Chapter 7

Pseudo-Slant Lightlike Submanifolds

7.1 Introduction

Various classes of lightlike submanifolds of indefinite Kaehler and indefinite Sasakian manifolds are defined according to the behaviour of distributions on these submanifolds with respect to the action of (1,1) type structure tensor field of the ambient manifolds. Such submanifolds have been studied in ([34], [37], [38], [54]). A. Carriazo defined and studied bi-slant submanifolds of almost Hermitian and almost contact metric manifolds and further gave the notion of pseudo-slant submanifolds ([16]). The theory of slant, contact Cauchy-Riemann lightlike submanifolds has been studied in ([35], [59]). The objective of this chapter is to introduce the notion of pseudo-slant lightlike submanifolds of indefinite Kaehler and Sasakian manifolds. In section 7.2, we study pseudo-slant lightlike submanifolds of an indefinite Kaehler manifold giving some examples. In section 7.3, we study pseudo-slant lightlike submanifolds of an indefinite Sasakian manifold giving some examples. Section 7.4 is devoted to the study of foliations determined by distributions on pseudo-slant lightlike submanifolds of the above manifolds.

7.2 Pseudo-Slant Lightlike Submanifolds of Indefinite Kaehler Manifolds

In this section, we introduce the notion of pseudo-slant lightlike submanifolds of indefinite Kaehler manifolds. At first, we state the following Lemmas, which help us to define pseudo-slant lightlike submanifolds of an indefinite Kaehler manifold.
Lemma 7.2.1: Let \( M \) be a \( r \)-lightlike submanifold of an indefinite Kaehler manifold \( \overline{M} \) of index \( 2q \). Suppose that \( \overline{\text{Rad}}TM \) is a distribution on \( M \) such that \( \text{Rad}TM \cap \overline{\text{Rad}}TM = \{0\} \). Then \( \overline{\text{ltr}}(TM) \) is a subbundle of the screen distribution \( S(TM) \) and \( \overline{\text{Rad}}TM \cap \overline{\text{ltr}}(TM) = \{0\} \).

Lemma 7.2.2: Let \( M \) be a \( q \)-lightlike submanifold of an indefinite Kaehler manifold \( \overline{M} \) of index \( 2q \). Suppose \( \overline{\text{Rad}}TM \) is a distribution on \( M \) such that \( \text{Rad}TM \cap \overline{\text{Rad}}TM = \{0\} \). Then any complementary distribution to \( \overline{\text{Rad}}TM \oplus \overline{\text{ltr}}(TM) \) in \( S(TM) \) is Riemannian.

The proofs of Lemma 7.2.1 and Lemma 7.2.2 follow as in Lemma 3.1 and Lemma 3.2 respectively of [54], so we omit them.

Definition 7.2.1. Let \( M \) be a \( q \)-lightlike submanifold of an indefinite Kaehler manifold \( \overline{M} \) of index \( 2q \) such that \( q < \dim(M) \). Then we say that \( M \) is a pseudo-slant lightlike submanifold of \( \overline{M} \) if the following conditions are satisfied:

(i) \( \overline{\text{Rad}}TM \) is a distribution on \( M \) such that \( \text{Rad}TM \cap \overline{\text{Rad}}TM = \{0\} \),

(ii) there exists non-degenerate orthogonal distributions \( D_1 \) and \( D_2 \) on \( M \) such that \( S(TM) = (\overline{\text{Rad}}TM \oplus \overline{\text{ltr}}(TM)) \oplus_{\text{orth}} D_1 \oplus_{\text{orth}} D_2 \),

(iii) the distribution \( D_1 \) is anti-invariant, i.e. \( \overline{J}D_1 \subset S(TM^\perp) \),

(iv) the distribution \( D_2 \) is slant with angle \( \theta(\neq \pi/2) \), i.e. for each \( x \in M \) and each non-zero vector \( X \in (D_2)_x \), the angle \( \theta \) between \( \overline{J}X \) and the vector subspace \( (D_2)_x \) is a constant \( (\neq \pi/2) \), which is independent of the choice of \( x \in M \) and \( X \in (D_2)_x \).

This constant angle \( \theta \) is called the slant angle of distribution \( D_2 \). A screen pseudo-slant lightlike submanifold is said to be proper if \( D_1 \neq \{0\} \), \( D_2 \neq \{0\} \) and \( \theta \neq 0 \).

From the above definition, we have the following decomposition

\[
(7.2.1) \quad TM = \text{Rad}TM \oplus_{\text{orth}} (\overline{\text{Rad}}TM \oplus \overline{\text{ltr}}(TM)) \oplus_{\text{orth}} D_1 \oplus_{\text{orth}} D_2.
\]

In particular, we have

(i) if \( D_1 = 0 \), then \( M \) is a slant lightlike submanifold,

(ii) if \( D_1 \neq 0 \) and \( \theta = 0 \), then \( M \) is a CR-lightlike submanifold.

Thus the above new class of lightlike submanifolds of an indefinite Kaehler manifold includes slant, Cauchy-Riemann lightlike submanifolds as its sub-cases which have been studied in ([32],[34]).

Let \((\mathbb{R}^{2m}_{2q}, \overline{g}, \overline{J})\) denote the manifold \( \mathbb{R}^{2m}_{2q} \) with its usual Kaehler structure given by
\[ \bar{g} = \frac{1}{4}(-\sum_{i=1}^{q} dx^i \otimes dx^i + dy^i \otimes dy^i + \sum_{i=q+1}^{m} dx^i \otimes dx^i + dy^i \otimes dy^i), \]
\[ \mathcal{J}(\sum_{i=1}^{m}(X_i \partial x_i + Y_i \partial y_i)) = \sum_{i=1}^{m}(Y_i \partial x_i - X_i \partial y_i), \]

where \((x^i, y^i)\) are the cartesian coordinates on \(\mathbb{R}_{2q}^m\). Now, we construct some examples of pseudo-slant lightlike submanifolds of an indefinite Kaehler manifold.

**Example 7.2.1.** Let \((\mathbb{R}_{12}^{1}, \bar{g}, \mathcal{J})\) be an indefinite Kaehler manifold, where \(\bar{g}\) is of signature \((-+, +, +, +, +, +, +, +, +, +, +, +)\) with respect to the canonical basis \(\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6\}\).

Suppose \(M\) is a submanifold of \(\mathbb{R}_{12}^{1}\) given by \(x^1 = y^2 = u_1, x^2 = u_2, y^1 = u_3, x^3 = y^4 = u_4, x^4 = y^3 = u_5, x^5 = u_6 \cos u_7, y^5 = u_6 \sin u_7, x^6 = \cos u_6, y^6 = \sin u_6\).

The local frame of \(TM\) is given by \(\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\}\), where
\begin{align*}
Z_1 &= 2(\partial x_1 + \partial y_2), \\
Z_2 &= 2\partial x_2, \\
Z_3 &= 2\partial y_1, \\
Z_4 &= 2(\partial x_3 + \partial y_4), \\
Z_5 &= 2(\partial x_4 + \partial y_3), \\
Z_6 &= 2(\cos u_7 \partial x_5 + \sin u_7 \partial y_5 - \sin u_6 \partial x_6 + \cos u_6 \partial y_6), \\
Z_7 &= 2(-u_6 \sin u_7 \partial x_5 + u_6 \cos u_7 \partial y_5).
\end{align*}

Hence \(\text{Rad}TM = \text{span} \{Z_1\}\) and \(S(TM) = \text{span} \{Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\}\).

Now \(ltr(TM)\) is spanned by \(N_1 = -\partial x_1 + \partial y_2\) and \(S(TM^\perp)\) is spanned by
\begin{align*}
W_1 &= 2(\partial x_3 - \partial y_4), \\
W_2 &= 2(\partial x_4 - \partial y_3), \\
W_3 &= 2(\cos u_7 \partial x_5 + \sin u_7 \partial y_5 + \sin u_6 \partial x_6 - \cos u_6 \partial y_6), \\
W_4 &= 2(u_6 \cos u_6 \partial x_6 + u_6 \sin u_6 \partial y_6).
\end{align*}

It follows that \(\overline{\mathcal{J}}Z_1 = Z_2 - Z_3\), which implies that \(\overline{\mathcal{J}}\text{Rad}TM\) is a distribution on \(M\). On the other hand, we can see that \(D_1 = \text{span} \{Z_4, Z_5\}\) such that \(\overline{\mathcal{J}}Z_4 = W_2, \overline{\mathcal{J}}Z_5 = W_1\), which implies that \(D_1\) is anti-invariant with respect to \(\mathcal{J}\) and \(D_2 = \text{span} \{Z_6, Z_7\}\) is a slant distribution with slant angle \(\pi/4\). Hence \(M\) is a pseudo-slant 2-lightlike submanifold of \(\mathbb{R}_{12}^{1}\).

**Example 7.2.2.** Let \((\mathbb{R}_{12}^{1}, \bar{g}, \mathcal{J})\) be an indefinite Kaehler manifold, where \(\bar{g}\) is of signature \((-+, +, +, +, +, +, +, +, +, +, +, +)\) with respect to the canonical basis \(\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6\}\).

