $P$. The elements $a$ and $b$ of a poset $P$ (also written as $(P, \leq)$) are called comparable if $a \leq b$ or $b \leq a$ holds. A poset is a chain if any
two elements of it are comparable.

Let $A$ be a subset of a poset $P$. We call an element $u$ of $P$ an \textit{upper bound} of $A$ if $a \leq u$ for all $a \in A$. Further, an upper bound $u$ of $A$ is said to be the \textit{least upper bound} (or supremum) of $A$, denoted by $\text{sup } A$, if every upper bound $u'$ of $A$ satisfies $u \leq u'$. Dually, we define an element $v$ of $P$ a \textit{lower bound} of $A$ if $v \leq a$ for all $a \in A$ and a lower bound $v$ of $A$ is called the \textit{greatest lower bound} (or infimum) of $A$, denoted by $\text{inf } A$, if every lower bound $v'$ of $P$ satisfies $v' \leq v$. The supremum of $P$ is often called the \textit{unit element} (or top element) of $P$ with notation $1$ and the infimum of $P$ is called \textit{zero element} (or bottom element) with notation $0$. A bounded poset $(P, \leq)$ is a poset which has $0$ and $1$.

2.1.2 [14, 53, 107] A \textit{lattice} is a poset such that any two elements of it possess supremum and infimum. We write $a \vee b$ for $\text{sup}\{a, b\}$ and $a \wedge b$ for $\text{inf}\{a, b\}$. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence in a lattice $L$. We call $a_n \uparrow a, (a \in L)$ if and only if $a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots$, $\bigvee_{n=1}^{\infty} a_n$ exists in $L$ and $\bigvee_{n=1}^{\infty} a_n = a$. Similarly, call $a_n \downarrow a, (a \in L)$ if and only if $a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots$, $\bigwedge_{n=1}^{\infty} a_n$ exists in $L$ and $\bigwedge_{n=1}^{\infty} a_n = a$. In these cases we also write $a = \lim_{n \to \infty} a_n$. If $a_n \uparrow a$, $b_n \uparrow b$ ($a_n, b_n, a, b, \in L$) and $a_n \leq b_n$, for all $n$, then we deduce that $a \leq b$. A lattice $L$ is called a \textit{bounded lattice} if $\text{inf } L (= 0)$ and $\text{sup } L (= 1)$ exist in $L$. In a bounded lattice, a \textit{complement} of an element $a$ is an element $b$ such that $a \wedge b = 0$ and $a \vee b = 1$.

The following properties hold in a lattice $L$, for any $a, b, c \in L$:

(i) \textit{idempotent}: $a \vee a = a$, $a \wedge a = a$;

(ii) \textit{commutative}: $a \vee b = b \vee a$, $a \wedge b = b \wedge a$;

(iii) \textit{associative}: $(a \vee b) \vee c = a \vee (b \vee c)$, $(a \wedge b) \wedge c = a \wedge (b \wedge c)$;

(iv) \textit{absorption}: $a \wedge (a \vee b) = a$, $a \vee (a \wedge b) = a$;

(v) $a \leq b \iff a \wedge b = a \iff a \vee b = b$;
(vi) \( b \leq c \implies a \land b \leq a \land c \) and \( a \lor b \leq a \lor c \);

A subset \( M \) of a lattice \( L \) is called a sublattice of \( L \) if \( M \) is closed under both operations \( \lor \) and \( \land \). Let \( L_1 \) and \( L_2 \) be lattices, and \( h : L_1 \to L_2 \) a map. Then \( h \) is a lattice homomorphism if and only if for any \( a, b \in L_1 \), \( h(a \lor b) = h(a) \lor h(b) \) and \( h(a \land b) = h(a) \land h(b) \).

2.1.3 [14, 32, 53, 107, 122] A lattice \( L \) is called

- modular if it satisfies the modular law, i.e. for any \( a, b, c \in L \),
  \( a \leq b \implies (a \lor c) \land b = a \lor (c \land b) \).
- distributive if it follows distributive laws, i.e. for any \( a, b, c \in L \),
  \( a \land (b \lor c) = (a \land b) \lor (a \land c) \) and \( a \lor (b \land c) = (a \lor b) \land (a \lor c) \).
- \( \sigma \)-lattice if every countable subset of \( L \) has supremum and infimum.
- complete if every subset of \( L \) has supremum and infimum.
- \( \sigma \)-continuous if for any sequence \( \{a_n\}_{n=1}^{\infty} \) in \( L \), \( a_n \uparrow a \) \( (a \in L) \) implies \( a_n \land b \uparrow a \land b \) for any \( b \in L \).
- complemented lattice if it is bounded and every element of \( L \) has a complement.
- Boolean algebra if it is a complemented distributive lattice.

