Chapter 9

Radical Transversal Screen Pseudo-Slant Lightlike Submanifolds

9.1 Introduction

The geometry of radical transversal, transversal, semi-transversal, generalized transversal lightlike submanifolds has been studied in ([52], [71]). In this chapter, we introduce the notion of radical transversal screen pseudo-slant lightlike submanifolds of indefinite Kaehler and Sasakian manifolds. In section 9.2, we introduce radical transversal screen pseudo-slant lightlike submanifolds of an indefinite Kaehler manifold giving some examples. In section 9.3, we introduce radical transversal screen pseudo-slant lightlike submanifolds of an indefinite Sasakian manifold giving some examples. Section 9.4 is devoted to the study of foliations determined by distributions on radical transversal screen pseudo-slant lightlike submanifolds of the above manifolds.

9.2 Radical Transversal Screen Pseudo-Slant Lightlike Submanifolds of Indefinite Kaehler Manifolds

In this section, we introduce the notion of radical transversal screen pseudo-slant lightlike submanifolds of indefinite Kaehler manifolds. At first, we state the following Lemma, which help us to define radical transversal screen pseudo-slant lightlike submanifolds of an indefinite Kaehler manifold.
Lemma 9.2.1: Let $M$ be a $2q$-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$, of index $2q$ such that $2q < \dim(M)$. Then the screen distribution $S(TM)$ on lightlike submanifold $M$ is Riemannian.

The proof of above Lemma follows as in Lemma 3.1 of [56], so we omit it.

Definition 9.2.1. Let $M$ be a $2q$-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$ of index $2q$ such that $2q < \dim(M)$. Then we say that $M$ is a radical transversal screen pseudo-slant lightlike submanifold if the following conditions are satisfied:

(i) $\mathcal{J} \text{Rad}TM = ltr(TM)$,

(ii) there exists non-degenerate orthogonal distributions $D_1$ and $D_2$ on $M$ such that $S(TM) = D_1 \oplus_{\text{orth}} D_2$,

(iii) the distribution $D_1$ is anti-invariant, i.e. $\mathcal{J} D_1 \subset S(TM^\perp)$,

(iv) the distribution $D_2$ is slant with angle $\theta(\neq \pi/2)$, i.e. for each $x \in M$ and each non-zero vector $X \in (D_2)_x$, the angle $\theta$ between $\mathcal{J} X$ and the vector subspace $(D_2)_x$ is a constant($\neq \pi/2$), which is independent of the choice of $x \in M$ and $X \in (D_2)_x$.

This constant angle $\theta$ is called the slant angle of distribution $D_2$. A radical transversal screen pseudo-slant lightlike submanifold is said to be proper if $D_1 \neq \{0\}$, $D_2 \neq \{0\}$ and $\theta \neq 0$. From the above definition, we have the following decomposition

$$TM = \text{Rad}TM \oplus_{\text{orth}} D_1 \oplus_{\text{orth}} D_2.$$ \hspace{1cm} (9.2.1)

Let $(\mathbb{R}^{2m}_2, \bar{g}, \mathcal{J})$ denote the manifold $\mathbb{R}^{2m}_2$ with its usual Kaehler structure given by

$$\bar{g} = \frac{1}{4}(- \sum_{i=1}^{q} dx^i \otimes dx^i + dy^i \otimes dy^i + \sum_{i=q+1}^{m} dx^i \otimes dx^i + dy^i \otimes dy^i),$$

$$\mathcal{J}(\sum_{i=1}^{m}(X_i \partial x_i + Y_i \partial y_i)) = \sum_{i=1}^{m}(Y_i \partial x_i - X_i \partial y_i),$$

where $(x^i, y^j)$ are the cartesian coordinates on $\mathbb{R}^{2m}_2$. Now, we construct some examples of radical transversal screen pseudo-slant lightlike submanifolds of an indefinite Kaehler manifold.

Example 9.2.1. Let $(\mathbb{R}^{12}_2, \bar{g}, \mathcal{J})$ be an indefinite Kaehler manifold, where $\bar{g}$ is of signature $(-, +, +, +, +, +, +, +, +, +, +, +, +)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6\}$.

Suppose $M$ is a submanifold of $\mathbb{R}^{12}_2$ given by $x^1 = y^2 = u_1$, $x^2 = y^1 = u_2$, $x^3 = u_3 \cos \beta$, $y^3 = u_3 \sin \beta$, $x^4 = u_4 \sin \beta$, $y^4 = u_4 \cos \beta$, $x^5 = u_5$, $y^5 = u_6$, $x^6 = k \cos u_6$, $y^6 = k \sin u_6$, where $k$ is any constant.
The local frame of $TM$ is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6\}$, where

- $Z_1 = 2(\partial x_1 + \partial y_2)$, $Z_2 = 2(\partial x_2 + \partial y_1)$,
- $Z_3 = 2(\cos \beta \partial x_3 + \sin \beta \partial y_3)$, $Z_4 = 2(\sin \beta \partial x_4 + \cos \beta \partial y_4)$,
- $Z_5 = 2(\partial x_5)$, $Z_6 = 2(\partial y_6 - k \sin u_6 \partial x_6 + k \cos u_6 \partial y_6)$.

Hence $\text{Rad}TM = \text{span} \{Z_1, Z_2\}$ and $S(TM) = \text{span} \{Z_3, Z_4, Z_5, Z_6\}$.

Now $ltr(TM)$ is spanned by $N_1 = -\partial x_1 + \partial y_2$, $N_2 = -\partial x_2 + \partial y_1$ and $S(TM^\perp)$ is spanned by

- $W_1 = 2(\sin \beta \partial x_3 - \cos \beta \partial y_3)$, $W_2 = 2(\cos \beta \partial x_4 - \sin \beta \partial y_4)$,
- $W_3 = 2(k \cos u_6 \partial x_6 + k \sin u_6 \partial y_6)$,
- $W_4 = 2(k^2 \partial y_6 + k \sin u_6 \partial x_6 - k \cos u_6 \partial y_6)$.

It follows that $\mathcal{J}Z_1 = -2N_2$, $\mathcal{J}Z_2 = -2N_1$, which implies that $\mathcal{J}\text{Rad}TM = ltr(TM)$. On the other hand, we can see that $D_1 = \text{span} \{Z_3, Z_4\}$ such that $\mathcal{J}Z_3 = W_1$, $\mathcal{J}Z_4 = W_2$, which implies that $D_1$ is anti-invariant with respect to $\mathcal{J}$ and $D_2 = \text{span} \{Z_5, Z_6\}$ is a slant distribution with slant angle $\theta = \arccos(1/\sqrt{1+k^2})$. Hence $M$ is a radical transversal screen pseudo-slant 2-lightlike submanifold of $\mathbb{R}^1_{12}$.

Example 9.2.2. Let $(\mathbb{R}^1_{12}, \mathcal{g}, \mathcal{J})$ be an indefinite Kaehler manifold, where $\mathcal{g}$ is of signature $(-, +, +, +, +, +, +, +, +, +, +)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6\}$.

Suppose $M$ is a submanifold of $\mathbb{R}^1_{12}$ given by $x^1 = u_1$, $y^1 = u_2$, $x^2 = -u_1 \cos \alpha + u_2 \sin \alpha$, $y^2 = u_1 \sin \alpha + u_2 \cos \alpha$, $x^3 = y^4 = u_3$, $x^4 = y^3 = u_4$, $x^5 = u_5 \cos u_6$, $y^5 = u_5 \sin u_6$, $x^6 = \cos u_5$, $y^6 = \sin u_5$.

The local frame of $TM$ is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6\}$, where

- $Z_1 = 2(\partial x_1 - \cos \alpha \partial x_2 + \sin \alpha \partial y_2)$, $Z_2 = 2(\partial y_1 + \sin \alpha \partial x_2 + \cos \alpha \partial y_2)$,
- $Z_3 = 2(\partial x_3 + \partial y_4)$, $Z_4 = 2(\partial x_4 + \partial y_3)$,
- $Z_5 = 2(\cos u_6 \partial x_5 + \sin u_6 \partial y_5 - \sin u_5 \partial x_6 + \cos u_5 \partial y_6)$,
- $Z_6 = 2(-u_5 \sin u_6 \partial x_5 + u_5 \cos u_6 \partial y_5)$.

Hence $\text{Rad}TM = \text{span} \{Z_1, Z_2\}$ and $S(TM) = \text{span} \{Z_3, Z_4, Z_5, Z_6\}$.