Suppose \(M\) is a submanifold of \(\mathbb{R}_{12}^{1}\) given by \(-x^1 = y^2 = u_1, x^2 = u_2, y^1 = u_3, x^3 = u_4 \cos \beta, y^3 = u_4 \sin \beta, x^4 = u_5 \sin \beta, y^4 = u_5 \cos \beta, x^5 = u_6 \cos \theta, y^5 = u_7 \cos \theta, x^6 = u_7 \sin \theta, y^6 = u_6 \sin \theta\).

The local frame of \(TM\) is given by \(\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\}\), where
\[ Z_1 = 2(-\partial x_1 + \partial y_2), \quad Z_2 = 2\partial x_2, \quad Z_3 = 2\partial y_1, \]
\[ Z_4 = 2(\cos \beta \partial x_3 + \sin \beta \partial y_3), \quad Z_5 = 2(\sin \beta \partial x_4 + \cos \beta \partial y_4), \]
\[ Z_6 = 2(\cos \theta \partial x_5 + \sin \theta \partial y_6), \quad Z_7 = 2(\sin \theta \partial x_6 + \cos \theta \partial y_5). \]

Hence \( \text{Rad}TM = \text{span} \{Z_1\} \) and \( S(TM) = \text{span} \{Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\} \).

Now \( \text{ltr}(TM) \) is spanned by \( N_1 = \partial x_1 + \partial y_2 \) and \( S(TM^\perp) \) is spanned by
\[ W_1 = 2(\sin \beta \partial x_3 - \cos \beta \partial y_3), \quad W_2 = 2(\cos \beta \partial x_4 - \sin \beta \partial y_4), \]
\[ W_3 = 2(\sin \theta \partial x_5 - \cos \theta \partial y_6), \quad W_4 = 2(\cos \theta \partial x_6 - \sin \theta \partial y_5). \]

It follows that \( \mathcal{J}Z_1 = Z_2 + Z_3 \), which implies that \( \mathcal{J}\text{Rad}TM \) is a distribution on \( M \). On the other hand, we can see that \( D_1 = \text{span} \{Z_4, Z_5\} \) such that \( \mathcal{J}Z_4 = W_1 \), \( \mathcal{J}Z_5 = W_2 \), which implies that \( D_1 \) is anti-invariant with respect to \( \mathcal{J} \) and \( D_2 = \text{span} \{Z_6, Z_7\} \) is a slant distribution with slant angle \( 2\theta \). Hence \( M \) is a pseudo-slant 2-lightlike submanifold of \( \mathbb{R}^4_2 \).

Now, for any vector field \( X \) tangent to \( M \), we put \( \mathcal{J}X = PX + FX \), where \( PX \) and \( FX \) are tangential and transversal parts of \( \mathcal{J}X \) respectively. We denote the projections on \( \text{Rad}TM, \mathcal{J}\text{Rad}TM, \mathcal{J}\text{ltr}(TM), D_1 \) and \( D_2 \) in \( TM \) by \( P_1, P_2, P_3, P_4, \) and \( P_5 \) respectively. Similarly, we denote the projections of \( tr(TM) \) on \( \text{ltr}(TM), \mathcal{J}(D_1) \) and \( D' \) by \( Q_1, Q_2 \) and \( Q_3 \) respectively, where \( D' \) is non-degenerate orthogonal complementary subbundle of \( \mathcal{J}(D_1) \) in \( S(TM^\perp) \). Then, for any \( X \in \Gamma(TM) \), we get
\[
(7.2.2) \quad X = P_1X + P_2X + P_3X + P_4X + P_5X.
\]

Now applying \( \mathcal{J} \) to (7.2.2), we have
\[
(7.2.3) \quad \mathcal{J}X = \mathcal{J}P_1X + \mathcal{J}P_2X + \mathcal{J}P_3X + \mathcal{J}P_4X + \mathcal{J}P_5X,
\]
which gives
\[
(7.2.4) \quad \mathcal{J}X = \mathcal{J}P_1X + \mathcal{J}P_2X + \mathcal{J}P_3X + \mathcal{J}P_4X + fP_5X + FP_5X,
\]
where \( fP_5X \) (resp. \( FP_5X \)) denotes the tangential (resp. transversal) component of \( \mathcal{J}P_5X \). Thus we get \( \mathcal{J}P_1X \in \Gamma(\mathcal{J}\text{Rad}TM), \mathcal{J}P_2X \in \Gamma(\text{Rad}TM), \mathcal{J}P_3X \in \Gamma(\text{ltr}(TM)), \mathcal{J}P_4X \in \Gamma(\mathcal{J}(D_1)) \subset \Gamma(S(TM^\perp)), fP_5X \in \Gamma(D_2) \) and \( FP_5X \in \Gamma(D') \). Also, for any \( W \in \Gamma(tr(TM)) \), we have
\[
(7.2.5) \quad W = Q_1W + Q_2W + Q_3W.
\]
Applying \( \mathcal{J} \) to (7.2.5), we obtain
\[
(7.2.6) \quad \mathcal{J}W = \mathcal{J}Q_1W + \mathcal{J}Q_2W + \mathcal{J}Q_3W,
\]
which gives

\[ 7.2.7 \quad \mathcal{J}W = \mathcal{J}Q_1W + \mathcal{J}Q_2W + BQ_3W + CQ_3W, \]

where \( BQ_3W \) (resp. \( CQ_3W \)) denotes the tangential (resp. transversal) component of \( \mathcal{J}Q_3W \). Thus we get \( \mathcal{J}Q_1W \in \Gamma(\mathcal{J}ltr(TM)) \), \( \mathcal{J}Q_2W \in \Gamma(D_1) \), \( BQ_3W \in \Gamma(D_2) \) and \( CQ_3W \in \Gamma(D^\prime) \).

Now, by using (1.4.3), (7.2.4), (7.2.7) and (1.3.7)-(1.3.9) and identifying the component of \( 7.2.12 \), we obtain

\[ 7.2.8 \quad P_1(\nabla_X \mathcal{J}P_1Y) + P_1(\nabla_X \mathcal{J}P_2Y) - P_1(A_{\mathcal{J}P_3}X) + P_1(\nabla_X fP_3Y) = P_1(A_{FP_3}X) + P_1(A_{\mathcal{J}P_3}X) + \mathcal{J}P_2\nabla_X Y, \]

\[ 7.2.9 \quad P_2(\nabla_X \mathcal{J}P_1Y) + P_2(\nabla_X \mathcal{J}P_2Y) - P_2(A_{\mathcal{J}P_3}X) + P_2(\nabla_X fP_3Y) = P_2(A_{FP_3}X) + P_2(A_{\mathcal{J}P_3}X) + \mathcal{J}P_1\nabla_X Y, \]

\[ 7.2.10 \quad P_3(\nabla_X \mathcal{J}P_1Y) + P_3(\nabla_X \mathcal{J}P_2Y) - P_3(A_{\mathcal{J}P_3}X) + P_3(\nabla_X fP_3Y) = P_3(A_{FP_3}X) + P_3(A_{\mathcal{J}P_3}X) + \mathcal{J}h^l(X,Y), \]

\[ 7.2.11 \quad P_4(\nabla_X \mathcal{J}P_1Y) + P_4(\nabla_X \mathcal{J}P_2Y) - P_4(A_{\mathcal{J}P_3}X) + P_4(\nabla_X fP_3Y) = P_4(A_{FP_3}X) + P_4(A_{\mathcal{J}P_3}X) + \mathcal{J}Q_2h^s(X,Y), \]

\[ 7.2.12 \quad P_5(\nabla_X \mathcal{J}P_1Y) + P_5(\nabla_X \mathcal{J}P_2Y) - P_5(A_{\mathcal{J}P_3}X) + P_5(\nabla_X fP_3Y) = P_5(A_{FP_3}X) + P_5(A_{\mathcal{J}P_3}X) + fP_5\nabla_X Y + BQ_3h^s(X,Y), \]

\[ 7.2.13 \quad h^l(X,\mathcal{J}P_1Y) + h^l(X,\mathcal{J}P_2Y) + D^l(X,\mathcal{J}P_4Y) + h^l(X,fP_3Y) = \mathcal{J}P_3\nabla_X Y - \nabla^l_X \mathcal{J}P_3Y - D^l(X,FP_3Y), \]

\[ 7.2.14 \quad Q_2h^s(X,\mathcal{J}P_1Y) + Q_2h^s(X,\mathcal{J}P_2Y) + Q_2h^s(X,fP_3Y) = Q_2\nabla^s_X FP_3Y - Q_2\nabla^s_X \mathcal{J}P_4Y - Q_2D^s(X,\mathcal{J}P_3Y) + \mathcal{J}P_4\nabla_X Y, \]

\[ 7.2.15 \quad Q_3h^s(X,\mathcal{J}P_1Y) + Q_3h^s(X,\mathcal{J}P_2Y) + Q_3h^s(X,fP_3Y) = CQ_3h^s(X,Y) - Q_3\nabla^s_X FP_3Y - Q_3\nabla^s_X \mathcal{J}P_4Y - Q_3D^s(X,FP_3Y) + FP_3\nabla_X Y. \]

**Theorem 7.2.2:** Let \( M \) be a q-lightlike submanifold of an indefinite Kaehler manifold \( \overline{M} \) of index 2q. Then \( M \) is a pseudo-slant lightlike submanifold of \( \overline{M} \) if and only if

(i) \( \mathcal{J}RadTM \) is a distribution on \( M \) such that \( RadTM \cap \mathcal{J}RadTM = \{0\} \),

(ii) the distribution \( D_1 \) is anti-invariant, i.e. \( \mathcal{J}D_1 \subset S(TM^\perp) \),

(iii) there exists a constant \( \lambda \in (0,1] \) such that \( P^2X = -\lambda X \).
Moreover, there also exists a constant \( \mu \in [0, 1) \) such that \( BFX = -\mu X \), for all \( X \in \Gamma(D_2) \), where \( D_1 \) and \( D_2 \) are non-degenerate orthogonal distributions on \( M \) such that \( S(TM) = (\mathcal{J} \text{Rad} TM \oplus \mathcal{J} \text{ltr}(TM)) \oplus_{\text{orth}} D_1 \oplus_{\text{orth}} D_2 \) and \( \lambda = \cos^2 \theta \), \( \theta \) is slant angle of \( D_2 \).