In every lattice the validity of one of the two distributive laws implies the validity of the other. Observe that every distributive lattice is a modular lattice, and also every complete lattice is a \( \sigma \)-lattice.

2.1.4 [53] Let \( (P, \leq) \) be a bounded poset. An orthocomplementation on \( P \) is a unary operation \( ' : P \to P \) satisfying, for any \( a, b, c \in P \),

(i) \( a \leq b \implies b' \leq a' \);
(ii) \( (a')' = a \);
(iii) \( a \lor a' \) exists and \( a \lor a' = 1 \).

An orthoposet is a bounded poset with an orthocomplementation. Let \( P \)
be an orthoposet. Then we obtain that $0' = 1$; $1' = 0$; for $a \in P$, $a \wedge a'$ exists and $a \land a' = 0$. Two elements $a, b \in P$ are called orthogonal if $a \leq b'$, and is denoted by $a \perp b$. For elements $a, b \in P$, following de Morgan laws hold:

(i) if $a \lor b \in P$, then $(a \lor b)' = a' \land b'$;
(ii) if $a \land b \in P$, then $(a \land b)' = a' \lor b'$.

An orthomodular poset (OMP for short) is an orthoposet $P$ satisfying for all $a, b, c \in P$, the following conditions:

(i) if $a \leq b'$, then $a \lor b$ exists in $P$;
(ii) (Orthomodular law) if $a \leq b$, then there exists $c \in P$ such that $c \leq a'$ and $b = a \lor c$ (equivalently if $a \leq b$, then $b = a \lor (a \lor b')'$).

As a consequence of the orthomodular law, we get $a \lor a' = 1$. The orthomodular law is a kind of distributivity, for $a \leq b$, we have $a \lor (a' \land b) = b = 1 \land b = (a \land a') \land (a \land b)$.

A subfamily $F$ of an OMP $P$, containing $0, 1$, is called suborthomodular poset (or sub-OMP) if it is closed under orthocomplementation and join of two orthogonal elements.

2.1.5 An ortholattice is an orthoposet which is also a lattice. An orthomodular lattice (OML for short) is an OMP that is also a lattice. In an OML $L$, complements of an element $a$ are precisely the elements $(b \land (b' \lor a')) \lor (b' \land a')$ for any $b \in L$. If an OML $L$ satisfies $a \land b = 0 \implies a \leq b'$, then $L$ is a Boolean algebra. An element $a$ of an orthomodular poset $P$ is called a central element if for every $b \in P$, $a = (a \land b) \lor (a \land b')$. The set $C(L)$ consisting of all central elements of an orthomodular lattice $L$ (called the centre of $L$) forms a Boolean algebra.

Let $L$ be an ortholattice. The following statements are equivalent:

(i) $L$ is a Boolean algebra,
(ii) $L$ is uniquely complemented, i.e. every element in $L$ has unique complement.

The most important example of an orthomodular lattice is the family $\mathbb{L}(H)$, consisting of all closed subspaces $M$ of a real or complex Hilbert space $H$; the partial order is set theoretic inclusion and $M^\perp$ denotes its orthogonal complement. Quantum logic is the study of OMPs, OMLs and their generalizations. From the probabilistic point of view, a quantum logic is a mathematical model of the set of random events of a quantum mechanical experiment.

2.2 Other quantum structures

In 1994, Kôpka and Chovanec [81] introduced an abstract partially ordered set called $D$-poset and in the same year Foulis and Bennett [38] introduced effect algebra, an equivalent structure to $D$-poset, for the study of foundations of quantum mechanics. The equivalence of $D$-posets and effect algebras is proved by Foulis and Bennett [38] and independently by Pulmannová [108] (also see [32]).

2.2.1 [31, 32, 49, 81] An effect algebra is a set $E$, with two distinguished elements $0$ and $1$, and a partially defined binary operation $\oplus$ such that for all $a, b, c \in E$ the following axioms hold:

(E1) if $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$ (commutative law);

(E2) if $b \oplus c$ is defined and $a \oplus (b \oplus c)$ is defined, then $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ (associative law);

(E3) for every $a \in E$, there exists a unique element $a^\perp \in E$ such that $a \oplus a^\perp$ exists and $a \oplus a^\perp = 1$ (orthosupplementation law);