Now $ltr(TM)$ is spanned by $N_1 = -\partial x_1 - \cos \alpha \partial x_2 + \sin \alpha \partial y_2$, $N_2 = -\partial y_1 + \sin \alpha \partial x_2 + \cos \alpha \partial y_2$ and $S(TM^\perp)$ is spanned by

- $W_1 = 2(\partial x_3 - \partial y_4)$, $W_2 = 2(\partial x_4 - \partial y_3)$,
- $W_3 = 2(\cos u_6 \partial x_5 + \sin u_6 \partial y_5 + \sin u_5 \partial x_6 - \cos u_5 \partial y_6)$,
- $W_4 = 2(u_5 \cos u_5 \partial x_6 + u_5 \sin u_5 \partial y_6)$.
It follows that $\overline{J}Z_1 = 2N_2$, $\overline{J}Z_2 = -2N_1$, which implies that $\overline{J}\text{Rad}TM = \text{ltr}(TM)$. On the other hand, we can see that $D_1 = \text{span}\{Z_3, Z_4\}$ such that $\overline{J}Z_3 = W_2$, $\overline{J}Z_4 = W_1$, which implies that $D_1$ is anti-invariant with respect to $\overline{J}$ and $D_2 = \text{span}\{Z_5, Z_6\}$ is a slant distribution with slant angle $\pi/4$. Hence $M$ is a radical transversal screen pseudo-slant 2-lightlike submanifold of $\mathbb{R}^{12}_2$.

Now, for any vector field $X$ tangent to $M$, we put $\overline{J}X = PX + FX$, where $PX$ and $FX$ are tangential and transversal parts of $\overline{J}X$ respectively. We denote the projections on $\text{Rad}TM$, $D_1$ and $D_2$ in $TM$ by $P_1$, $P_2$ and $P_3$ respectively. Similarly, we denote the projections of $tr(TM)$ on $ltr(TM)$, $\overline{J}(D_1)$ and $D'$ by $Q_1$, $Q_2$ and $Q_3$ respectively, where $D'$ is non-degenerate orthogonal complementary subbundle of $\overline{J}(D_1)$ in $S(TM^\perp)$. Then, for any $X \in \Gamma(TM)$, we get

\begin{equation}
X = P_1X + P_2X + P_3X. \tag{9.2.2}
\end{equation}

Now applying $\overline{J}$ to (9.2.2), we have

\begin{equation}
\overline{J}X = \overline{J}P_1X + \overline{J}P_2X + \overline{J}P_3X, \tag{9.2.3}
\end{equation}

which gives

\begin{equation}
\overline{J}X = \overline{J}P_1X + \overline{J}P_2X + fP_3X + FP_3X, \tag{9.2.4}
\end{equation}

where $fP_3X$ (resp. $FP_3X$) denotes the tangential (resp. transversal) component of $\overline{J}P_3X$. Thus we get $\overline{J}P_1X \in \Gamma(ltr(TM))$, $\overline{J}P_2X \in \Gamma(\overline{J}(D_1)) \subset \Gamma(S(TM^\perp))$, $fP_3X \in \Gamma(D_2)$ and $FP_3X \in \Gamma(D')$. Also, for any $W \in \Gamma(tr(TM))$, we have

\begin{equation}
W = Q_1W + Q_2W + Q_3W. \tag{9.2.5}
\end{equation}

Applying $\overline{J}$ to (9.2.5), we obtain

\begin{equation}
\overline{J}W = \overline{J}Q_1W + \overline{J}Q_2W + \overline{J}Q_3W, \tag{9.2.6}
\end{equation}

which gives

\begin{equation}
\overline{J}W = \overline{J}Q_1W + \overline{J}Q_2W + BQ_3W + CQ_3W, \tag{9.2.7}
\end{equation}

where $BQ_3W$ (resp. $CQ_3W$) denotes the tangential (resp. transversal) component of $\overline{J}Q_3W$. Thus we get $\overline{J}Q_1W \in \Gamma(\text{Rad}TM)$, $\overline{J}Q_2W \in \Gamma(D_1)$, $BQ_3W \in \Gamma(D_2)$ and $CQ_3W \in \Gamma(D')$.

Now, by using (1.4.3), (9.2.4), (9.2.7) and (1.3.7)-(1.3.9) and identifying the components on $\text{Rad}TM$, $D_1$, $D_2$, $ltr(TM)$, $\overline{J}(D_1)$ and $D'$, we obtain

\begin{equation}
P_1(A_{\overline{J}P_2X}Y) + P_1(A_{\overline{J}P_1X}Y) + P_1(A_{FP_3X}Y) = P_1(\nabla_X fP_3Y) - \overline{J}h'(X,Y), \tag{9.2.8}
\end{equation}

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\[ P_2(A_{\tilde{J}P_2}X) + P_2(A_{\tilde{J}P_3}X) + P_2(A_{FP_3}X) = P_2(\nabla_X f P_3Y) - \tilde{J}Q_2 h^s(X,Y), \]

(9.2.10) \[ P_3(A_{\tilde{J}P_2}X) + P_3(A_{\tilde{J}P_3}X) + P_3(A_{FP_3}X) = P_3(\nabla_X f P_3Y) - BQ_3 h^s(X,Y) - f P_3 \nabla_X Y, \]

(9.2.11) \[ \tilde{J}P_1 \nabla_X Y = \nabla_X \tilde{J}P_1 Y + D^l(X, \tilde{J}P_2 Y) + h^l(X, f P_3 Y) + D^l(X, FP_3 Y), \]

(9.2.12) \[ Q_2 \nabla_X \tilde{J}P_2 Y + Q_2 \nabla_X F P_3 Y = \tilde{J}P_2 \nabla_X Y - Q_2 D^s(X, \tilde{J}P_1 Y) - Q_2 h^s(X, f P_3 Y), \]

(9.2.13) \[ Q_3 \nabla_X \tilde{J}P_2 Y + Q_3 \nabla_X F P_3 Y - F P_3 \nabla_X Y = CQ_3 h^s(X,Y) - Q_3 h^s(X, f P_3 Y) - Q_3 D^s(X, \tilde{J}P_1 Y). \]

**Theorem 9.2.2:** Let \( M \) be a 2-q-lightlike submanifold of an indefinite Kaehler manifold \( \overline{M} \). Then \( M \) is a radical transversal screen pseudo-slant lightlike submanifold of \( \overline{M} \) if and only if

(i) \( \tilde{J} ltr(TM) \) is a distribution on \( M \) such that \( \tilde{J} ltr(TM) = \text{RadTM} \),

(ii) distribution \( D_1 \) is anti-invariant with respect to \( \tilde{J} \), i.e. \( \tilde{J} D_1 \subset S(TM^\perp) \),

(iii) there exists a constant \( \lambda \in (0, 1] \) such that \( P^2 X = -\lambda X \).

Moreover, there also exists a constant \( \mu \in [0, 1) \) such that \( BFX = -\mu X \), for all \( X \in \Gamma(D_2) \), where \( D_1 \) and \( D_2 \) are non-degenerate orthogonal distributions on \( M \) such that \( S(TM) = D_1 \oplus_{\text{orth}} D_2 \) and \( \lambda = \cos^2 \theta \), \( \theta \) is slant angle of \( D_2 \).

**Proof.** Let \( M \) be a radical transversal screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold \( \overline{M} \). Then distribution \( D_1 \) is anti-invariant with respect to \( \tilde{J} \) and \( \tilde{J} \text{RadTM} = ltr(TM) \). Thus \( \tilde{J}X \in \Gamma(ltr(TM)) \), for all \( X \in \Gamma(\text{RadTM}) \). Hence \( \tilde{J}(\tilde{J}X) \in \Gamma(\tilde{J}(\tilde{J}ltr(TM))) \), which implies \( -X \in \Gamma(\tilde{J}(\tilde{J}ltr(TM))) \), for all \( X \in \Gamma(\text{RadTM}) \), which proves (i) and (ii).

Now for any \( X \in \Gamma(D_2) \), we have \( |PX| = |\tilde{J}X| \cos \theta \), which implies

(9.2.14) \[ \cos \theta = \frac{|PX|}{|\tilde{J}X|}. \]

In view of (9.2.14), we get \( \cos^2 \theta = \frac{|PX|^2}{|\tilde{J}X|^2} = \frac{g(PX,PX)}{g(\tilde{J}X,\tilde{J}X)} = \frac{g(X,P^2X)}{g(X,\tilde{J}^2X)} \), which gives

(9.2.15) \[ g(X, P^2 X) = \cos^2 \theta g(X, \tilde{J}^2 X). \]
Since $M$ is a radical transversal screen pseudo-slant lightlike submanifold, $\cos^2 \theta = \lambda(\text{constant}) \in (0, 1]$ and therefore from (9.2.15), we get
\[ g(X, P^2 X) = \lambda g(X, \bar{J}^2 X) = g(X, \lambda \bar{J}^2 X), \]
which implies
\[ g(X, (P^2 - \lambda \bar{J}^2) X) = 0. \]
(9.2.16)

Since $(P^2 - \lambda \bar{J}^2) X \in \Gamma(D_2)$ and the induced metric $g = g|_{D_2 \times D_2}$ is non-degenerate (positive definite), from (9.2.16), we have $(P^2 - \lambda \bar{J}^2) X = 0$, which implies
\[ P^2 X = \lambda \bar{J}^2 X = -\lambda X. \]
(9.2.17)

Now, for any vector field $X \in \Gamma(D_2)$, we have
\[ \bar{J}X = PX + FX, \]
where $PX$ and $FX$ are tangential and transversal parts of $\bar{J}X$ respectively. Applying $\bar{J}$ to (9.2.18) and taking tangential component, we get
\[ -X = P^2 X + BFX. \]
(9.2.19)

From (9.2.17) and (9.2.19), we get
\[ BFX = -\mu X, \]
where $1 - \lambda = \mu(\text{constant}) \in [0, 1)$. This proves (iii).