**Proof.** Let \( M \) be a pseudo-slant lightlike submanifold of an indefinite Kaehler manifold \( \overline{M} \). Then distribution \( D_1 \) is anti-invariant with respect to \( \mathcal{J} \) and \( \mathcal{J} \text{Rad} TM \) is a distribution on \( M \) such that \( \text{Rad} TM \cap \mathcal{J} \text{Rad} TM = \{0\} \).

Now for any \( X \in \Gamma(D_2) \), we have \( |PX| = |JX| \cos \theta \), which implies

\[
(7.2.16) \quad \cos \theta = \frac{|PX|}{|JX|}.
\]

In view of (7.2.16), we get \( \cos^2 \theta = \frac{|PX|^2}{|JX|^2} = \frac{g(PX, PX)}{g(JX, JX)} = \frac{g(X, P^2 X)}{g(X, J^2 X)} \), which gives

\[
(7.2.17) \quad g(X, P^2 X) = \cos^2 \theta g(X, J^2 X).
\]

Since \( M \) is a pseudo-slant lightlike submanifold, \( \cos^2 \theta = \lambda(\text{constant}) \in (0, 1] \), from (7.2.17), we get \( g(X, P^2 X) = \lambda g(X, J^2 X) = g(X, \lambda J^2 X) \), which implies

\[
(7.2.18) \quad g(X, (P^2 - \lambda J^2) X) = 0.
\]

Since \( (P^2 - \lambda J^2) X \in \Gamma(D_2) \) and the induced metric \( g = g|_{D_2 \times D_2} \) is non-degenerate (positive definite), from (7.2.18), we have \( (P^2 - \lambda J^2) X = 0 \), which implies

\[
(7.2.19) \quad P^2 X = \lambda J^2 X = -\lambda X.
\]

Now, for any vector field \( X \in \Gamma(D_2) \), we have

\[
(7.2.20) \quad JX = PX + FX,
\]

where \( PX \) and \( FX \) are tangential and transversal parts of \( JX \) respectively. Applying \( J \) to (7.2.20) and taking tangential component, we get

\[
(7.2.21) \quad -X = P^2 X + BFX.
\]

From (7.2.19) and (7.2.21), we get

\[
(7.2.22) \quad BFX = -\mu X, \quad \forall X \in \Gamma(D_2),
\]

where \( 1 - \lambda = \mu(\text{constant}) \in [0, 1) \). This proves (iii).
Conversely suppose that conditions (i), (ii) and (iii) are satisfied. From (7.2.21), for any \( X \in \Gamma(D_2) \), we get
\[
\tag{7.2.23} -X = P^2 X - \mu X,
\]
which implies
\[
\tag{7.2.24} P^2 X = -\lambda X,
\]
where \( 1 - \mu = (constant) \in (0, 1] \).

Now \( \cos \theta = \frac{g(JX,PX)}{|JX||PX|} = -g(X,JPX) \), which gives
\[
\tag{7.2.25} \cos \theta = -\lambda = \frac{g(JX,JX)}{|JX||PX|}.
\]
From above equation, we get
\[
\cos \theta = \lambda \frac{|JX|}{|PX|}.
\]
Therefore (7.2.16) and (7.2.25) give \( \cos^2 \theta = \lambda (constant) \).

Hence \( M \) is a pseudo-slant lightlike submanifold.

**Corollary 7.2.1:** Let \( M \) be a pseudo-slant lightlike submanifold of an indefinite Kaehler manifold \( \overline{M} \) with slant angle \( \theta \), then for any \( X,Y \in \Gamma(D_2) \), we have

(i) \( g(PX,PY) = \cos^2 \theta g(X,Y) \),

(ii) \( g(FX,FY) = \sin^2 \theta g(X,Y) \).

The proof of above Corollary follows by using similar steps as in proof of Corollary 3.1 of [54].

**Theorem 7.2.3:** Let \( M \) be a pseudo-slant lightlike submanifold of an indefinite Kaehler manifold \( \overline{M} \). Then \( \text{RadTM} \) is integrable if and only if

(i) \( P_1(\nabla_X \overline{JP}_1 Y) = P_1(\nabla_Y \overline{JP}_1 X) \) and \( P_5(\nabla_X \overline{JP}_1 Y) = P_5(\nabla_Y \overline{JP}_1 X) \),

(ii) \( Q_2 h^s(Y,\overline{JP}_1 X) = Q_2 h^s(X,\overline{JP}_1 Y) \) and \( h^l(Y,\overline{JP}_1 X) = h^l(X,\overline{JP}_1 Y) \),

(iii) \( Q_3 h^s(Y,\overline{JP}_1 X) = Q_3 h^s(X,\overline{JP}_1 Y) \), for all \( X,Y \in \Gamma(\text{RadTM}) \).

**Proof.** Let \( M \) be a pseudo-slant lightlike submanifold of an indefinite Kaehler manifold \( \overline{M} \). Let \( X,Y \in \Gamma(\text{RadTM}) \). From (7.2.8), we have
\[
P_1(\nabla_X \overline{JP}_1 Y) = \overline{JP}_2 \nabla_X Y, \quad \text{which gives } P_1(\nabla_X \overline{JP}_1 Y) - P_1(\nabla_Y \overline{JP}_1 X) = \overline{JP}_2[X,Y].
\]
From (7.2.12), we get \( P_5(\nabla_X \overline{JP}_1 Y) = fP_5 \nabla_X Y + Bh^s(X,Y) \), which gives \( P_5(\nabla_X \overline{JP}_1 Y) - P_5(\nabla_Y \overline{JP}_1 X) = fP_5[X,Y] \). From (7.2.13), we get \( h^l(X,\overline{JP}_1 Y) = \overline{JP}_3 \nabla_X Y \), which implies \( h^l(X,\overline{JP}_1 Y) - h^l(Y,\overline{JP}_1 X) = \overline{JP}_3[X,Y] \). From (7.2.14), we have \( Q_2 h^s(X,\overline{JP}_1 Y) = \overline{JP}_4 \nabla_X Y \), which gives \( Q_2 h^s(X,\overline{JP}_1 Y) - Q_2 h^s(Y,\overline{JP}_1 X) = \overline{JP}_4[X,Y] \). In view of (7.2.15), we obtain \( Q_3 h^s(X,\overline{JP}_1 Y) = Ch^s(X,Y) + fP_5 \nabla_X Y \), which gives \( Q_3 h^s(X,\overline{JP}_1 Y) - Q_3 h^s(Y,\overline{JP}_1 X) = fP_5[X,Y] \). This concludes the theorem.
Theorem 7.2.4: Let $M$ be a pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then $D_1$ is integrable if and only if

(i) $P_1(A_{\overline{J}P_1}X) = P_1(A_{\overline{J}P_1}X)$ and $P_2(A_{\overline{J}P_1}X) = P_2(A_{\overline{J}P_1}X)$,

(ii) $D^l(Y, \overline{J}P_1X) = D^l(X, \overline{J}P_1Y)$ and $Q_3\nabla^*_Y \overline{J}P_1X = Q_3\nabla^*_X \overline{J}P_1Y$,

(iii) $P_3(A_{\overline{J}P_1}X) = P_3(A_{\overline{J}P_1}X)$, for all $X, Y \in \Gamma(D_1)$.