(E4) if $a \oplus 1$ is defined, then $a = 0$ (zero-one law).
Since $\oplus$ is not defined for every pair of elements of $E$, so it is a partial binary operation. The element $a^\perp$ in $(E3)$ is called the orthosupplement of $a$. Occasionally, we also say that $(E, \oplus, 0, 1)$ is an effect algebra. Two elements $a, b \in E$ are called orthogonal, and written as $a \perp b$, if $a \oplus b$ exists. Define a binary relation $\leq$ on $E$ as follows: $a \leq b$ if and only if there exists $c \in E$ such that $c \oplus a = b$ (generalized orthomodular law); so effect algebras can be considered as quantum logics. The relation $\leq$ is a partial ordering on $E$, with 0 as the smallest element of $E$. Since in an effect algebra $E, a \lor a^\perp = 1 \ (a \in E)$ need not be satisfied, effect algebras are also referred to as unsharp quantum logics. If $(E, \leq)$ is a lattice, we say that effect algebra is a lattice effect algebra or a $D$-lattice. The subfamily $F$ of an effect algebra $E$ is called a subeffect algebra if it is an effect algebra in its own right with respect to the induced partially defined binary operation $\oplus$ on $F$.

In every effect algebra $E$, a dual operation $\ominus$ to $\oplus$ can be defined as follows:

$$(\ast) \quad a \ominus c \text{ exists and equals } b \text{ if and only if } b \oplus c \text{ exists and equals } a.$$ 

The system $(E, \leq, \ominus, 1)$ is called a difference poset or a $D$-poset. An alternative axiomatic definition of a $D$-poset $(E, \leq, \ominus, 1)$ may be seen in the forthcoming Section 3.2; a dual operation $\ominus$ to $\ominus$ is given by:

$$(\ast\ast) \quad a \oplus b \text{ exists and equals } c \text{ if and only if } c \ominus b \text{ exists and equals } a.$$ 

Interestingly, effect algebras are categorically equivalent to $D$-posets [32]. An OMP may be regarded as an effect algebra (or as a $D$-poset) by defining $a \oplus b = a \lor b$ precisely when $a \leq b^\perp$ (or $b \ominus a = b \land a^\prime$ precisely when $a \leq b$).

For every $a \in E$, we have $a^\perp = 1 \ominus a$. It is clear that $1^\perp = 0; (a^\perp)^\perp = a; a \ominus 0 = a; a \leq b \implies b^\perp \leq a^\perp; a \perp b$ if and only if $a \leq b^\perp; (a \oplus b)^\perp = a^\perp \ominus b = b^\perp \ominus a$, where $a, b \in E$. 
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2.2.2 [3, 4, 5, 32, 60, 61, 66, 67, 68, 69] Assume that $a, b, c$ are elements of an effect algebra $E$. Then we have the following properties:

(i) If $a \leq b$, then $b \ominus a \leq b$ and $b \ominus (b \ominus a) = a$.

(ii) If $a \leq b \leq c$, then $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

(iii) If $a \leq b \leq c$, then $b \ominus a \leq c \ominus a$ and $(c \ominus a) \ominus (b \ominus a) = c \ominus b$.

(iv) If $a \leq b^\perp$ and $a \ominus b \leq c$, then $c \ominus (a \ominus b) = (c \ominus a) \ominus b = (c \ominus b) \ominus a$.

(v) If $a \leq b \leq c^\perp$, then $a \ominus c \leq b \ominus c$ and $(b \ominus c) \ominus (a \ominus c) = (b \ominus a)$.

(vi) If $a \leq b \leq c$, then $a \ominus (c \ominus b) = c \ominus (b \ominus a)$.

(vii) If $a \leq b^\perp \leq c^\perp$, then $a \ominus (b \ominus c) = (a \ominus b) \ominus c$.

2.2.3 [6, 31, 32, 49, 81] Let $E$ be an effect algebra. For $a_1, a_2, \ldots, a_n \in E$ ($n \in \mathbb{N}$), we inductively define $a_1 \oplus a_2 \oplus \cdots \oplus a_n = (a_1 \oplus a_2 \oplus \cdots \oplus a_{n-1}) \oplus a_n$, provided that the right hand side exists. The definition is independent on a permutation of the elements. A finite subset $\{a_1, a_2, \ldots, a_n\}$ of $E$ is said to be an orthogonal set if $a_1 \oplus a_2 \oplus \cdots \oplus a_n$ exists. A sequence $\{a_n\}_{n=1}^{\infty}$ is called an orthogonal sequence if $\bigoplus_{i=1}^{n} a_i$ exists for all $n$. In an orthogonal sequence $\{a_n\}_{n=1}^{\infty}$, $a_i \perp a_j$ for $i, j \in \mathbb{N}$ with $i \neq j$. If moreover, $\sup_n \bigoplus_{i=1}^{n} a_i$ exists, the sum $\bigoplus_{n=1}^{\infty} a_n$ of an orthogonal sequence $\{a_n\}_{n=1}^{\infty}$ in $E$ is defined as $\sup_n \bigoplus_{i=1}^{n} a_i$.