Conversely suppose that conditions (i), (ii) and (iii) are satisfied. In view of (i), we have $\bar{J}N \in \Gamma(\text{RadTM})$, for all $N \in \Gamma(ltr(TM))$. Hence $\bar{J}(\bar{J}N) \in \Gamma(\bar{J}(\text{RadTM}))$, which implies $-N \in \Gamma(\bar{J}(\text{RadTM}))$, for all $N \in \Gamma(ltr(TM))$. Thus $\bar{J}\text{RadTM} = ltr(TM)$. From (9.2.19), for any $X \in \Gamma(D_2)$, we get
\[ -X = P^2 X - \mu X, \]
which implies
\[ P^2 X = -\lambda X, \]
where $1 - \mu = \lambda(\text{constant}) \in (0, 1]$.

Now $\cos \theta = \frac{g(JX, PX)}{|JX||PX|} = -\frac{g(X, JPX)}{|JX||PX|} = -\frac{g(X, P^2 X)}{|JX||PX|} = -\lambda \frac{g(X, \bar{J}^2 X)}{|JX||PX|} = \lambda \frac{g(JX, JX)}{|JX||PX|}$. From above equation, we get
\[ \cos \theta = \lambda \frac{|JX|}{|PX|}. \]
(9.2.23)

Therefore (9.2.14) and (9.2.23) give $\cos^2 \theta = \lambda(\text{constant})$. Hence $M$ is a radical transversal screen pseudo-slant lightlike submanifold.
Corollary 9.2.1: Let $M$ be a radical transversal screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$ with slant angle $\theta$, then for any $X, Y \in \Gamma(D_2)$, we have

(i) $g(\overline{PX}, \overline{PY}) = \cos^2 \theta g(X, Y)$,

(ii) $g(\overline{FX}, \overline{FY}) = \sin^2 \theta g(X, Y)$.

The proof of above Corollary follows by using similar steps as in proof of Corollary 3.2 of [56].

Theorem 9.2.3: Let $M$ be a radical transversal screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then $\text{RadTM}$ is integrable if and only if

(i) $Q_2 D^s(X, \overline{JP_1} Y) = Q_2 D^s(X, \overline{JP_1} Y),$

(ii) $Q_3 D^s(Y, \overline{JP_1} X) = Q_3 D^s(X, \overline{JP_1} Y),$

(iii) $P_3 A_{\overline{JP_1} X} Y = P_3 A_{\overline{JP_1} Y} X$, for all $X, Y \in \Gamma(\text{RadTM})$.

Proof. Let $M$ be a radical transversal screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Let $X, Y \in \Gamma(\text{RadTM})$. In view of (9.2.12), we obtain $Q_2 D^s(X, \overline{JP_1} Y) = \overline{JP_2} \nabla_X Y$, which gives $Q_2 D^s(X, \overline{JP_1} Y) - Q_2 D^s(Y, \overline{JP_1} X) = \overline{JP_2}[X, Y]$. From (9.2.13), we get $Q_3 D^s(X, \overline{JP_1} Y) = CQ_3 h^s(X, Y) + FP_3 \nabla_X Y$, which gives $Q_3 D^s(X, \overline{JP_1} Y) - Q_3 D^s(Y, \overline{JP_1} X) = FP_3[X, Y]$. Also from (9.2.10), we get $P_3 A_{\overline{JP_1} X} Y + BQ_3 h^s(X, Y) = -f P_3 \nabla_X Y$. Thus we have $P_3 A_{\overline{JP_1} X} Y - P_3 A_{\overline{JP_1} Y} X = f P_3[X, Y]$. This concludes the theorem.

Theorem 9.2.4: Let $M$ be a radical transversal screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then $D_1$ is integrable if and only if

(i) $Q_3(\nabla^s_X \overline{JP_2} X) = Q_3(\nabla^s_X \overline{JP_2} Y)$ and $P_3 A_{\overline{JP_2} X} Y = P_3 A_{\overline{JP_2} Y} X$,

(ii) $D^l(X, \overline{JP_2} Y) = D^l(Y, \overline{JP_2} X)$, for all $X, Y \in \Gamma(D_1)$.

Proof. Let $M$ be a radical transversal screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Let $X, Y \in \Gamma(D_1)$. From (9.2.11), $D^l(X, \overline{JP_2} Y) = \overline{JP_1} \nabla_X Y$, which gives $D^l(X, \overline{JP_2} Y) - D^l(Y, \overline{JP_2} X) = \overline{JP_1}[X, Y]$. In view of (9.2.10), we obtain $P_3 A_{\overline{JP_2} Y} X + BQ_3 h^s(X, Y) = -f P_3 \nabla_X Y$, which implies $P_3 A_{\overline{JP_2} X} Y - P_3 A_{\overline{JP_2} Y} X = f P_3[X, Y]$. Also from (9.2.13), we have $Q_3(\nabla^s_X \overline{JP_2} Y) - CQ_3 h^s(X, Y) = FP_3 \nabla_X Y$, which gives $Q_3(\nabla^s_X \overline{JP_2} Y) - Q_3(\nabla^s_Y \overline{JP_2} X) = FP_3[X, Y]$. This proves the theorem.
Theorem 9.2.5: Let $M$ be a radical transversal screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then $D_2$ is integrable if and only if

(i) $D^l(X, FP_3Y) - h^l(Y, fP_3X) = D^l(Y, FP_3X) - h^l(X, fP_3Y),$

(ii) $Q_2(\nabla^s_X FP_3Y - h^s(Y, fP_3X)) = Q_2(\nabla^s_Y FP_3X - h^s(X, fP_3Y)),$

for all $X, Y \in \Gamma(D_2)$.

Proof. Let $M$ be a radical transversal screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Let $X, Y \in \Gamma(D_2)$. From (9.2.11), we have $h^l(X, fP_3Y) + D^l(X, FP_3Y) = \overline{J}P_1\nabla_X Y$, which gives $h^l(X, fP_3Y) - h^l(Y, fP_3X) + D^l(X, FP_3Y) - D^l(Y, FP_3X) = \overline{J}P_1[X, Y].$

From (9.2.12), we get $Q_2\nabla^s_X FP_3Y + Q_2h^s(X, fP_3Y) = \overline{J}P_2\nabla_X Y$, which implies $Q_2\nabla^s_X FP_3Y - Q_2\nabla^s_Y FP_3X + Q_2h^s(X, fP_3Y) - Q_2h^s(Y, fP_3X) = \overline{J}P_2[X, Y].$ This completes the proof.

Theorem 9.2.6: Let $M$ be a radical transversal screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then the induced connection $\nabla$ is a metric connection if and only if $BQ_3D^s(X, N) = fP_3A_N X$ and $\overline{J}Q_2D^s(X, N) = 0$, for all $X \in \Gamma(TM)$, $N \in \Gamma(ltr(TM)).$

Proof. Let $M$ be a radical transversal screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then the induced connection $\nabla$ on $M$ is a metric connection if and only if $Rad(TM)$ is parallel distribution with respect to $\nabla$ ([32]). From (1.4.3), for any $X \in \Gamma(TM)$ and $N \in \Gamma(ltr(TM))$, we have

\begin{equation}
\nabla_X JN = J \nabla_X N.
\end{equation}

From (1.3.7), (1.3.8) and (9.2.24), we obtain

\begin{equation}
\nabla_X JN = JQ_2D^s(X, N) + \overline{J}Q_3D^s(X, N)
+ J\nabla^l_X N - JA_N X.
\end{equation}

On comparing tangential components of both sides of above equation, we get

\begin{equation}
\nabla_X JN = \overline{J}Q_2D^s(X, N) + BQ_3D^s(X, N)
+ J\nabla^l_X N - fP_3A_N X,
\end{equation}

which completes the proof.
9.3 Radical Transversal Screen Pseudo-Slant Lightlike Submanifolds of Indefinite Sasakian Manifolds

In this section, we introduce the notion of radical transversal screen pseudo-slant lightlike submanifolds of indefinite Sasakian manifolds. At first, we state the following Lemma, which help us to define radical transversal screen pseudo-slant lightlike submanifolds of an indefinite Sasakian manifold.

**Lemma 9.3.1:** Let $M$ be a $2q$-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$, of index $2q$ such that $2q < \dim(M)$ with structure vector field tangent to $M$. Then the screen distribution $S(TM)$ on lightlike submanifold $M$ is Riemannian.

The proof of above Lemma follows as in Lemma 4.1 of [59], so we omit it.