Proof. Let $M$ be a pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Let $X, Y \in \Gamma(D_1)$. In view of (7.2.8), we have $P_1(A_{\overline{J}P_1}X) + \overline{J}P_2\nabla_XY = 0$, which gives $P_1(A_{\overline{J}P_1}X) - P_1(A_{\overline{J}P_1}X) = \overline{J}P_2[X, Y]$. From (7.2.9), we get $P_2(A_{\overline{J}P_1}X) + \overline{J}P_1\nabla_XY = 0$, which gives $P_2(A_{\overline{J}P_1}X) - P_2(A_{\overline{J}P_1}X) = \overline{J}P_2[X, Y]$. From (7.2.12), we obtain $P_3(A_{\overline{J}P_1}X) + fP_3\nabla_XY + BQ_3h^s(X, Y) = 0$, which implies $P_3(A_{\overline{J}P_1}X) - P_3(A_{\overline{J}P_1}X) = fP_3[X, Y]$. In view of (7.2.13), we have $D^l(X, \overline{J}P_1Y) = \overline{J}P_3\nabla_XY$, which gives $D^l(X, \overline{J}P_1Y) - D^l(Y, \overline{J}P_1X) = \overline{J}P_3[X, Y]$. Also from (7.2.15), we obtain $Q_3\nabla^*_Y \overline{J}P_1X = CQ_3h^s(X, Y) + FP_5\nabla_XY$, which implies $Q_3\nabla^*_Y \overline{J}P_1X - Q_3\nabla^*_X \overline{J}P_1Y = FP_5[X, Y]$. Thus, we obtain the required results.

Theorem 7.2.5: Let $M$ be a pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then $D_2$ is integrable if and only if

(i) $P_1(\nabla_XfP_5Y - \nabla_YfP_5X) = P_1(A_{FP_5}X - A_{FP_5}X)$,

(ii) $P_2(\nabla_XfP_5Y - \nabla_YfP_5X) = P_2(A_{FP_5}X - A_{FP_5}X)$,

(iii) $h^l(X, fP_5Y) - h^l(Y, fP_5X) = D^l(Y, FP_5X) - D^l(X, FP_5Y)$,

(iv) $Q_2(\nabla^*_XFP_5Y - \nabla^*_YFP_5X) = Q_2(h^s(X, fP_5Y) - h^s(Y, fP_5X))$, for all $X, Y \in \Gamma(D_2)$.

Proof. Let $M$ be a pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Let $X, Y \in \Gamma(D_2)$. In view of (7.2.8), we have $P_1(\nabla_XfP_5Y - \nabla_YfP_5X) = \overline{J}P_2\nabla_XY$, which gives $P_1(\nabla_XfP_5Y - \nabla_YfP_5X) - P_1(A_{FP_5}X - A_{FP_5}X) = \overline{J}P_2[X, Y]$. From (7.2.9), we get $P_2(\nabla_XfP_5Y - \nabla_YfP_5X) = \overline{J}P_1\nabla_XY$, which gives $P_2(\nabla_XfP_5Y - \nabla_YfP_5X) - P_2(A_{FP_5}X - A_{FP_5}X) = \overline{J}P_1[X, Y]$. Also from (7.2.13), we get $h^l(X, fP_5Y) + D^l(X, FP_5Y) = \overline{J}P_3\nabla_XY$, which implies $h^l(X, fP_5Y) - h^l(Y, fP_5X) + D^l(X, FP_5Y) - D^l(Y, FP_5X) = \overline{J}P_3[X, Y]$. In view of (7.2.14), we have $Q_2h^s(X, fP_5Y) - Q_2\nabla^*_XFP_5Y = \overline{J}P_4\nabla_XY$, which gives $Q_2(\nabla^*_XFP_5Y - \nabla^*_YFP_5Y) + Q_2(h^s(X, fP_5Y) - h^s(Y, fP_5X)) = \overline{J}P_4[X, Y]$. This proves the theorem.
7.3 Pseudo-Slant Lightlike Submanifolds of Indefinite Sasakian Manifolds

In this section, we introduce the notion of pseudo-slant lightlike submanifolds of indefinite Sasakian manifolds. At first, we state the following Lemmas, which help us to define pseudo-slant lightlike submanifolds of an indefinite Sasakian manifold.

**Lemma 7.3.1:** Let $M$ be a $r$-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ of index $2q$ with structure vector field tangent to $M$. Suppose that $\phi\text{Rad}TM$ is a distribution on $M$ such that $\text{Rad}TM \cap \phi\text{Rad}TM = \{0\}$. Then $\phi\text{ltr}(TM)$ is a subbundle of the screen distribution $S(TM)$ and $\phi\text{Rad}TM \cap \phi\text{ltr}(TM) = \{0\}$.

**Lemma 7.3.2:** Let $M$ be a $q$-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ of index $2q$ with structure vector field tangent to $M$. Suppose $\phi\text{Rad}TM$ is a distribution on $M$ such that $\text{Rad}TM \cap \phi\text{Rad}TM = \{0\}$. Then any complementary distribution to $\phi\text{Rad}TM \oplus \phi\text{ltr}(TM)$ in $S(TM)$ is Riemannian.

The proofs of Lemma 7.3.1 and Lemma 7.3.2 follow as in Lemma 3.1 and Lemma 3.2 respectively of [59], so we omit them.

**Definition 7.3.1.** Let $M$ be a $q$-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ of index $2q$ such that $q < \text{dim}(M)$ with structure vector field tangent to $M$. Then we say that $M$ is a pseudo-slant lightlike submanifold of $\overline{M}$ if the following conditions are satisfied:

(i) $\phi\text{Rad}TM$ is a distribution on $M$ such that $\text{Rad}TM \cap \phi\text{Rad}TM = \{0\}$,

(ii) there exists non-degenerate orthogonal distributions $D_1$ and $D_2$ on $M$ such that $S(TM) = (\phi\text{Rad}TM \oplus \phi\text{ltr}(TM)) \oplus_{\text{orth}} D_1 \oplus_{\text{orth}} D_2 \oplus_{\text{orth}} \{V\},$

(iii) the distribution $D_1$ is anti-invariant, i.e. $\phi D_1 \subset S(TM^\perp),$

(iv) the distribution $D_2$ is slant with angle $\theta(\neq \pi/2)$, i.e. for each $x \in M$ and each non-zero vector $X \in (D_2)_x$, the angle $\theta$ between $\phi X$ and the vector subspace $(D_2)_x$ is a constant($\neq \pi/2$), which is independent of the choice of $x \in M$ and $X \in (D_2)_x$.

This constant angle $\theta$ is called the slant angle of distribution $D_2$. A screen pseudo-slant lightlike submanifold is said to be proper if $D_1 \neq \{0\}$, $D_2 \neq \{0\}$ and $\theta \neq 0$. From the above definition, we have the following decomposition

\[
TM = \text{Rad}TM \oplus_{\text{orth}} (\phi\text{Rad}TM \oplus \phi\text{ltr}(TM)) \oplus_{\text{orth}} D_1 \oplus_{\text{orth}} D_2 \oplus_{\text{orth}} \{V\}.
\]
In particular, we have

(i) if $D_1 = 0$, then $M$ is a slant lightlike submanifold,

(ii) if $D_1 \neq 0$ and $\theta = 0$, then $M$ is a contact CR-lightlike submanifold.

Thus the above new class of lightlike submanifolds of an indefinite Sasakian manifold includes slant, contact Cauchy-Riemann lightlike submanifolds as its sub-cases which have been studied in ([35], [59]).

Let $(\mathbb{R}^{2m+1}_{2q}, \bar{g}, \phi, \eta, V)$ denote the manifold $\mathbb{R}^{2m+1}_{2q}$ with its usual Sasakian structure given by

\[ \eta = \frac{1}{2}(dz - \sum_{i=1}^{m} y^i dx^i), \quad V = 2\partial z, \]

\[ \bar{g} = \eta \otimes \eta + \frac{1}{4}(- \sum_{i=1}^{q} dx^i \otimes dx^i + dy^i \otimes dy^i + \sum_{i=q+1}^{m} dx^i \otimes dx^i + dy^i \otimes dy^i), \]

\[ \phi(\sum_{i=1}^{m} (X_i \partial x_i + Y_i \partial y_i) + Z \partial z) = \sum_{i=1}^{m} (Y_i \partial x_i - X_i \partial y_i) + \sum_{i=q+1}^{m} Y_i y^i \partial z, \]

where $(x^i, y^i, z)$ are the cartesian coordinates on $\mathbb{R}^{2m+1}_{2q}$. Now we construct some examples of pseudo-slant lightlike submanifolds of an indefinite Sasakian manifold.

**Example 7.3.1.** Let $(\mathbb{R}^{13}_{2}, \bar{g})$ be an indefinite Sasakian manifold, where $\bar{g}$ is of signature $(-, +, +, +, +, +, - , +, +, +, +, +, +, +)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6, \partial z\}$. Suppose $M$ is a submanifold of $\mathbb{R}^{13}_{2}$ given by $x^1 = y^2 = u_1, x^2 = u_2, y^1 = u_3, x^3 = y^4 = u_4, x^4 = y^3 = u_5, x^5 = u_6 \cos u_7, y^5 = u_6 \sin u_7, x^6 = \cos u_6, y^6 = \sin u_6, z = u_8$.