An effect algebra $E$ is called $\sigma$-complete if every orthogonal sequence in $E$ has its sum. A complete effect algebra is an effect algebra such that there exists $\bigoplus_{a \in A} a$ for all orthogonal subsets $A$ of $E$.

Assume that $a, b, b_n$ ($n \in \mathbb{N}$) are elements of a $D$-lattice $E$. Then we have the following properties.

(i) If $b_n \downarrow b$ and $a \perp b_n$ for each $n$, then $a \ominus b_n \downarrow a \ominus b$.

(ii) If $b_n \downarrow b$ and $a \geq b_n$ for each $n$, then $a \ominus b_n \uparrow a \ominus b$.

(iii) If $b_n \downarrow b$ and $a \leq b_n$ for each $n$, then $b_n \ominus a \downarrow b \ominus a$.

(iv) If $b_n \uparrow b$ and $a \geq b_n$ for each $n$, then $a \ominus b_n \downarrow a \ominus b$. 
Let $P, Q$ be effect algebras. A mapping $\psi : P \to Q$ is called a *morphism* (of effect algebras) if the following conditions\(^\dagger\) are satisfied:

(i) $\psi(1) = 1$;

(ii) if $a, b \in P, a \perp b$, then $\psi(a) \perp \psi(b)$ and $\psi(a \oplus b) = \psi(a) \oplus \psi(b)$.

Let $E, F$ be $D$-posets. A mapping $\varphi : E \to F$ is called a *morphism* (of $D$-posets) if the following conditions are satisfied:

(i) $\varphi(1) = 1$;

(ii) if $a, b \in E, a \leq b$, then $\varphi(a) \leq \varphi(b)$ and $\varphi(b \ominus a) = \varphi(b) \ominus \varphi(a)$.

A morphism $\varphi$ of effect algebras or $D$-posets is called a *$\wedge$-preserving morphism* if $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$ whenever $a \wedge b$ exists. A morphism $\varphi$ such that $a \perp b$ if and only if $\varphi(a) \perp \varphi(b)$ is called a *monomorphism*. A bijective morphism $\varphi$ such that $\varphi^{-1}$ is also a morphism is an *isomorphism*. Observe that an isomorphism is necessarily a $\wedge$-preserving morphism, and a surjective monomorphism is an isomorphism.

Let $(E, \leq, \ominus, 1)$ be an effect algebra. If $\ominus$ is defined by 2.2.1(\(*\)), then $(E, \leq, \ominus, 1)$ is a $D$-poset, and the partial ordering $\leq$ on $E$ coincides with the partial ordering induced by $\ominus$ on $E$. Moreover, to every morphism $\psi$ of effect algebras there is a unique morphism $\varphi$ between corresponding $D$-posets such that $\psi = \varphi$.

Let $(E, \leq, \ominus, 1)$ be a $D$-poset. If $\ominus$ is defined by 2.2.1(\(**\)) , then $(E, \ominus, 1)$ is an effect algebra, and the partial ordering induced by $\ominus$ on $E$ coincides with $\leq$ on $E$. Moreover, to every morphism $\varphi$ of $D$-posets there is a unique morphism $\psi$ between corresponding effect algebras such that $\varphi = \psi$.

\[2.2.5\] [31, 32, 49, 51, 81] An effect algebra $E$ satisfies the *Riesz decom-

\(^\dagger\) Notice that in the expression “$\psi(1) = 1$”, 1 in the left hand side is the unit element of $P$, while 1 in the right hand side is the unit element of $Q$. The same happens for the other symbols such as $\perp, \ominus, \oplus$, etc.
position property, (RDP for short) if for all \( a, b_1, b_2 \in E \) with \( a \leq b_1 \oplus b_2 \) there exist \( a_1, a_2 \in E \) with \( a_1 \leq b_1 \) and \( a_2 \leq b_2 \) such that \( a = a_1 \oplus a_2 \). An effect algebra \( E \) has RDP if and only if, for \( x_1, x_2, y_1, y_2 \in E \) with \( x_1 \oplus x_2 = y_1 \oplus y_2 \), there exist \( c_{11}, c_{12}, c_{21}, c_{22} \in E \) such that \( x_1 = c_{11} \oplus c_{12}, x_2 = c_{21} \oplus c_{22}, y_1 = c_{11} \oplus c_{21} \) and \( y_2 = c_{12} \oplus c_{22} \).

An effect algebra \( E \) satisfies coherence law if, for \( a, b, c \in E \), with \( a \perp b, b \perp c, c \perp a \), we have \( a \oplus b \oplus c \in E \). An effect algebra is an orthomodular poset if and only if it satisfies the coherence law.