**Definition 9.3.1.** Let $M$ be a $2q$-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ of index $2q$ such that $2q < \dim(M)$ with structure vector field tangent to $M$. Then we say that $M$ is a radical transversal screen pseudo-slant lightlike submanifold of $\overline{M}$ if the following conditions are satisfied:

(i) $\phi \text{Rad}(TM) = \text{ltr}(TM)$,

(ii) there exists non-degenerate orthogonal distributions $D_1$ and $D_2$ on $M$ such that $S(TM) = D_1 \oplus_{\text{orth}} D_2 \oplus_{\text{orth}} \{V\}$,

(iii) the distribution $D_1$ is anti-invariant, i.e. $\phi D_1 \subset S(TM^\perp)$,

(iv) the distribution $D_2$ is slant with angle $\theta(\neq \pi/2)$, i.e. for each $x \in M$ and each non-zero vector $X \in (D_2)_x$, the angle $\theta$ between $\phi X$ and the vector subspace $(D_2)_x$ is a constant($\neq \pi/2$), which is independent of the choice of $x \in M$ and $X \in (D_2)_x$.

This constant angle $\theta$ is called the slant angle of distribution $D_2$. A radical transversal screen pseudo-slant lightlike submanifold is said to be proper if $D_1 \neq \{0\}$, $D_2 \neq \{0\}$ and $\theta \neq 0$. From the above definition, we have the following decomposition

\[(9.3.1)\quad TM = \text{Rad}(TM) \oplus_{\text{orth}} D_1 \oplus_{\text{orth}} D_2 \oplus_{\text{orth}} \{V\}.

In particular, we have

(i) if $D_2 = 0$, then $M$ is a transversal lightlike submanifold,
(ii) if $D_1 = 0$ and $\theta = 0$, then $M$ is a radical transversal lightlike submanifold,

(iii) if $D_1 \neq 0$ and $\theta = 0$, then $M$ is a generalized transversal lightlike submanifold.

Thus the above new class of lightlike submanifolds of an indefinite Sasakian manifold includes radical transversal, transversal, generalized transversal lightlike submanifolds as its sub-cases which have been studied in ([71], [80]).

Let $(\mathbb{R}^{2m+1}_2, \bar{g}, \phi, \eta, V)$ denote the manifold $\mathbb{R}^{2m+1}_2$ with its usual Sasakian structure given by

$$\eta = \frac{1}{2}(dz - \sum_{i=1}^{m} y^i dx^i), \quad V = 2\partial z,$$

$$\bar{g} = \eta \otimes \eta + \frac{1}{4}(-\sum_{i=1}^{q} dx^i \otimes dx^i + dy^i \otimes dy^i + \sum_{i=q+1}^{m} dx^i \otimes dx^i + dy^i \otimes dy^i),$$

$$\phi(\sum_{i=1}^{m}(X_i \partial x_i + Y_i \partial y_i) + Z \partial z) = \sum_{i=1}^{m}(Y_i \partial x_i - X_i \partial y_i) + \sum_{i=1}^{m} Y_i y^i \partial z,$$

where $(x^i, y^i, z)$ are the cartesian coordinates on $\mathbb{R}^{2m+1}_2$. Now we construct some examples of radical transversal screen pseudo-slant lightlike submanifolds of an indefinite Sasakian manifold.

**Example 9.3.1.** Let $(\mathbb{R}^{13}_2, \bar{g})$ be an indefinite Sasakian manifold, where $\bar{g}$ is of signature $(-, +, +, +, +, - , +, +, +, +, +, +, +, +)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6, \partial z\}$.

Suppose $M$ is a submanifold of $\mathbb{R}^{13}_2$ given by $x^1 = u_1$, $y^1 = u_2$, $x^2 = -u_1 \cos \alpha + u_2 \sin \alpha$, $y^2 = u_1 \sin \alpha + u_2 \cos \alpha$, $x^3 = y^4 = u_3$, $x^4 = y^3 = u_4$, $x^5 = u_5$, $y^5 = u_6$, $x^6 = k \cos u_6$, $y^6 = k \sin u_6$, $z = u_7$, where $k$ is any constant.

The local frame of $TM$ is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\}$, where

$$Z_1 = 2(\partial x_1 - \cos \alpha \partial x_2 + \sin \alpha \partial y_2 + y^1 \partial z - \cos \alpha y^2 \partial z),$$

$$Z_2 = 2(\partial y_1 + \sin \alpha \partial x_2 + \cos \alpha \partial y_2 + \sin \alpha y^2 \partial z),$$

$$Z_3 = 2(\partial x_3 + \partial y_4 + y^3 \partial z), \quad Z_4 = 2(\partial x_4 + \partial y_3 + y^4 \partial z),$$

$$Z_5 = 2(\partial x_5 + y^5 \partial z),$$

$$Z_6 = 2(\partial y_5 - k \sin u_6 \partial x_6 + k \cos u_6 \partial y_6 - k \sin u_6 y^6 \partial z), \quad Z_7 = V = 2\partial z.$$  

Hence $\text{Rad}TM = \text{span} \{Z_1, Z_2\}$ and $S(TM) = \text{span} \{Z_3, Z_4, Z_5, Z_6, V\}$.

Now $\text{ltr}(TM)$ is spanned by $N_1 = -\partial x_1 - \cos \alpha \partial x_2 + \sin \alpha \partial y_2 - y^1 \partial z - \cos \alpha y^2 \partial z$, $N_2 = -\partial y_1 + \sin \alpha \partial x_2 + \cos \alpha \partial y_2 + \sin \alpha y^2 \partial z$ and $S(TM^\bot)$ is spanned by

$$W_1 = 2(\partial x_3 - \partial y_4 + y^3 \partial z), \quad W_2 = 2(\partial x_4 - \partial y_3 + y^4 \partial z),$$

$$W_3 = 2(k \cos u_6 \partial x_6 + k \sin u_6 \partial y_6 + k \cos u_6 y^6 \partial z),$$

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It follows that $\phi Z_1 = 2N_2$, $\phi Z_2 = -2N_1$, which implies that $\phi \text{Rad}TM = \text{ltr}(TM)$. On the other hand, we can see that $D_1 = \text{span} \{Z_3, Z_4\}$ such that $\phi Z_3 = W_2$, $\phi Z_4 = W_1$, which implies that $D_1$ is anti-invariant with respect to $\phi$ and $D_2 = \text{span} \{Z_5, Z_6\}$ is a slant distribution with slant angle $\theta = \arccos(1/\sqrt{1+K^2})$. Hence $M$ is a radical transversal screen pseudo-slant 2-lightlike submanifold of $\mathbb{R}^{13}_2$.

**Example 9.3.2.** Let $(\mathbb{R}^{13}_2, \bar{g})$ be an indefinite Sasakian manifold, where $\bar{g}$ is of signature $(-, +, +, +, +, +, -, +, +, +, +, +)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6, \partial z\}$.

Suppose $M$ is a submanifold of $\mathbb{R}^{13}_2$ given by $x^1 = y^2 = u_1$, $x^2 = y^1 = u_2$, $x^3 = u_3 \cos \beta$, $y^3 = u_3 \sin \beta$, $x^4 = u_4 \sin \beta$, $y^4 = u_4 \cos \beta$, $x^5 = u_5 \cos u_6$, $y^5 = u_5 \sin u_6$, $x^6 = \cos u_5$, $y^6 = \sin u_5$, $z = u_7$.

The local frame of $TM$ is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\}$, where

\[
Z_1 = 2(\partial x_1 + \partial y_2 + y^1 \partial z), \quad Z_2 = 2(\partial x_2 + \partial y_1 + y^2 \partial z),
\]

\[
Z_3 = 2(\cos \beta \partial x_3 + \sin \beta \partial y_3 + y^3 \cos \beta \partial z),
\]

\[
Z_4 = 2(\sin \beta \partial x_4 + \cos \beta \partial y_4 + y^4 \sin \beta \partial z),
\]

\[
Z_5 = 2(\cos u_6 \partial x_5 + \sin u_6 \partial y_5 - \sin u_5 \partial x_6 + \cos u_5 \partial y_6 + \cos u_6 y^5 \partial z - \sin u_5 y^6 \partial z),
\]

\[
Z_6 = 2(-u_5 \sin u_6 \partial x_5 + u_5 \cos u_6 \partial y_5 - u_5 \sin u_6 y^5 \partial z), \quad Z_7 = V = 2\partial z.
\]

Hence $\text{Rad}TM = \text{span} \{Z_1, Z_2\}$ and $S(TM) = \text{span} \{Z_3, Z_4, Z_5, Z_6, V\}$.