The local frame of $TM$ is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8\}$, where

\[ Z_1 = 2(\partial x_1 + \partial y_2 + y^1 \partial z), \quad Z_2 = 2(\partial x_2 + y^2 \partial z), \quad Z_3 = 2\partial y_1, \]

\[ Z_4 = 2(\partial x_3 + \partial y_4 + y^3 \partial z), \quad Z_5 = 2(\partial x_4 + \partial y_3 + y^4 \partial z), \]

\[ Z_6 = 2(\cos u_7 \partial x_5 + \sin u_7 \partial y_5 - u_6 \partial y_6 + \cos u_7 y^5 \partial z - \sin u_6 y^6 \partial z), \]

\[ Z_7 = 2(u_6 \sin u_7 \partial x_5 + u_6 \cos u_7 \partial y_5 - u_6 \sin u_7 y^5 \partial z), \]

\[ Z_8 = V = 2\partial z. \]

Hence $\text{Rad} TM = \text{span} \{Z_1\}$ and $S(TM) = \text{span} \{Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, V\}$.

Now $\text{ltr}(TM)$ is spanned by $N_1 = -\partial x_1 + \partial y_2 - y^1 \partial z$ and $S(TM^\perp)$ is spanned by

\[ W_1 = 2(\partial x_3 - \partial y_4 + y^3 \partial z), \quad W_2 = 2(\partial x_4 - \partial y_3 + y^4 \partial z), \]

\[ W_3 = 2(\cos u_7 \partial x_5 + \sin u_7 \partial y_5 + \sin u_6 \partial x_6 - \cos u_6 \partial y_6 + \cos u_7 y^5 \partial z + \sin u_6 y^6 \partial z), \]

\[ W_4 = 2(u_6 \cos u_6 \partial x_6 + u_6 \sin u_6 \partial y_6 + u_6 \cos u_6 y^6 \partial z). \]
It follows that $\phi Z_1 = Z_2 - Z_3$, which implies $\phi \text{Rad} TM$ is a distribution on $M$. On the other hand, we can see that $D_1 = \text{span} \{Z_4, Z_5\}$ such that $\phi Z_4 = W_2$, $\phi Z_5 = W_1$, which implies $D_1$ is anti-invariant with respect to $\phi$ and $D_2 = \text{span} \{Z_6, Z_7\}$ is a slant distribution with slant angle $\pi/4$. Hence $M$ is a pseudo-slant 2-lightlike submanifold of $\mathbb{R}_2^{13}$.

**Example 7.3.2.** Let $(\mathbb{R}_2^{13}, \overline{g})$ be an indefinite Sasakian manifold, where $\overline{g}$ is of signature $(-, +, +, +, +, +, +, +, +, +, +, +, +, +)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6, \partial z\}$.

Suppose $M$ is a submanifold of $\mathbb{R}_2^{13}$ given by $-x^1 = y^2 = u_1$, $x^2 = u_2$, $y^1 = u_3$, $x^3 = u_4 \cos \beta$, $y^3 = u_4 \sin \beta$, $x^4 = u_5 \sin \beta$, $y^4 = u_5 \cos \beta$, $x^5 = u_6$, $y^5 = u_7$, $x^6 = k \cos u_7$, $y^6 = k \sin u_7$, $z = u_8$, where $k$ is any constant.

The local frame of $TM$ is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\}$, where

- $Z_1 = 2(-\partial x_1 + \partial y_2 - y^1 \partial z)$,
- $Z_2 = 2(\partial x_2 + y^2 \partial z)$,
- $Z_3 = 2\partial y_1$,
- $Z_4 = 2(\cos \beta \partial x_3 + \sin \beta \partial y_3 + y^3 \cos \beta \partial z)$,
- $Z_5 = 2(\sin \beta \partial x_4 + \cos \beta \partial y_4 + y^4 \sin \beta \partial z)$,
- $Z_6 = 2(\partial x_5 + y^5 \partial z)$,
- $Z_7 = 2(\partial y_5 - k \sin u_7 \partial x_6 + k \cos u_7 \partial y_6 - k \sin u_7 y^6 \partial z)$,

$Z_8 = V = 2\partial z$.

Hence $\text{Rad} TM = \text{span} \{Z_1\}$ and $S(TM) = \text{span} \{Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, V\}$.

Now $\text{ltr}(TM)$ is spanned by $N_1 = \partial x_1 + \partial y_2 + y^1 \partial z$ and $S(TM^\perp)$ is spanned by

- $W_1 = 2(\sin \beta \partial x_3 - \cos \beta \partial y_3 + y^3 \sin \beta \partial z)$,
- $W_2 = 2(\cos \beta \partial x_4 - \sin \beta \partial y_4 + y^4 \cos \beta \partial z)$,
- $W_3 = 2(k \cos u_7 \partial x_6 + k \sin u_7 \partial y_6 + k \cos u_7 y^6 \partial z)$,
- $W_4 = 2(k^2 \partial y_5 + k \sin u_7 \partial x_6 - k \cos u_7 \partial y_6 + k \sin u_7 y^6 \partial z)$.

It follows that $\phi Z_1 = Z_2 + Z_3$, which implies $\phi \text{Rad} TM$ is a distribution on $M$. On the other hand, we can see that $D_1 = \text{span} \{Z_4, Z_5\}$ such that $\phi Z_4 = W_1$, $\phi Z_5 = W_2$, which implies $D_1$ is anti-invariant with respect to $\phi$ and $D_2 = \text{span} \{Z_6, Z_7\}$ is a slant distribution with slant angle $\theta = \arccos(1/\sqrt{1 + k^2})$. Hence $M$ is a pseudo-slant 2-lightlike submanifold of $\mathbb{R}_2^{13}$.

Now, for any vector field $X$ tangent to $M$, we put $\phi X = PX + FX$, where $PX$ and $FX$ are tangential and transversal parts of $\phi X$ respectively. We denote the projections on $\text{Rad} TM$, $\phi \text{Rad} TM$, $\phi \text{ltr}(TM)$, $D_1$ and $D_2$ in $TM$ by $P_1$, $P_2$, $P_3$, $P_4$, and $P_5$ respectively. Similarly, we denote
the projections of $tr(TM)$ on $ltr(TM)$, $\phi(D_1)$ and $D'$ by $Q_1$, $Q_2$ and $Q_3$ respectively, where $D'$ is non-degenerate orthogonal complementary subbundle of $\phi(D_1)$ in $S(TM^\perp)$. Then, for any $X \in \Gamma(TM)$, we get

$$
(7.3.2) \quad X = P_1X + P_2X + P_3X + P_4X + P_5X + \eta(X)V.
$$

Now applying $\phi$ to $(7.3.2)$, we have

$$
(7.3.3) \quad \phi X = \phi P_1X + \phi P_2X + \phi P_3X + \phi P_4X + \phi P_5X,
$$

which gives

$$
(7.3.4) \quad \phi X = \phi P_1X + \phi P_2X + \phi P_3X + \phi P_4X + fP_5X + FP_5X,
$$

where $fP_5X$ (resp. $FP_5X$) denotes the tangential (resp. transversal) component of $\phi P_5X$. Thus we get $\phi P_1X \in \Gamma(\phi \text{Rad}TM)$, $\phi P_2X \in \Gamma(\text{Rad}TM)$, $\phi P_3X \in \Gamma(ltr(TM))$, $\phi P_4X \in \Gamma(D_1) \subset \Gamma(S(TM^\perp))$, $fP_5X \in \Gamma(D_2)$ and $FP_5X \in \Gamma(D')$. Also, for any $W \in \Gamma(tr(TM))$, we have

$$
(7.3.5) \quad W = Q_1W + Q_2W + Q_3W.
$$

Applying $\phi$ to $(7.3.5)$, we obtain

$$
(7.3.6) \quad \phi W = \phi Q_1W + \phi Q_2W + \phi Q_3W,
$$

which gives

$$
(7.3.7) \quad \phi W = \phi Q_1W + \phi Q_2W + BQ_3W + CQ_3W,
$$

where $BQ_3W$ (resp. $CQ_3W$) denotes the tangential (resp. transversal) component of $\phi Q_3W$. Thus we get $\phi Q_1W \in \Gamma(\phi ltr(TM))$, $\phi Q_2W \in \Gamma(D_1)$, $BQ_3W \in \Gamma(D_2)$ and $CQ_3W \in \Gamma(D')$.