An element \( a \in E \) is sharp if \( a \land a^\perp = 0 \) (or equivalently, if \( a \lor a^\perp = 1 \)). The set of all sharp elements of an effect algebra \( E \) is denoted by \( E_s \). An effect algebra \( E \) is called an orthoalgebra if \( a \perp a \) implies \( a = 0 \), where \( a \in E \). An effect algebra \( E \) is an orthoalgebra if and only if \( E = E_s \). It may be observed that an orthomodular poset is an orthoalgebra. Indeed, if \( a \perp a \), then together with \( a \perp a^\perp \), it gives \( a \oplus 1 \) which entails \( a = 0 \).

A very important relation from the point of view of physical applications is the compatibility relation. Two elements \( a \) and \( b \) in an effect algebra \( E \) are called compatible (written as \( a \leftrightarrow b \)), if there are elements \( a_1, b_1, c \in E \) such that \( a = a_1 \oplus c, b = b_1 \oplus c \) and \( a_1 \oplus b_1 \oplus c \in E \). An effect algebra \( E \) is called an MV-algebra if \( E \) is lattice ordered and for every \( a, b \in E \), \( a \leftrightarrow b \) holds. MV-algebras, introduced by Chang [23], play an important role in many-valued logic. A lattice effect algebra with RDP is an MV-algebra. An effect algebra \( E \) turns out to be a Boolean algebra if \( E \) is an MV-algebra and an orthoalgebra at the same time, equivalently, if \( E \) is an orthomodular lattice and for every \( a, b \in E \), \( a \leftrightarrow b \) holds.

If Boole, OML, OMP, OA, MV and EA represent the class of all Boolean algebras, orthomodular lattices, orthomodular posets, orthoalgebras, MV-algebras and effect algebras, respectively, we have Boole \( \subseteq \) OML \( \subseteq \) OMP.
\( \subseteq \text{OA} \subseteq \text{EA}; \text{Boole} \subseteq \text{MV} \subseteq \text{EA}. \)

2.2.6 [24, 28, 45] Following are some examples of quantum structures:

(i) Let \( H = (H, \langle \cdot, \cdot \rangle) \) be a Hilbert space and let \( B(H) \) be the set of all bounded (i.e., continuous) linear operators on \( H \). An element \( x \in B(H) \) is said to be an \textit{effect} if it is self-adjoint (i.e., \( x^* = x \), where \( x^* \) is the adjoint operator of \( x \), given by \( \langle x^*P, Q \rangle = \langle P, xQ \rangle \) for all \( P, Q \in H \)) and \( O \leq x \leq I \), where \( O, I \) are the zero operator and the identity operator on \( H \) respectively. If \( \mathbb{E}(H) \) is the set of all effects on \( H \), and we define, for \( x, y \in \mathbb{E}(H) \), \( x \perp y \) if and only if \( x + y \leq I \), and \( x \oplus y := x + y \) whenever \( x \perp y \), then \( (\mathbb{E}(H), \oplus, O, I) \) is an effect algebra which is not an orthoalgebra. This effect algebra \( \mathbb{E}(H) \) plays an important role for unsharp measurements in quantum mechanics, and can be considered as standard effect algebra.

(ii) Let \( H \) be a Hilbert space. An element \( x \in B(H) \) is called a \textit{projection} operator if \( x \) is self-adjoint and idempotent. Let \( \mathbb{P}(H) \) be the set of all projection operators on \( H \). Define for \( x_1, x_2 \in \mathbb{P}(H) \), \( x_1 \perp x_2 \) if and only if \( x_1x_2 = x_2x_1 = O \) and \( x_1 \oplus x_2 := x_1 + x_2 \) whenever \( x_1 \perp x_2 \). Then \( (\mathbb{P}(H), \oplus, O, I) \) is a complete effect algebra, which is also a complete orthomodular lattice, but is not a Boolean algebra if dimension of \( H \) is greater than or equal to 2.

(iii) Let \( \mathbb{L}(H) \) be the set of all closed subspaces of \( H \). For \( M_1, M_2 \in \mathbb{L}(H) \), define \( M_1 \perp M_2 \) if \( M_1 \cap M_2 = \{0\} \) (0 is the additive identity in \( H \)) and \( M_1 \oplus M_2 \) the smallest closed subspace of \( \mathbb{L}(H) \) containing \( M_1 \) and \( M_2 \). Then \( (\mathbb{L}(H), \oplus, \{0\}, H) \) is a complete effect algebra, which is also a complete orthomodular lattice, and \( \mathbb{P}(H) \) and \( \mathbb{L}(H) \) are isomorphic quantum logics.