Now $\text{ltr}(TM)$ is spanned by $N_1 = -\partial x_1 + \partial y_2 - y^1 \partial z$, $N_2 = -\partial x_2 + \partial y_1 - y^2 \partial z$ and $S(TM^-)$ is spanned by

\[
W_1 = 2(\sin \beta \partial x_3 - \cos \beta \partial y_3 + y^3 \sin \beta \partial z),
\]

\[
W_2 = 2(\cos \beta \partial x_4 - \sin \beta \partial y_4 + y^4 \cos \beta \partial z),
\]

\[
W_3 = 2(\cos u_6 \partial x_5 + \sin u_6 \partial y_5 + \sin u_5 \partial x_6 - \cos u_5 \partial y_6 + \cos u_6 y^5 \partial z + \sin u_5 y^6 \partial z),
\]

\[
W_4 = 2(u_5 \cos u_5 \partial x_6 + u_5 \sin u_5 \partial y_6 + u_5 \cos u_5 y^6 \partial z).
\]

It follows that $\phi Z_1 = -2N_2$, $\phi Z_2 = -2N_1$, which implies that $\phi \text{Rad}TM = \text{ltr}(TM)$. On the other hand, we can see that $D_1 = \text{span} \{Z_3, Z_4\}$ such that $\phi Z_3 = W_1$, $\phi Z_4 = W_2$, which implies that $D_1$ is anti-invariant with respect to $\phi$ and $D_2 = \text{span} \{Z_5, Z_6\}$ is a slant distribution with slant angle $\pi/4$. Hence $M$ is a radical transversal screen pseudo-slant 2-lightlike submanifold of $\mathbb{R}^{13}_2$. 

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Now, for any vector field $X$ tangent to $M$, we put $\phi X = PX + FX$, where $PX$ and $FX$ are tangential and transversal parts of $\phi X$ respectively. We denote the projections on $RadTM$, $D_1$ and $D_2$ in $TM$ by $P_1$, $P_2$ and $P_3$ respectively. Similarly, we denote the projections of $tr(TM)$ on $ltr(TM)$, $\phi(D_1)$ and $D'$ by $Q_1$, $Q_2$ and $Q_3$ respectively, where $D'$ is non-degenerate orthogonal complementary subbundle of $\phi(D_1)$ in $S(TM^\perp)$. Then, for any $X \in \Gamma(TM)$, we get

\[(9.3.2)\]

\[X = P_1X + P_2X + P_3X + \eta(X)V.\]

Now applying $\phi$ to (9.3.2), we have

\[(9.3.3)\]

\[\phi X = \phi P_1X + \phi P_2X + \phi P_3X,\]

which gives

\[(9.3.4)\]

\[\phi X = \phi P_1X + \phi P_2X + fP_3X + FP_3X,\]

where $fP_3X$ (resp. $FP_3X$) denotes the tangential (resp. transversal) component of $\phi P_3X$. Thus we get $\phi P_1X \in \Gamma(ltr(TM))$, $\phi P_2X \in \Gamma(\phi D_1) \subset \Gamma(S(TM^\perp))$, $fP_3X \in \Gamma(D_2)$ and $FP_3X \in \Gamma(D')$. Also, for any $W \in \Gamma(tr(TM))$, we have

\[(9.3.5)\]

\[W = Q_1W + Q_2W + Q_3W.\]

Applying $\phi$ to (9.3.5), we obtain

\[(9.3.6)\]

\[\phi W = \phi Q_1W + \phi Q_2W + \phi Q_3W,\]

which gives

\[(9.3.7)\]

\[\phi W = \phi Q_1W + \phi Q_2W + BQ_3W + CQ_3W,\]

where $BQ_3W$ (resp. $CQ_3W$) denotes the tangential (resp. transversal) component of $\phi Q_3W$. Thus we get $\phi Q_1W \in \Gamma(RadTM)$, $\phi Q_2W \in \Gamma(D_1)$, $BQ_3W \in \Gamma(D_2)$ and $CQ_3W \in \Gamma(D')$.

Now, by using (1.5.6), (9.3.4), (9.3.7) and (1.3.7)-(1.3.9) and identifying the components on $RadTM$, $D_1$, $D_2$, $ltr(TM)$, $\phi(D_1)$, $D'$ and \{V\}, we obtain

\[(9.3.8)\]

\[P_1(A_{\phi P_2Y}X) + P_1(A_{\phi P_1Y}X) + P_1(A_{FP_3Y}X) = P_1(\nabla_X f P_3Y) - \phi h^i(X,Y) + \eta(Y)P_1X,\]

\[(9.3.9)\]

\[P_2(A_{\phi P_2Y}X) + P_2(A_{\phi P_1Y}X) + P_2(A_{FP_3Y}X) = P_2(\nabla_X f P_3Y) - \phi Q_2h^s(X,Y) + \eta(Y)P_2X,\]
\[ P_3(A_{\phi P_2 Y} X) + P_3(A_{\phi P_1 Y} X) + P_3(A_{FP_3 Y} X) = P_3(\nabla_X f P_3 Y) - BQ_3 h^s(X, Y) - f P_3 \nabla_X Y + \eta(Y) P_3 X, \]

\[ \phi P_1 \nabla_X Y = h(f P_3 Y) + D(f(X, FP_3 Y)) + \nabla_X^l \phi P_2 Y + D(f(X, \phi P_2 Y)), \]

\[ Q_2 \nabla_X^s \phi P_2 Y + Q_2 \nabla_X^s FP_3 Y = \phi P_2 \nabla_X Y - Q_2 D^s(X, \phi P_1 Y) - 2 h^s(X, f P_3 Y), \]

\[ Q_3 \nabla_X^s \phi P_2 Y + Q_3 \nabla_X^s FP_3 Y - FP_3 \nabla_X Y = CQ_3 h^s(X, Y) - Q_3 h^s(X, f P_3 Y) - Q_3 D^s(X, \phi P_1 Y), \]

\[ \bar{g}(\phi X, \phi Y) V = \eta(\nabla_X f P_3 Y) - \eta(A_{\phi P_1 Y} X) - \eta(A_{\phi P_2 Y} X) - \eta(A_{FP_3 Y} X). \]

**Theorem 9.3.2:** Let \( M \) be a 2q-lightlike submanifold of an indefinite Sasakian manifold \( \overline{M} \) with structure vector field tangent to \( M \). Then \( M \) is a radical transversal screen pseudo-slant lightlike submanifold of \( \overline{M} \) if and only if

(i) \( \phi \text{ltr}(TM) \) is a distribution on \( M \) such that \( \phi \text{ltr}(TM) = \text{RadTM} \),

(ii) distribution \( D_1 \) is anti-invariant with respect to \( \phi \), i.e. \( \phi D_1 \subset S(TM^\perp) \),

(iii) there exists a constant \( \lambda \in (0, 1] \) such that \( P^2 X = -\lambda X \).

Moreover, there also exists a constant \( \mu \in [0, 1) \) such that \( BFX = -\mu X \), for all \( X \in \Gamma(D_2) \), where \( D_1 \) and \( D_2 \) are non-degenerate orthogonal distributions on \( M \) such that \( S(TM) = D_1 \oplus_{\text{orth}} D_2 \oplus_{\text{orth}} \{V\} \) and \( \lambda = \cos^2 \theta \), \( \theta \) is slant angle of \( D_2 \).

**Proof.** Let \( M \) be a radical transversal screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold \( \overline{M} \). Then distribution \( D_1 \) is anti-invariant with respect to \( \phi \) and \( \phi \text{RadTM} = \text{ltr}(TM) \). Thus \( \phi X \in \Gamma(\text{ltr}(TM)) \), for all \( X \in \Gamma(\text{RadTM}) \). Hence \( \phi(\phi X) \in \Gamma(\phi(\text{ltr}(TM))) \), which implies \( -X \in \Gamma(\phi(\text{ltr}(TM))) \), for all \( X \in \Gamma(\text{RadTM}) \), which proves (i) and (ii).

Now for any \( X \in \Gamma(D_2) \), we have \( |PX| = |\phi X| \cos \theta \), which implies

\[ \cos \theta = \frac{|PX|}{|\phi X|}. \]

In view of (9.3.15), we get \( \cos^2 \theta = \frac{|PX|^2}{|\phi X|^2} = \frac{g(PX, PX)}{g(\phi X, \phi X)} = \frac{g(X, P^2 X)}{g(X, \phi^2 X)} \), which gives

\[ g(X, P^2 X) = \cos^2 \theta g(X, \phi^2 X). \]
Since $M$ is a radical transversal screen pseudo-slant lightlike submanifold, $\cos^2 \theta = \lambda(\text{constant}) \in (0, 1]$ and therefore from (9.3.16), we get
\[ g(X, P^2X) = \lambda g(X, \phi^2X) = g(X, \lambda \phi^2X), \]
which implies
\[ (9.3.17) \quad g(X, (P^2 - \lambda \phi^2)X) = 0. \]
Since $(P^2 - \lambda \phi^2)X \in \Gamma(D_2)$ and the induced metric $g = g|_{D_2 \times D_2}$ is non-degenerate (positive definite), from (9.3.17), we have $(P^2 - \lambda \phi^2)X = 0,$ which implies
\[ (9.3.18) \quad P^2X = \lambda \phi^2X = -\lambda X. \]

Now, for any vector field $X \in \Gamma(D_2),$ we have
\[ (9.3.19) \quad \phi X = PX + FX, \]
where $PX$ and $FX$ are tangential and transversal parts of $\phi X$ respectively. Applying $\phi$ to (9.3.19) and taking tangential component, we get
\[ (9.3.20) \quad -X = P^2X + BFX. \]
From (9.3.18) and (9.3.20), we get
\[ (9.3.21) \quad BFX = -\mu X, \]
where $1 - \lambda = \mu(\text{constant}) \in [0, 1).$ This proves (iii).