Now, by using $(1.5.6)$, $(7.3.4)$, $(7.3.7)$ and $(1.3.7)-(1.3.9)$ and identifying the components on $\text{Rad}TM$, $\phi \text{Rad}TM$, $\phi ltr(TM)$, $D_1$, $D_2$, $ltr(TM)$, $\phi(D_1)$, $D'$ and $\{V\}$, we obtain

$$
(7.3.8) \quad P_1(\nabla_X\phi P_1Y) + P_1(\nabla_X\phi P_2Y) - P_1(A_{\phi P_4Y}X) + P_1(\nabla_X fP_5Y) = P_1(A_{FP_3Y}X) + P_1(A_{\phi P_3Y}X) + \phi P_2\nabla_XY - \eta(Y)P_1X,
$$

$$
(7.3.9) \quad P_2(\nabla_X\phi P_1Y) + P_2(\nabla_X\phi P_2Y) - P_2(A_{\phi P_4Y}X) + P_2(\nabla_X fP_5Y) = P_2(A_{FP_3Y}X) + P_2(A_{\phi P_3Y}X) + \phi P_1\nabla_XY - \eta(Y)P_2X,
$$

$$
(7.3.10) \quad P_3(\nabla_X\phi P_1Y) + P_3(\nabla_X\phi P_2Y) - P_3(A_{\phi P_4Y}X) + P_3(\nabla_X fP_5Y) = P_3(A_{FP_3Y}X) + P_3(A_{\phi P_3Y}X) + \phi h^l(X,Y) - \eta(Y)P_3X,
$$

$$
(7.3.11) \quad P_4(\nabla_X\phi P_1Y) + P_4(\nabla_X\phi P_2Y) - P_4(A_{\phi P_4Y}X) + P_4(\nabla_X fP_5Y) = P_4(A_{FP_3Y}X) + P_4(A_{\phi P_3Y}X) + \phi Q_2 h^s(X,Y) - \eta(Y)P_4X,
$$

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\begin{align*}
P_5(\nabla_X \phi P_1 Y) + P_5(\nabla_X \phi P_2 Y) - P_5(A_{\phi P_4 Y} X) + P_5(\nabla_X f P_3 Y) \\
(7.3.12) & = P_5(A_{FP_3 Y} X) + P_5(A_{\phi P_3 Y} X) + f P_3 \nabla_X Y \\
& + B Q_3 h^s(X, Y) - \eta(Y) P_3 X,
\end{align*}

\begin{align*}
h^l(X, \phi P_1 Y) + h^l(X, \phi P_2 Y) + D^l(X, \phi P_4 Y) + h^l(X, f P_3 Y) \\
(7.3.13) & = \phi P_3 \nabla_X Y - \nabla_X \phi P_3 Y - D^l(X, FP_3 Y),
\end{align*}

\begin{align*}
& Q_2 h^s(X, \phi P_1 Y) + Q_2 h^s(X, \phi P_2 Y) + Q_2 h^s(X, f P_3 Y) \\
& = Q_2 \nabla_X FP_3 Y - Q_2 \nabla_X \phi P_4 Y \\
& - Q_2 D^s(X, \phi P_3 Y) + \phi P_4 \nabla_X Y,
\end{align*}

\begin{align*}
& Q_3 h^s(X, \phi P_1 Y) + Q_3 h^s(X, \phi P_2 Y) + Q_3 h^s(X, f P_3 Y) \\
& = C Q_3 h^s(X, Y) - Q_3 \nabla_X FP_3 Y - Q_3 \nabla_X \phi P_4 Y \\
& - Q_3 D^s(X, \phi P_3 Y) + FP_3 \nabla_X Y,
\end{align*}

\begin{align*}
& \eta(\nabla_X \phi P_1 Y) + \eta(\nabla_X \phi P_2 Y) - \eta(A_{\phi P_4 Y} X) + \eta(\nabla_X f P_3 Y) \\
& = \eta(A_{FP_3 Y} X) + \eta(A_{FP_3 Y} X) + g(X, Y) V.
\end{align*}

**Theorem 7.3.2:** Let $M$ be a q-lightlike submanifold of an indefinite Sasakian manifold $M$ of index $2q$ with structure vector field tangent to $M$. Then $M$ is a pseudo-slant lightlike submanifold of $M$ if and only if

(i) $\phi \text{Rad}_TM$ is a distribution on $M$ such that $\text{Rad}_TM \cap \phi \text{Rad}_TM = \{0\}$,

(ii) the distribution $D_1$ is an anti-invariant distribution, i.e. $\phi D_1 \subset S(TM^\perp)$,

(iii) there exists a constant $\lambda \in (0, 1]$ such that $P^2 X = -\lambda X$.

Moreover, there also exists a constant $\mu \in [0, 1)$ such that $BF_X = -\mu X$, for all $X \in \Gamma(D_2)$, where $D_1$ and $D_2$ are non-degenerate orthogonal distributions on $M$ such that $S(TM) = (\phi \text{Rad}_TM \oplus \text{oltr}(TM)) \oplus \text{orth} D_1 \oplus \text{orth} D_2 \oplus \text{orth} \{V\}$ and $\lambda = \cos^2 \theta$, $\theta$ is slant angle of $D_2$.

**Proof.** Let $M$ be a pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $M$. Then distribution $D_1$ is anti-invariant with respect to $\phi$ and $\phi \text{Rad}_TM$ is a distribution on $M$ such that $\text{Rad}_TM \cap \phi \text{Rad}_TM = \{0\}$.

Now for any $X \in \Gamma(D_2)$, we have $|PX| = |\phi X| \cos \theta$, which implies

\begin{equation}
(7.3.17) \quad \cos \theta = \frac{|PX|}{|\phi X|}.
\end{equation}

In view of (7.3.17), we get $\cos^2 \theta = \frac{|PX|^2}{|\phi X|^2} = \frac{g(PX, PX)}{g(\phi X, \phi X)} = \frac{g(X, P^2 X)}{g(X, \phi^2 X)}$, which gives

\begin{equation}
(7.3.18) \quad g(X, P^2 X) = \cos^2 \theta g(X, \phi^2 X).
\end{equation}
Since $M$ is a pseudo-slant lightlike submanifold and $\cos^2 \theta = \lambda(\text{constant}) \in (0, 1]$. Then from (7.3.18), we get $g(X, P^2X) = g(X, \lambda \phi^2 X)$, which implies

(7.3.19) \quad g(X, (P^2 - \lambda \phi^2)X) = 0.

Since $(P^2 - \lambda \phi^2)X \in \Gamma(D_2)$ and the induced metric $g = g|_{D_2 \times D_2}$ is non-degenerate (positive definite), from (7.3.19), we have $(P^2 - \lambda \phi^2)X = 0$, which implies

(7.3.20) \quad P^2X = \lambda \phi^2 X = -\lambda X.

Now, for any vector field $X \in \Gamma(D_2)$, we have

(7.3.21) \quad \phi X = PX + FX,

where $PX$ and $FX$ are tangential and transversal parts of $\phi X$ respectively.

Applying $\phi$ to (7.3.21) and taking tangential component we get

(7.3.22) \quad -X = P^2X + BFX.

From (7.3.20) and (7.3.22), we get

(7.3.23) \quad BFX = -\mu X,

where $1 - \lambda = \mu(\text{constant}) \in [0, 1)$.

This proves (iii).

Conversely suppose that conditions (i), (ii) and (iii) are satisfied. From (7.3.22), for any $X \in \Gamma(D_2)$, we get

(7.3.24) \quad -X = P^2X - \mu X,

which implies

(7.3.25) \quad P^2X = -\lambda X,

where $1 - \mu = \lambda(\text{constant}) \in (0, 1]$. 

Now $\cos \theta = \frac{g(\phi X, PX)}{|\phi X||PX|} = -\frac{g(X, \phi^2 PX)}{|\phi X||PX|} = -\lambda \frac{g(X, P^2X)}{|\phi X||PX|} = \lambda \frac{g(\phi X, \phi X)}{|\phi X||PX|}$.

From above equation, we get

(7.3.26) \quad \cos \theta = \lambda \frac{|\phi X|}{|PX|}.

Therefore (7.3.17) and (7.3.26) give $\cos^2 \theta = \lambda(\text{constant})$.

Hence $M$ is a pseudo-slant lightlike submanifold.
Corollary 7.3.1: Let $M$ be a pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ with slant angle $\theta$, then for any $X, Y \in \Gamma(D_2)$, we have

(i) $g(PX, PY) = \cos^2 \theta (g(X, Y) - \eta(X)\eta(Y))$,

(ii) $g(FX, FY) = \sin^2 \theta (g(X, Y) - \eta(X)\eta(Y))$.

The proof of above Corollary follows by using similar steps as in proof of Corollary 3.2 of [56].

Lemma 7.3.3: Let $M$ be a pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then for any $X, Y \in \Gamma(TM - \{V\})$, we have

(i) $g(\nabla_X Y, V) = \overline{g}(Y, \phi X)$,

(ii) $g([X, Y], V) = 2\overline{g}(X, \phi Y)$.

Proof. Let $M$ be a pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Since $\nabla$ is a metric connection, from (1.3.7) and (1.5.7), for any $X, Y \in \Gamma(TM - \{V\})$, we have

\[ g(\nabla_X Y, V) = \overline{g}(Y, \phi X). \]  

From (1.5.5) and (7.3.27), we have $g([X, Y], V) = 2\overline{g}(X, \phi Y)$.

Theorem 7.3.3: Let $M$ be a pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ with structure vector field tangent to $M$. Then $\text{Rad}TM$ is integrable if and only if

(i) $P_1(\nabla_X \phi P_1 Y) = P_1(\nabla_Y \phi P_1 X)$ and $P_5(\nabla_X \phi P_1 Y) = P_5(\nabla_Y \phi P_1 X)$,

(ii) $Q_2 h^s(Y, \phi P_1 X) = Q_2 h^s(X, \phi P_1 Y)$ and $h^l(Y, \phi P_1 X) = h^l(X, \phi P_1 Y)$,

(iii) $Q_3 h^s(Y, \phi P_1 X) = Q_3 h^s(X, \phi P_1 Y)$, for all $X, Y \in \Gamma(\text{Rad}TM)$.