(iv) Let \( (G, +, 0, \leq) \) be a partially ordered commutative group with 0 as its additive identity. Let \( u \in G \) be such that \( 0 \leq u \) and \( 0 \neq u \). Take \( E = \{ x \in G : 0 \leq x \leq u \} \). For \( x, y \in E \), let us define \( x \perp y \) if and only if \( x + y \in E \), and \( x \oplus y := x + y \) whenever \( x \perp y \). Then \( (E, \oplus, 0, u) \) is an effect
algebra and it is called an *interval effect algebra* which is not, in general, an orthoalgebra. For instance, if $G = \mathbb{R}$ with the usual ordering and $u = 1$, then $([0, 1], \oplus, 0, 1)$ is an effect algebra but not an orthoalgebra because $\frac{1}{2} \leq 1 \oplus \frac{1}{2}$ and $\frac{1}{2} \neq 0$.

(v) Let $X$ be a nonempty set. Then $(\mathcal{P}(X), \oplus, \emptyset, X)$ is an effect algebra, where $\mathcal{P}(X)$ denotes the power set of $X$, and $\oplus$ is defined as follows: $A \oplus B = A \cup B$, whenever $A$ and $B$ are disjoint (here $A, B \in \mathcal{P}(X)$).

(vi) Let $X = \{ A \subseteq \mathbb{N} : A$ is finite or $A^c$ is finite$\}$. For $A, B \in X$, we say that $A \perp B$ if $A \cap B = \emptyset$ and $A \oplus B := A \cup B$ whenever $A \perp B$. Then $P = (X, \oplus, \emptyset, \mathbb{N})$ is a $D$-poset.

(vii) Let $X$ be a nonempty set and $\mathcal{A} \subseteq [0, 1]^X$. Let the constant function $1 \in \mathcal{A}$. Then $\mathcal{A}$ with partial binary operation $\ominus$ given by: for $f, g \in \mathcal{A}$, $g \ominus f$ is defined if and only if $f \leq g$ and $g \ominus f := g - f$, is a $D$-poset or an effect algebra (here $\leq$ and $-$ are pointwise partial order and difference of real functions, respectively).

(viii) Let $E = \{0, a, b, 1\}$. Let us define $a \oplus a = b \oplus b = 1$ and $0 \oplus x = x \oplus 0 = x$ for all $x \in E$. Then $(E, \oplus, 0, 1)$ is an effect algebra (known as the *diamond*).

(ix) Let $E = [0, 1] \times [0, 1]$. For $(x_1, x_2), (y_1, y_2) \in E$, $(x_1, x_2) \oplus (y_1, y_2)$ is defined if and only if $x_1 + y_1 \leq 1$ and $x_2 + y_2 \leq 1$. Then $(E, \oplus, 0, 1)$ is an effect algebra.

2.3 Measures on quantum structures: Basic definitions and facts

Throughout this section, let $E$ be a bounded poset with largest element $1$ and smallest element $0$. Let $\mu : E \rightarrow [0, \infty)$ (or $[0, 1]$) be a function.
2.3.1 The function $\mu$ is called *monotone* if $b \leq a$ implies $\mu(b) \leq \mu(a)$, where $a, b \in E$; $\mu$ is called *continuous from below* (continuous from above, respectively) if following condition holds:

$$a_n \uparrow a \ (a, a_n \in E) \implies \lim_{n \to \infty} \mu(a_n) = \mu(a).$$

$$(a_n \downarrow a \ (a, a_n \in E) \implies \lim_{n \to \infty} \mu(a_n) = \mu(a), \text{ respectively.})$$

If $\mu$ is monotone and continuous from below (continuous from above, respectively), then $\mu$ is termed as a lower semicontinuous function, or an *lsc-function* in brief (an upper semicontinuous function, or a *usc-function* in brief). If $\mu$ is an lsc-function as well as a usc-function, then it is called a *continuous function*.

2.3.2 In case $E$ is a lattice, the function $\mu$ is called *submodular* (supermodular, respectively) if

$$\mu(a \lor b) + \mu(a \land b) \leq \mu(a) + \mu(b), \text{ for all } a, b \in E$$

$$(\mu(a \lor b) + \mu(a \land b) \geq \mu(a) + \mu(b), \text{ for all } a, b \in E, \text{ respectively});$$

$\mu$ is called *modular* if it is both supermodular and submodular, i.e. $\mu(a \lor b) + \mu(a \land b) = \mu(a) + \mu(b)$, for all $a, b \in E$.