Conversely suppose that conditions (i), (ii) and (iii) are satisfied. In view of (i), we have $\phi N \in \Gamma(\text{RadTM}),$ for all $N \in \Gamma(ltr(TM)).$ Hence $\phi(\phi N) \in \Gamma(\phi(\text{RadTM})), which implies $N \in \Gamma(\phi(\text{RadTM})).$ Thus $\phi \text{RadTM} = ltr(TM).$ From (9.3.20), for any $X \in \Gamma(D_2),$ we get
\[ (9.3.22) \quad -X = P^2X - \mu X, \]
which implies
\[ (9.3.23) \quad P^2X = -\lambda X, \]
where $1 - \mu = \lambda(\text{constant}) \in (0, 1].$

Now
\[ \cos \theta = \frac{g(\phi X, PX)}{||\phi X|| \cdot ||PX||} = -\frac{g(X, \phi PX)}{||\phi X|| \cdot ||PX||} = -\lambda \frac{g(X, P^2X)}{||\phi X|| \cdot ||PX||} = \lambda \frac{g(\phi X, \phi X)}{||\phi X|| \cdot ||PX||}. \]

From above equation, we get
\[ (9.3.24) \quad \cos \theta = \lambda \frac{||\phi X||}{||PX||}. \]

Therefore (9.3.15) and (9.3.24) give $\cos^2 \theta = \lambda(\text{constant}).$

Hence $M$ is a radical transversal screen pseudo-slant lightlike submanifold.
**Corollary 9.3.1:** Let $M$ be a radical transversal screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\tilde{M}$ with slant angle $\theta$, then for any $X, Y \in \Gamma(D_2)$, we have

(i) $g(\nabla_X Y, V) = \cos^2 \theta (g(X, Y) - \eta(X)\eta(Y))$,

(ii) $g([X, Y], V) = 2\eta(X)\phi Y$.

The proof of above Corollary follows by using similar steps as in proof of Corollary 3.2 of [56].

**Lemma 9.3.2:** Let $M$ be a radical transversal screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\tilde{M}$. Then we have

(i) $g(\nabla_X Y, V) = \overline{g}(Y, \phi X)$,

(ii) $g([X, Y], V) = 2\overline{g}(X, \phi Y)$, for all $X, Y \in \Gamma(TM - \{V\})$.

**Proof.** Let $M$ be a radical transversal screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\tilde{M}$. Since $\nabla$ is a metric connection, from (1.5.7), for any $X, Y \in \Gamma(TM - \{V\})$, we have

(9.3.25) $g(\nabla_X Y, V) = \overline{g}(Y, \phi X)$.

From (1.5.5) and (9.3.25), we have $g([X, Y], V) = 2\overline{g}(X, \phi Y)$.

**Theorem 9.3.3:** Let $M$ be a radical transversal screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\tilde{M}$ with structure vector field tangent to $M$. Then $\text{Rad}TM \oplus \{V\}$ is integrable if and only if

(i) $Q_2 D^s(Y, \phi P_1 X) = Q_2 D^s(X, \phi P_1 Y)$,

(ii) $Q_3 D^s(Y, \phi P_1 X) = Q_3 D^s(X, \phi P_1 Y)$,

(iii) $P_3 A_{\phi P_1 Y} X = P_3 A_{\phi P_1 Y} X$, for all $X, Y \in \Gamma(\text{Rad}TM \oplus \{V\})$.

**Proof.** Let $M$ be a radical transversal screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\tilde{M}$. Let $X, Y \in \Gamma(\text{Rad}TM \oplus \{V\})$. From (9.3.12), we have $Q_2 D^s(X, \phi P_1 Y) = \phi P_2 \nabla_X Y$, which gives $Q_2 D^s(X, \phi P_1 Y) - Q_2 D^s(Y, \phi P_1 X) = \phi P_2 [X, Y]$. In view of (9.3.13), we get $Q_3 D^s(X, \phi P_1 Y) = CQ_3 D^s(X, Y) + f P_3 \nabla_X Y$, which implies $Q_3 D^s(X, \phi P_1 Y) - Q_3 D^s(Y, \phi P_1 X) = f P_3 [X, Y]$. Also from (9.3.10), we have $P_3 A_{\phi P_1 Y} X + BQ_3 D^s(X, Y) = -f P_3 \nabla_X Y$. Thus we have $P_3 A_{\phi P_1 Y} X = f P_3 [X, Y]$. This proves the theorem.
**Theorem 9.3.4:** Let $M$ be a radical transversal screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ with structure vector field tangent to $M$. Then $D_1$ is integrable if and only if

1. $Q_3(\nabla^s_X\phi P_2X) = Q_3(\nabla^s_X\phi P_2Y)$ and $P_3A_{\phi P_2X}Y = P_3A_{\phi P_2Y}X$,
2. $D_l(X, \phi P_2Y) = D_l(Y, \phi P_2X)$, for all $X, Y \in \Gamma(D_1)$.

**Proof.** Let $M$ be a radical transversal screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Let $X, Y \in \Gamma(D_1)$. From (9.3.11), we have $D_l(X, \phi P_2Y) = \phi P_1\nabla_XY$, which gives $D_l(X, \phi P_2Y) - D_l(Y, \phi P_2X) = \phi P_1[X,Y]$. In view of (9.3.10), we get $P_3A_{\phi P_2X}Y + BQ_3h^s(X, Y) = -P_3\nabla_XY$, which implies $P_3A_{\phi P_2X}Y - P_3A_{\phi P_2Y}X = fP_3[X,Y]$. Also from (9.3.13), we have $Q_3(\nabla^s_X\phi P_2Y) - CQ_3h^s(X, Y) = fP_3\nabla_XY$, which gives $Q_3(\nabla^s_X\phi P_2Y) - Q_3(\nabla^s_Y\phi P_2X) = fP_3[X,Y]$. This concludes the theorem.

**Theorem 9.3.5:** Let $M$ be a radical transversal screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ with structure vector field tangent to $M$. Then $D_2 \oplus \{V\}$ is integrable if and only if

1. $D_l(X, FP_3X) - h^l(X, fP_3Y) - h^l(Y, fP_3X) = \phi P_1[Y, fP_3X]$,
2. $Q_2(\nabla^s_XFP_3Y - h^s(Y, fP_3X)) = Q_2(\nabla^s_XFP_3X - h^s(X, fP_3Y))$, for all $X, Y \in \Gamma(D_2 \oplus \{V\})$.

**Proof.** Let $M$ be a radical transversal screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Let $X, Y \in \Gamma(D_2 \oplus \{V\})$. From (9.3.11), we have $h^l(X, fP_3Y) + D_l(X, FP_3Y) = \phi P_1\nabla_XY$, which gives $h^l(X, fP_3Y) - h^l(Y, fP_3X) = \phi P_1[X,Y]$. In view of (9.3.12), we get $Q_2\nabla^s_XFP_3Y + Q_2h^s(X, fP_3Y) = \phi P_2\nabla_XY$, which implies $Q_2\nabla^s_XFP_3Y - Q_2\nabla^s_YFP_3X + Q_2h^s(X, fP_3Y) = Q_2h^s(Y, fP_3X) = \phi P_2[X,Y]$. Thus, we obtain the required results.

**Theorem 9.3.6:** Let $M$ be a radical transversal screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ with structure vector field tangent to $M$. Then the induced connection $\nabla$ on $M$ is not a metric connection.

**Proof.** Let $M$ be a radical transversal screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Suppose that the induced connection $\nabla$ on $M$ is a metric connection. Then $\nabla_X\phi N \in \Gamma(RadTM)$ for all $X \in \Gamma(TM)$ and $N \in \Gamma(ltr(TM))$. From (1.3.7), (1.3.8) and (1.5.6), for any $X \in \Gamma(TM)$ and $N \in \Gamma(ltr(TM))$, we have

\begin{align}
\nabla_X\phi N + h^l(X, \phi N) + h^s(X, \phi N) = -\phi A_NX + \phi \nabla^l_XN + \phi Q_2D^s(X, N) + \phi Q_3D^s(X, N) + \bar{g}(X, N)V.
\end{align}
Now, on comparing tangential components of both sides of (9.3.26), we get
\[
\nabla_X \phi N = -f P_3 A_N X + \phi \nabla^l_X N + \phi Q_2 D^s(X, N) + B Q_3 D^s(X, N) + \bar{g}(X, N) V,
\]
(9.3.27)

Since \( TM = \text{Rad}TM \oplus_{\text{orth}} D_1 \oplus_{\text{orth}} D_2 \oplus_{\text{orth}} \{ V \} \), from (9.3.27), we obtain
\[
\nabla_X \phi N - \phi \nabla^l_X N = 0, \quad B Q_3 D^s(X, N) - f P_3 A_N X = 0,
\]
(9.3.28)
\[
\phi Q_2 D^s(X, N) = 0, \quad \bar{g}(X, N) V = 0.
\]
(9.3.29)

Now taking \( X = \xi \in \Gamma(\text{Rad}(TM)) \) in (9.3.29), we get \( \bar{g}(\xi, N) V = 0 \).
Thus \( V = 0 \), which is a contradiction. Hence \( M \) does not have a metric connection.