Proof. Let $M$ be a pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Let $X, Y \in \Gamma(\text{Rad}TM)$. From (7.3.8), we have $P_1(\nabla_X \phi P_1 Y) = \phi P_2 \nabla_X Y$, which gives $P_1(\nabla_X \phi P_1 Y) - P_1(\nabla_Y \phi P_1 X) = \phi P_2[X, Y]$. In view of (7.3.12), we obtain $P_5(\nabla_X \phi P_1 Y) = f P_5 \nabla_X Y + B h^s(X, Y)$, which implies $P_5(\nabla_X \phi P_1 Y) - P_5(\nabla_Y \phi P_1 X) = f P_5[X, Y]$. From (7.3.13), we have $h^l(X, \phi P_1 Y) = \phi P_3 \nabla_X Y$, which gives $h^l(Y, \phi P_1 X) - h^l(Y, \phi P_1 X) = \phi P_3[X, Y]$. Also from (7.3.14), we get $Q_2 h^s(X, \phi P_1 Y) = \phi P_4 \nabla_X Y$, which gives $Q_2 h^s(X, \phi P_1 Y) - Q_2 h^s(Y, \phi P_1 X) = \phi P_4[X, Y]$. In view of (7.3.15), we obtain $Q_3 h^s(X, \phi P_1 Y) = C h^s(X, Y) + F P_5 \nabla_X Y$, which implies $Q_3 h^s(X, \phi P_1 Y) - Q_3 h^s(Y, \phi P_1 X) = F P_5[X, Y]$. This concludes the theorem.
**Theorem 7.3.4:** Let $M$ be a pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ with structure vector field tangent to $M$. Then $D_1$ is integrable if and only if

1. $P_1(A_{\phi P_1}X) = P_1(A_{\phi P_2}X)$ and $P_2(A_{\phi P_1}X) = P_2(A_{\phi P_2}X)$,
2. $D^l(Y, \phi P_4X) = D^l(X, \phi P_4X)$ and $P_5(A_{\phi P_1}X) = P_5(A_{\phi P_2}X)$,
3. $Q_3\nabla^s_Y \phi P_4X = Q_3\nabla^s_X \phi P_4Y$, for all $X, Y \in \Gamma(D_1)$.

**Proof.** Let $M$ be a pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Let $X, Y \in \Gamma(D_1)$. In view of (7.3.8), we have $P_1(A_{\phi P_1}X) + \phi P_2 \nabla_X Y = 0$, which gives $P_1(A_{\phi P_1}X) - P_1(A_{\phi P_2}X) = \phi P_2[X, Y]$. From (7.3.9), we get $P_2(A_{\phi P_1}X) + \phi P_1 \nabla_X Y = 0$, which gives $P_2(A_{\phi P_1}X) - P_2(A_{\phi P_2}X) = \phi P_1[X, Y]$. In view of (7.3.12), we obtain $P_3(A_{\phi P_1}X) + f P_5 \nabla_X Y + BQ_3 h^s(X, Y) = 0$, which implies $P_3(A_{\phi P_1}X) - P_3(A_{\phi P_2}X) = f P_5[X, Y]$. From (7.3.13), we get $D^l(X, \phi P_4Y) = \phi P_3 \nabla_X Y$, which gives $D^l(X, \phi P_4Y) - D^l(Y, \phi P_4X) = \phi P_3[X, Y]$. Also from (7.3.15), we have $Q_3 \nabla^s_X \phi P_4Y = CQ_3 h^s(X, Y) + F P_5 \nabla_X Y$, which gives $Q_3 \nabla^s_X \phi P_4Y - Q_3 \nabla^s_Y \phi P_4X = F P_5[X, Y]$. This completes the proof.

**Theorem 7.3.5:** Let $M$ be a pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ with structure vector field tangent to $M$. Then $D_2 \oplus \{V\}$ is integrable if and only if

1. $P_1(\nabla_X f P_5 Y - \nabla_Y f P_5 X) = P_1(A_{FP_3}X - A_{FP_3}Y)$,
2. $P_2(\nabla_X f P_5 Y - \nabla_Y f P_5 X) = P_2(A_{FP_3}X - A_{FP_3}Y)$,
3. $h^l(X, f P_5 Y) - h^l(Y, f P_5 X) = D^l(Y, F P_5 X) - D^l(X, F P_5 Y)$,
4. $Q_2(\nabla^s_X F P_3 Y - \nabla^s_Y F P_3 X) = Q_2(h^s(X, f P_5 Y) - h^s(Y, f P_5 X))$, for all $X, Y \in \Gamma(D_2 \oplus \{V\})$.

**Proof.** Let $M$ be a pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Let $X, Y \in \Gamma(D_2 \oplus \{V\})$. In view of (7.3.8), we obtain $P_1(\nabla_X f P_5 Y - \nabla_Y f P_5 X) - P_1(A_{FP_3}X - A_{FP_3}Y) = \phi P_2 \nabla_X Y$, which gives $P_1(\nabla_X f P_5 Y - \nabla_Y f P_5 X) - P_1(A_{FP_3}X - A_{FP_3}Y) = \phi P_2[X, Y]$. From (7.3.9), we get $P_2(\nabla_X f P_5 Y) - P_2(A_{FP_3}X) = \phi P_1 \nabla_X Y$, which gives $P_2(\nabla_X f P_5 Y - \nabla_Y f P_5 X) - P_2(A_{FP_3}X - A_{FP_3}Y) = \phi P_1[X, Y]$. From (7.3.13), we obtain $h^l(X, f P_5 Y) + D^l(Y, F P_5 X) = \phi P_3 \nabla_X Y$, which implies $h^l(X, f P_5 Y) - h^l(Y, f P_5 X) + D^l(Y, F P_5 X) - D^l(X, F P_5 Y) = \phi P_3[X, Y]$. In view of (7.3.14), we have $Q_2 h^s(X, f P_5 Y) - Q_2 \nabla^s_X F P_3 Y = \phi P_3 \nabla_X Y$, which implies $Q_2(\nabla^s_X F P_3 X - \nabla^s_Y F P_3 X) + Q_2(h^s(X, f P_5 Y) - h^s(Y, f P_5 X)) = \phi P_3[X, Y]$. Thus, the theorem is completed.
Theorem 7.3.6: Let $M$ be a pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ with structure vector field $V$ tangent to $M$. Then induced connection $\nabla$ is not a metric connection.

Proof. Let $M$ be a pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Suppose that the induced connection is a metric connection. Then $\nabla_X \phi P^2 Y \in \Gamma(RadTM)$ and $h^i(X,Y) = 0$ for all $X,Y \in \Gamma(TM)$. Thus for any $Z \in \Gamma(\phi RadTM)$ and $W \in \Gamma(\phi ltr(TM))$, from (1.5.6), we have

\[
(7.3.28) \quad \nabla_W \phi Z - \phi \nabla_W Z = \overline{g}(Z,W)V.
\]

In view of (1.3.7), (7.3.28) and taking tangential components, we get

\[
(7.3.29) \quad \nabla_W \phi Z - \phi P_1 \nabla_W Z - \phi P_2 \nabla_W Z - \phi Q_2 h^s(Z,W) = fP_5 \nabla_W Z + BQ_3 h^s(Z,W) + \overline{g}(Z,W)V.
\]

Since $TM = RadTM \oplus_{orth}(\phi RadTM \oplus \phi ltr(TM)) \oplus_{orth} D_1 \oplus_{orth} D_2 \oplus_{orth} V$, from (7.3.29), we obtain

\[
(7.3.30) \quad \nabla_W \phi Z - \phi P_2 \nabla_W Z = 0, \quad \phi P_1 \nabla_W Z = 0, \quad \phi Q_2 h^s(Z,W) = 0,
\]

\[
(7.3.31) \quad fP_5 \nabla_W Z - BQ_3 h^s(Z,W) = 0, \quad \overline{g}(Z,W)V = 0.
\]

Now taking $W = \phi N$ and $Z = \phi \xi$ in (7.3.31), we get $\overline{g}(N, \xi)V = 0$.

Thus $V = 0$, which is a contradiction. Hence $M$ does not have a metric connection.

7.4 Foliations Determined by Distributions

In this section, we obtain necessary and sufficient conditions for foliations determined by distributions on a pseudo-slant lightlike submanifold of indefinite Kaehler and Sasakian manifolds to be totally geodesic.

Definition 7.4.1. A pseudo-slant lightlike submanifold $M$ of an indefinite Kaehler (Sasakian) manifold $\overline{M}$ is said to be mixed geodesic if its second fundamental form $h$ satisfies $h(X,Y) = 0$, for all $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$. Thus $M$ is a mixed geodesic pseudo-slant lightlike submanifold if $h^i(X,Y) = 0$ and $h^s(X,Y) = 0$, for all $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$.