2.3.3 In case $E$ is a $D$-poset (or an effect algebra), $\mu$ is called

- *superadditive* if for $a, b \in E$, $b \leq a$ implies $\mu(a \ominus b) \leq \mu(a) - \mu(b)$; equivalently if for $a, b \in E$, $a \perp b$ implies $\mu(a \oplus b) \geq \mu(a) + \mu(b)$;

- *subadditive* if for $a, b \in E$, $b \leq a$ implies $\mu(a \ominus b) \geq \mu(a) - \mu(b)$; equivalently if for $a, b \in E$, $a \perp b$ implies $\mu(a \oplus b) \leq \mu(a) + \mu(b)$;

- a *measure* (or finitely additive) if $a, b \in E$, $b \leq a$ implies $\mu(a \ominus b) = \mu(a) - \mu(b)$; equivalently if for $a, b \in E$, $a \perp b$ implies $\mu(a \oplus b) = \mu(a) + \mu(b)$.

- a $\sigma$-additive (or countably additive), if for every orthogonal sequence $\{a_n\}_{n=1}^{\infty}$ in $E$ such that $\bigoplus_{n=1}^{\infty} a_n$ exists in $E$, we have $\mu(\bigoplus_{n=1}^{\infty} a_n) =$
\[\sum_{n=1}^{\infty} \mu(a_n).\]

- **exhaustive**, if for any orthogonal sequence \(\{a_n\}_{n=1}^{\infty}\) in \(E\), \(\lim_{n \to \infty} \mu(a_n) = 0\).

It may be noted that if \(\mu\) is a measure, then \(\mu\) is monotone, and also \(\mu(0) = 0\). In case \(\mu\) is a \([0,1]\)-valued measure and \(\mu(1) = 1\), \(\mu\) is called a **probability measure** (or a **state**) on \(F\). Also every \(\sigma\)-additive function \(\mu\) satisfying \(\mu(0) = 0\) is finitely additive (or a measure).

2.3.4 Let \(E\) be an effect algebra. The family \(\mathcal{M}_s(E)\) of all real-valued subadditive functions on \(E\) is closed under finite meet, i.e. if \(\mu, \nu \in \mathcal{M}_s(E)\), then \(\mu \land \nu \in \mathcal{M}_s(E)\) and for \(a \in E\), \((\mu \land \nu)(a) = \inf \{\mu(b) + \nu(a \ominus b) : b \leq a, b \in E\}\). (Definitions of subadditive and superadditive functions that are given in 2.3.3 are valid for real-valued functions also.) For, let \(a_1, a_2 \in E\) such that \(a_1 \oplus a_2\) exists. Then \((\mu \land \nu)(a_1) + (\mu \land \nu)(a_2) = \inf \{\mu(b_1) + \nu(a_1 \ominus b_1) : b_1 \leq a_1, b_1 \in E\} + \inf \{\mu(b_2) + \nu(a_2 \ominus b_2) : b_2 \leq a_2, b_2 \in E\} = \inf \{\mu(b_1) + \nu(b_2) + \nu(a_1 \ominus b_1) + \nu(a_2 \ominus b_2) : b_1 \leq a_1, b_2 \leq a_2, b_1, b_2 \in E\} \geq \inf \{\mu(b_1 + b_2) + \nu((a_1 \ominus b_2) \ominus (b_1 \ominus b_2)) : b_1 \leq a_1, b_2 \leq a_2, b_1, b_2 \in E\} \geq \inf \{\mu(b) + \nu((a_1 \oplus a_2) \ominus b) : b \leq a_1 \oplus a_2, b \in E\} = (\mu \land \nu)(a_1 \oplus a_2).

The family \(\mathcal{M}^s(E)\) of all real-valued superadditive functions on \(E\) is closed under finite join, i.e. if \(\mu, \nu \in \mathcal{M}^s(E)\), then \(\mu \lor \nu \in \mathcal{M}_s(E)\) and it can be similarly verified that for \(a \in E\), \((\mu \lor \nu)(a) = \sup \{\mu(b) + \nu(a \ominus b) : b \leq a, b \in E\}\).

The family \(\mathcal{M}(E)\) of all real-valued finitely additive functions (or measures) on \(E\) with Riesz decomposition property is a Banach lattice with norm \(\|\mu\| = |\mu|(1)\), where \(\mu \in \mathcal{M}(E)\) (cf. [29]).

2.3.5 [68] Let \(\mu\) be a measure on a \(D\)-poset \(E\). An element \(a \in E\) with \(\mu(a) \neq 0\) is called a \(\mu\)-atom of \(E\), if for any \(b \in E\) with \(b \leq a\), either \(\mu(b) = 0\) or \(\mu(a \ominus b) = 0\).