### 9.4 Foliations Determined by Distributions

In this section, we obtain necessary and sufficient conditions for foliations determined by distributions on a radical transversal screen pseudo-slant lightlike submanifold of indefinite Kaehler and Sasakian manifolds to be totally geodesic.

**Definition 9.4.1.** A radical transversal screen pseudo-slant lightlike submanifold \( M \) of an indefinite Kaehler (Sasakian) manifold \( \overline{M} \) is said to be mixed geodesic screen pseudo-slant lightlike submanifold if its second fundamental form \( h \) satisfies \( h(X, Y) = 0 \), for all \( X \in \Gamma(D_1) \) and \( Y \in \Gamma(D_2) \). Thus \( M \) is a mixed geodesic radical transversal screen pseudo-slant lightlike submanifold if \( h^l(X, Y) = 0 \) and \( h^s(X, Y) = 0 \), for all \( X \in \Gamma(D_1) \) and \( Y \in \Gamma(D_2) \).

**Theorem 9.4.2:** Let \( M \) be a radical transversal screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold \( \overline{M} \). Then \( \text{Rad}TM \) defines a totally geodesic foliation if and only if \( P_1 \nabla_X f P_3 Z = P_1 A_{\overline{J}P_2 Z} X + P_1 A_{FP_3 Z} X \), for all \( X \in \Gamma(\text{Rad}TM) \) and \( Z \in \Gamma(S(TM)) \).

**Proof.** Let \( M \) be a radical transversal screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold \( \overline{M} \). The distribution \( \text{Rad}TM \) defines a totally geodesic foliation if and only if \( \nabla_X Y \in \Gamma(\text{Rad}TM) \), for all \( X, Y \in \Gamma(\text{Rad}TM) \). Since \( \nabla \) is a metric connection, using (1.3.7), (1.4.2), (1.4.3) and (9.2.4), for any \( X, Y, Z \in \Gamma(\text{Rad}TM) \), \( Z \in \Gamma(S(TM)) \), we get \( \bar{g}(\nabla_X Y, Z) = -\bar{g}(\nabla_X (\overline{J}P_2 Z + f P_3 Z + FP_3 Z), \overline{J}Y) \). From (1.3.7), (1.3.9) and above equation, we get \( \bar{g}(\nabla_X Y, Z) = -\bar{g}(\nabla_X f P_3 Z - A_{\overline{J}P_2 Z} X - A_{FP_3 Z} X, \overline{J}Y) \), which implies \( \bar{g}(\nabla_X Y, Z) = -\bar{g}(P_1 \nabla_X f P_3 Z - P_1 A_{\overline{J}P_2 Z} X - P_1 A_{FP_3 Z} X, \overline{J}Y) \). Thus, we obtain the required results.
Theorem 9.4.3: Let $M$ be a radical transversal screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then $D_1$ defines a totally geodesic foliation if and only if

(i) $\overline{g}(h^s(X,fZ),\overline{J}Y) = -\overline{g}(\nabla^s_X FZ,\overline{J}Y)$,

(ii) $h^s(X,\overline{J}N)$ has no component in $\overline{J}(D_1)$,

for all $X,Y \in \Gamma(D_1)$, $Z \in \Gamma(D_2)$ and $N \in \Gamma(ltr(TM))$.

Proof. Let $M$ be a radical transversal screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. The distribution $D_1$ defines a totally geodesic foliation if and only if $\nabla_X Y \in \Gamma(D_1)$, for all $X,Y \in \Gamma(D_1)$. Since $\overline{\nabla}$ is a metric connection, from (1.3.7), (1.4.2) and (1.4.3), for any $X,Y \in \Gamma(D_1)$ and $Z \in \Gamma(D_2)$, we obtain $\overline{g}(\nabla_X Y,Z) = -\overline{g}(\nabla_X JZ,\overline{J}Y)$, which implies $\overline{g}(\nabla_X Y,Z) = -\overline{g}(h^s(X,fZ)+\nabla^s_X FZ,\overline{J}Y)$. Now from (1.3.7), (1.4.2) and (1.4.3), for any $X,Y \in \Gamma(D_1)$ and $N \in \Gamma(ltr(TM))$, we have $\overline{g}(\nabla_X Y,N) = -\overline{g}(\overline{J}Y,\nabla_X \overline{J}N)$, which gives $\overline{g}(\nabla_X Y, N) = -\overline{g}(\overline{J}Y, h^s(X,\overline{J}N))$. Thus, we obtain the required results.

Theorem 9.4.4: Let $M$ be a radical transversal screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then $D_2$ defines a totally geodesic foliation if and only if

(i) $\overline{g}(fY,A^*_Y X) = \overline{g}(FY,\nabla^s_X JZ)$,

(ii) $\overline{g}(fY,A^*_Y X) = \overline{g}(FY,h^s(X,\overline{J}N))$,

for all $X,Y \in \Gamma(D_2)$, $Z \in \Gamma(D_1)$ and $N \in \Gamma(ltr(TM))$.

Proof. Let $M$ be a radical transversal screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. The distribution $D_2$ defines a totally geodesic foliation if and only if $\nabla_X Y \in \Gamma(D_2)$, for all $X,Y \in \Gamma(D_2)$. Since $\overline{\nabla}$ is a metric connection, from (1.3.7), (1.4.2) and (1.4.3), for any $X,Y \in \Gamma(D_2)$ and $Z \in \Gamma(D_1)$, we obtain $\overline{g}(\nabla_X Y,Z) = -\overline{g}(\overline{J}Y,\nabla_X JZ)$, which implies $\overline{g}(\nabla_X Y,Z) = -\overline{g}(fY,A^*_Y X)-\overline{g}(FY,\nabla^s_X JZ)$. Now, from (1.3.7), (1.4.2) and (1.4.3), for all $X,Y \in \Gamma(D_2)$ and $N \in \Gamma(ltr(TM))$, we have $\overline{g}(\nabla_X Y,N) = -\overline{g}(\overline{J}Y,\nabla_X \overline{J}N)$, which gives $\overline{g}(\nabla_X Y, N) = \overline{g}(fY,A^*_Y X) - \overline{g}(FY,h^s(X,\overline{J}N))$. This proves the theorem.

Theorem 9.4.5: Let $M$ be a mixed geodesic radical transversal screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then $D_1$ defines a totally geodesic foliation if and only if $\nabla^s_X FZ$ and $h^s(X,\overline{J}N)$ have no components in $\overline{J}(D_1)$, for all $X \in \Gamma(D_1)$, $Z \in \Gamma(D_2)$ and $N \in \Gamma(ltr(TM))$.

Proof. Since $M$ is a mixed geodesic radical transversal screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$, we have
\[ h^s(X, Z) = 0, \text{ for all } X \in \Gamma(D_1) \text{ and } Z \in \Gamma(D_2). \] Now the proof follows from theorem 9.4.3.

**Theorem 9.4.6:** Let \( M \) be a mixed geodesic radical transversal screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold \( \overline{M} \). Then the induced connection \( \nabla \) on \( S(TM) \) is a metric connection if and only if

(i) \( \overline{g}(fW, A_{\overline{J}\xi}Z) = \overline{g}(FW, D^s(Z, \overline{J}\xi)), \)

(ii) \( D^s(X, \overline{J}\xi) \) has no component in \( \overline{J}(D_1) \),

for all \( X \in \Gamma(D_1), Z,W \in \Gamma(D_2) \) and \( \xi \in \Gamma(RadTM) \).

**Proof.** Let \( M \) be a mixed geodesic radical transversal screen pseudo-slant lightlike submanifold of an indefinite Kaehler manifold \( \overline{M} \). Then \( h^l(X, Z) = 0, \text{ for all } X \in \Gamma(D_1) \text{ and } Z \in \Gamma(D_2). \) Since \( \nabla \) is a metric connection, using (1.3.7), (1.4.2) and (1.4.3), for any \( X,Y \in \Gamma(D_1) \) and \( \xi \in \Gamma(RadTM) \), we have \( \overline{g}(h^l(X,Y), \xi) = -g(\overline{J}Y, \nabla_X \overline{J}\xi), \) which gives \( \overline{g}(h^l(X,Y), \xi) = -g(\overline{J}Y, D^s(X, \overline{J}\xi)). \) In view of (1.3.7), (1.4.2) and (1.4.3), for any \( Z,W \in \Gamma(D_2) \) and \( \xi \in \Gamma(RadTM) \), we obtain \( \overline{g}(h^l(Z,W), \xi) = -\overline{g}(\overline{J}W, \nabla_Z \overline{J}\xi). \) Thus \( \overline{g}(h^l(Z,W), \xi) = \overline{g}(fW, A_{\overline{J}\xi}Z) - \overline{g}(FW, D^s(Z, \overline{J}\xi)). \) This completes the theorem.

**Theorem 9.4.7:** Let \( M \) be a radical transversal screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold \( \overline{M} \) with structure vector field tangent to \( M \). Then \( RadTM \oplus \{V\} \) defines a totally geodesic foliation if and only if \( P_1 \nabla_X fP_3Z = P_1A_{\phi P_2Z}X + P_1A_{FP_3Z}X, \) for any \( X,Y \in \Gamma(RadTM \oplus \{V\}) \) and \( Z \in \Gamma(D_1 \oplus D_2). \)

**Proof.** Let \( M \) be a radical transversal screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold \( \overline{M} \). It is easy to see that \( RadTM \oplus \{V\} \) defines a totally geodesic foliation if and only if \( \nabla_X Y \in \Gamma(RadTM \oplus \{V\}), \) for all \( X,Y \in \Gamma(RadTM \oplus \{V\}). \) Since \( \nabla \) is a metric connection, using (1.3.7), (1.5.2), (1.5.6) and (9.3.4), for any \( X,Y \in \Gamma(RadTM \oplus \{V\}) \) and \( Z \in \Gamma(D_1 \oplus D_2), \) we get \( \overline{g}(\nabla_X Y, Z) = -\overline{g}(\nabla_X (\phi P_2Z + fP_3Z + FP_3Z), \phi Y). \) From (1.3.7), (1.3.9) and above equation, we get \( \overline{g}(\nabla_X Y, Z) = -\overline{g}(\nabla_X fP_3Z - A_{\phi P_2Z}X - A_{FP_3Z}X, \phi Y), \) which implies \( \overline{g}(\nabla_X Y, Z) = -\overline{g}(P_1 \nabla_X fP_3Z - P_1A_{\phi P_2Z}X - P_1A_{FP_3Z}X, \phi Y). \) This concludes the theorem.

**Theorem 9.4.8:** Let \( M \) be a radical transversal screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold \( \overline{M} \) with structure vector field tangent to \( M \). Then \( D_1 \) defines a totally geodesic foliation if and only if
\(\overline{\nabla}\) defines a totally geodesic foliation if and only if
\[
\overline{\nabla}(\nabla_X Y, Z) = -\overline{\nabla}(\nabla^*_X FZ, \phi Y),
\]
for all \(X, Y \in \Gamma(D_1), Z \in \Gamma(D_2)\) and \(N \in \Gamma(ltr(TM))\).

**Proof.** Let \(M\) be a radical transversal screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold \(\overline{M}\). The distribution \(D_1\) defines a totally geodesic foliation if and only if \(\nabla_X Y \in \Gamma(D_1)\), for all \(X, Y \in \Gamma(D_1)\). Since \(\overline{\nabla}\) is a metric connection, from (1.3.7), (1.5.2) and (1.5.6), for any \(X, Y \in \Gamma(D_1)\) and \(Z \in \Gamma(D_2)\), we obtain
\[
\overline{g}(\nabla_X Y, Z) = -\overline{g}(\nabla^*_X FZ, \phi Y),
\]
which gives
\[
\overline{g}(\nabla_X Y, Z) = -\overline{g}(h^*(X, fZ) + \nabla^*_X FZ, \phi Y).
\]
Now from (1.3.7), (1.5.2) and (1.5.6), for any \(X, Y \in \Gamma(D_1)\) and \(N \in \Gamma(ltr(TM))\), we have
\[
\overline{g}(\nabla_X Y, N) = -\overline{g}(\phi Y, \overline{\nabla}_X \phi N),
\]
which implies \(\overline{g}(\nabla_X Y, N) = -\overline{g}(\phi Y, h^*(X, \phi N))\). Thus, we obtain the required results.

**Theorem 9.4.9:** Let \(M\) be a radical transversal screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold \(\overline{M}\) with structure vector field tangent to \(M\). Then \(D_2 \oplus \{V\}\) defines a totally geodesic foliation if and only if
\[
(i) \ \overline{g}(fY, A_{\phi N}^* X) = \overline{g}(FY, \nabla_X^* \phi Z),
\]
\[
(ii) \ \overline{g}(fY, A_{\phi N}^* X) = \overline{g}(FY, h^*(X, \phi N)),
\]
for all \(X, Y \in \Gamma(D_2 \oplus \{V\}), Z \in \Gamma(D_1)\) and \(N \in \Gamma(ltr(TM))\).

**Proof.** Let \(M\) be a radical transversal screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold \(\overline{M}\). The distribution \(D_2 \oplus \{V\}\) defines a totally geodesic foliation if and only if \(\nabla_X Y \in \Gamma(D_2 \oplus \{V\})\), for all \(X, Y \in \Gamma(D_2 \oplus \{V\})\). Since \(\overline{\nabla}\) is a metric connection, from (1.3.7), (1.5.2) and (1.5.6), for any \(X, Y \in \Gamma(D_2 \oplus \{V\})\) and \(Z \in \Gamma(D_1)\), we obtain
\[
\overline{g}(\nabla_X Y, Z) = -\overline{g}(\phi Y, \overline{\nabla}_X \phi Z),
\]
which gives
\[
\overline{g}(\nabla_X Y, Z) = \overline{g}(fY, A_{\phi N}^* X) - \overline{g}(FY, \nabla_X^* \phi Z).
\]
Now, from (1.3.7), (1.5.2) and (1.5.6), for any \(X, Y \in \Gamma(D_2 \oplus \{V\})\) and \(N \in \Gamma(ltr(TM))\), we have
\[
\overline{g}(\nabla_X Y, N) = -\overline{g}(\phi Y, \overline{\nabla}_X \phi N),
\]
which implies \(\overline{g}(\nabla_X Y, N) = \overline{g}(fY, A_{\phi N}^* X) - \overline{g}(FY, h^*(X, \phi N))\). Thus, the theorem is completed.

**Theorem 9.4.10:** Let \(M\) be a mixed geodesic radical transversal screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold \(\overline{M}\) with structure vector field tangent to \(M\). Then \(D_1\) defines a totally geodesic foliation if and only if \(\nabla^*_X FZ\) and \(h^*(X, \phi N)\) have no components in \(\phi(D_1)\), for all \(X \in \Gamma(D_1), Z \in \Gamma(D_2)\) and \(N \in \Gamma(ltr(TM))\).

**Proof.** Since \(M\) is a mixed geodesic radical transversal screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold \(\overline{M}\), we have \(h^*(X, Z) = 0\), for all \(X \in \Gamma(D_1)\) and \(Z \in \Gamma(D_2)\). Now the proof follows from theorem 9.4.8.
**Theorem 9.4.11:** Let $M$ be a mixed geodesic radical transversal screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ with structure vector field tangent to $M$. Then the induced connection $\nabla$ on $D_1 \oplus D_2$ is a metric connection if and only if

(i) $D^s(X, \phi \xi)$ has no component in $\phi(D_1)$,

(ii) $\bar{g}(fW, A_{\phi \xi} Z) = \bar{g}(FW, D^s(Z, \phi \xi))$,

for all $X, \in \Gamma(D_1)$, $Z, W \in \Gamma(D_2)$ and $\xi \in \Gamma(\text{RadTM})$.

**Proof.** Let $M$ be a mixed geodesic radical transversal screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then $h^l(X, Z) = 0$, for all $X \in \Gamma(D_1)$ and $Z \in \Gamma(D_2)$. Since $\overline{\nabla}$ is a metric connection, from (1.3.7), (1.5.2) and (1.5.6), for any $X, Y \in \Gamma(D_1)$ and $\xi \in \Gamma(\text{RadTM})$, we have $\bar{g}(h^l(X, Y), \xi) = -g(\phi Y, \nabla_X \phi \xi)$, which gives $\bar{g}(h^l(X, Y), \xi) = -g(\phi Y, D^s(X, \phi \xi))$. From (1.3.7), (1.5.2) and (1.5.6), for any $Z, W \in \Gamma(D_2)$ and $\xi \in \Gamma(\text{RadTM})$, we get $\bar{g}(h^l(Z, W), \xi) = -\bar{g}(\phi W, \nabla_Z \phi \xi)$. Thus $\bar{g}(h^l(Z, W), \xi) = \bar{g}(fW, A_{\phi \xi} Z) - \bar{g}(FW, D^s(Z, \phi \xi))$. This concludes the theorem.