2.3.6 [99, 100] Let \(L\) be an OML. A function \(p : L \times L \to [0,1]\) is called
an \( s \)-map if the following conditions hold:

(i) \( p(1, 1) = 1 \);

(ii) if \( a \perp b \), then \( p(a, b) = 0 \);

(iii) if \( a \perp b \), then for any \( c \in L \),

\[
p(a \lor b, c) = p(a, c) + p(b, c) \quad \text{and} \quad p(c, a \lor b) = p(c, a) + p(c, b).
\]

If \( p \) is an \( s \)-map on an OML \( L \), then for \( a, b, c \in L \), \( p \) satisfies following properties:

(i) If \( a \leftrightarrow b \), then \( p(a, b) = p(a \land b, a \land b) = p(b, a) \).

(ii) If \( a \leq b \), then \( p(a, b) = p(a, a) \).

(iii) If \( a \leq b \), then \( p(a, c) \leq p(b, c) \); \( p(c, a) \leq p(c, b) \).

(iv) \( p(a, b) \leq p(b, b) \).

(v) If \( \nu(a) = p(a, a) \), then \( \nu \) is a state on \( L \).

2.3.7 [18, 139] A (classical) dynamical system is a quadruple \( \Phi = (X, \Sigma, \mu, \phi) \), where \( (X, \Sigma, \mu) \) is a probability measure space and \( \phi : X \rightarrow X \) is a measure preserving transformation, i.e. if \( a \in \Sigma \), then \( \mu(a) = \mu(\phi(a)) \).

Let \( (X, \Sigma, \mu) \) be a probability measure space and \( J \) be a finite or countable set of indices. A collection of measurable subsets \( \mathcal{A} = \{A_\alpha \in \Sigma : \alpha \in J\} \) is called a measurable partition of \( X \) if \( \mu(X \setminus \bigcup_{\alpha \in J} A_\alpha) = 0 \) and \( \mu(A_{\alpha_1} \cap A_{\alpha_2}) = 0 \) for \( \alpha_1 \neq \alpha_2, \alpha_1, \alpha_2 \in J \). If \( \mathcal{A} \) is a finite partition of the probability measure space \( (X, \Sigma, \mu) \), then the collection of all elements of \( \Sigma \) which are unions of \( \mathcal{A} \) is a finite subalgebra of \( \Sigma \). Let us denote it by \( C(\mathcal{A}) \).

On the other hand, if \( \Sigma_1 \) is a finite subalgebra of \( \Sigma \), say \( \Sigma_1 = \{C_i : i = 1, 2, \ldots, k\} \), then the nonempty sets of the form \( B_1 \cap B_2 \cap \cdots \cap B_k \), where \( B_i = C_i \) or \( C_i^c \), forms a finite partition of \( (X, \Sigma, \mu) \), denoted by \( \mathcal{A}(\Sigma_1) \). We have \( C(\mathcal{A}(\Sigma_1)) = \Sigma_1 \) and \( \mathcal{A}(C(\zeta)) = \zeta \) (here \( \zeta \) is a finite partition of the probability measure space). Thus, there is a one-to-one correspondence between finite
partitions and finite subalgebras of $\Sigma$.

2.3.8 [18] Let $\Phi = (X, \Sigma, \mu, \phi)$ be a classical dynamical system and let $\mathcal{A} = \{A_i : i = 1, 2, \ldots, k\}$ be a finite measurable partition of $X$. Then the entropy of $\mathcal{A}$ is

$$H(\mathcal{A}) := -\sum_{i=1}^{k} \mu(A_i) \log \mu(A_i);$$

the entropy of $\phi$ with respect to $\mathcal{A}$ is $h(\phi, \xi) := \lim_{k \to \infty} \frac{1}{k} H(\bigvee_{i=0}^{k-1} \phi^{-i}(\xi)), k \in \mathbb{N}$; and the entropy of $\Phi$ is $h(\Phi) := \sup h(\phi, \mathcal{A})$, where supremum is taken over all finite measurable partitions of $X$.

An alternative description of the theory of entropy may be given by using the notion of atoms [18]. The concept of entropy of partitions with the use of integrals may be found in [18, 56, 76].

Let $(X, \Sigma, \mu)$ and $(X, \Sigma, \nu)$ be probability measures spaces and $\mathcal{A} = \{A_i : i = 1, 2, \ldots, n\}$ be a measurable partition of $X$. Then relative entropy of $\mu$ with respect to $\nu$, denoted by $D(\mu\|\nu)$, is

$$D(\mu\|\nu) := \sum_{A_i \in \mathcal{A}} \mu(A_i) \log \frac{\mu(A_i)}{\nu(A_i)}.$$ 

Note that relative entropy is not necessarily symmetric, i.e. $D(\mu\|\nu) \neq D(\nu\|\mu)$, in general.

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