Theorem 7.4.2: Let $M$ be a pseudo-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then $RadTM$ defines a totally geodesic foliation if and only if

\[
\overline{g}(\nabla_X \overline{J} P^2 Z + \nabla_X f P^5 Z, \overline{J} Y) = \overline{g}(A_{\overline{J} P^2 Z} X + A_{\overline{J} P^4 Z} X + A_{FP^2 Z} X, \overline{J} Y),\quad \text{for all } X,Y \in \Gamma(RadTM) \text{ and } Z \in \Gamma(S(TM)).
\]
Proof. Let \( M \) be a pseudo-slant lightlike submanifold of an indefinite Kaehler manifold \( \overline{M} \). It is easy to see that \( \text{Rad}TM \) defines a totally geodesic foliation if and only if \( \nabla_X Y \in \Gamma(\text{Rad}TM) \), for all \( X, Y \in \Gamma(\text{Rad}TM) \). Since \( \nabla \) is a metric connection, using (1.3.7), (1.4.2), (1.4.3) and (7.2.4), for any \( X, Y, Z \in \Gamma(\text{Rad}TM) \) and \( Z \in \Gamma(S(TM)) \), we get
\[
\overline{g}(\nabla_X Y, Z) = -\overline{g}(\nabla_X (\overline{J}_P Z + \overline{J}_P Z + \overline{J}_P Z + fP_5 Z + fP_5 Z), \overline{J}Y),
\]
which gives
\[
\overline{g}(\nabla_X Y, Z) = \overline{g}(A_{\overline{J}_P Z} X + A_{fP_5 Z} X + A_{\overline{J}_P Z} X - \nabla_X \overline{J}_P Z - \nabla_X fP_5 Z, \overline{J}Y).
\]
This completes the proof.

Theorem 7.4.3: Let \( M \) be a pseudo-slant lightlike submanifold of an indefinite Kaehler manifold \( \overline{M} \). Then \( D_1 \) defines a totally geodesic foliation if and only if

(i) \( \overline{g}(\nabla_X^s FZ, \overline{J}Y) = -\overline{g}(h^s(X, fZ), \overline{J}Y), \)

(ii) \( h^s(X, \overline{J}N) \) and \( D^s(X, \overline{J}W) \) have no components in \( \overline{J}(D_1) \),

for all \( X, Y \in \Gamma(D_1), Z \in \Gamma(D_2), N \in \Gamma(\text{ltr}(TM)), W \in \Gamma(\overline{J}\text{ltr}(TM)) \).

Proof. Let \( M \) be a pseudo-slant lightlike submanifold of an indefinite Kaehler manifold \( \overline{M} \). The distribution \( D_1 \) defines a totally geodesic foliation if and only if \( \nabla_X Y \in \Gamma(D_1) \), for all \( X, Y \in \Gamma(D_1) \). Since \( \nabla \) is a metric connection, using (1.3.7), (1.4.2) and (1.4.3), for any \( X, Y, Z \in \Gamma(D_1) \) and \( Z \in \Gamma(D_2) \), we obtain
\[
\overline{g}(\nabla_X Y, Z) = -\overline{g}(\nabla_X \overline{J}Z, \overline{J}Y),
\]
which implies
\[
\overline{g}(\nabla_X Y, Z) = \overline{g}(\nabla_X^s FZ + h^s(X, fZ), \overline{J}Y). \]
From (1.3.7), (1.4.2) and (1.4.3), for any \( X, Y \in \Gamma(D_1) \) and \( N \in \Gamma(\text{ltr}(TM)) \), we have
\[
\overline{g}(\nabla_X Y, N) = -\overline{g}(\overline{J}Y, \nabla_X \overline{J}N),
\]
thus
\[
\overline{g}(\nabla_X Y, N) = -\overline{g}(\overline{J}Y, h^s(X, \overline{J}N)).
\]
Now, from (1.3.7), (1.4.2) and (1.4.3), for any \( X, Y \in \Gamma(D_1) \) and \( W \in \Gamma(\overline{J}\text{ltr}(TM)) \), we obtain
\[
\overline{g}(\nabla_X Y, W) = -\overline{g}(\overline{J}Y, \nabla_X \overline{J}W),
\]
which implies
\[
\overline{g}(\nabla_X Y, W) = \overline{g}(\overline{J}Y, D^s(X, \overline{J}W)).
\]
This concludes the theorem.

Theorem 7.4.4: Let \( M \) be a pseudo-slant lightlike submanifold of an indefinite Kaehler manifold \( \overline{M} \). Then \( D_2 \) defines a totally geodesic foliation if and only if

(i) \( \overline{g}(A_{\overline{J}Z} X, fY) = \overline{g}(\nabla_X^s \overline{J}Z, FY), \)

(ii) \( \overline{g}(fY, \nabla_X \overline{J}N) = -\overline{g}(FY, h^s(X, \overline{J}N)), \)

(iii) \( \overline{g}(fY, A_{\overline{J}W} X) = \overline{g}(FY, D^s(X, \overline{J}W)), \)

for all \( X, Y \in \Gamma(D_2), Z \in \Gamma(D_1), N \in \Gamma(\text{ltr}(TM)), W \in \Gamma(\overline{J}\text{ltr}(TM)) \).

Proof. Let \( M \) be a pseudo-slant lightlike submanifold of an indefinite Kaehler manifold \( \overline{M} \). The distribution \( D_2 \) defines a totally geodesic foliation if and only if \( \nabla_X Y \in \Gamma(D_2) \), for all \( X, Y \in \Gamma(D_2) \). Since \( \nabla \) is a metric connection, using (1.3.7), (1.4.2) and (1.4.3), for any \( X, Y \in \Gamma(D_2) \) and \( Z \in \Gamma(D_1) \), we get
\[
\overline{g}(\nabla_X Y, Z) = -\overline{g}(\nabla_X \overline{J}Z, \overline{J}Y),
\]
which gives
\( \bar{g}(\nabla_X Y, Z) = \bar{g}(A_{\nabla_Z} X, fY) - \bar{g}(\nabla_X \bar{J} Z, FY) \). From (1.3.7), (1.4.2) and (1.4.3), for any \( X, Y \in \Gamma(D_2) \) and \( N \in \Gamma(ltr(TM)) \), we get \( \bar{g}(\nabla_X Y, N) = -\bar{g}(\bar{J} Y, \nabla_X \bar{J} N) \), which gives \( \bar{g}(\nabla_X Y, N) = -\bar{g}(fY, \nabla_X \bar{J} N) - \bar{g}(FY, h^s(X, \bar{J} N)) \). Now, from (1.3.7), (1.4.2) and (1.4.3), for any \( X, Y \in \Gamma(D_2) \) and \( W \in \Gamma(\bar{J} ltr(TM)) \), we have \( \bar{g}(\nabla_X Y, W) = -\bar{g}(\bar{J} Y, \nabla_X \bar{J} W) \), which gives \( \bar{g}(\nabla_X Y, W) = \bar{g}(fY, A_{\nabla W} X) - \bar{g}(FY, D^s(X, \bar{J} W)) \). Thus, we obtain the required results.

**Theorem 7.4.5:** Let \( M \) be a mixed geodesic pseudo-slant lightlike submanifold of an indefinite Kaehler manifold \( \bar{M} \). Then \( D_1 \) defines a totally geodesic foliation if and only if \( \nabla^s X F Z, h^s(X, \bar{J} N) \) and \( D^s(X, \bar{J} W) \) have no components in \( \bar{J}(D_1) \), for all \( X \in \Gamma(D_1), Z \in \Gamma(D_2), N \in \Gamma(ltr(TM)) \) and \( W \in \Gamma(\bar{J} ltr(TM)) \).

**Proof.** Since \( M \) is a mixed geodesic pseudo-slant lightlike submanifold of an indefinite Kaehler manifold \( \bar{M} \), we have \( h^s(X, Z) = 0 \), for all \( X \in \Gamma(D_1) \) and \( Z \in \Gamma(D_2) \). Now the proof follows from theorem 7.4.3.

**Theorem 7.4.6:** Let \( M \) be a pseudo-slant lightlike submanifold of an indefinite Sasakian manifold \( \bar{M} \) with structure vector field tangent to \( M \). Then \( Rad(TM) \) defines a totally geodesic foliation if and only if \( \bar{g}(\nabla_X \phi P_2 Z + \nabla_X f P_5 Z, \phi Y) = \bar{g}(A_{\phi P_2} X + A_{\phi P_5} X + A_{FP_5} Z, \phi Y) \), for all \( X, Y \in \Gamma(Rad(TM)) \) and \( Z \in \Gamma(S(TM)) \).

**Proof.** Let \( M \) be a pseudo-slant lightlike submanifold of an indefinite Sasakian manifold \( \bar{M} \). It is easy to see that \( Rad(TM) \) defines a totally geodesic foliation if and only if \( \nabla_X Y \in \Gamma(Rad(TM)) \), for all \( X, Y \in \Gamma(Rad(TM)) \). Since \( \nabla \) is a metric connection, from (1.3.7), (1.5.2), (1.5.6) and (7.3.4), for any \( X, Y \in \Gamma(Rad(TM)) \) and \( Z \in \Gamma(S(TM)) \), we get
\[
\bar{g}(\nabla_X Y, Z) = -\bar{g}(\nabla_X (\phi P_2 Z + \phi P_5 Z + \phi P_4 Z + f P_5 Z + FP_5 Z), \phi Y),
\]
thus \( \bar{g}(\nabla_X Y, Z) = \bar{g}(A_{\phi P_2} X + A_{\phi P_5} X + A_{FP_5} Z - \nabla_X \phi P_2 Z - \nabla_X f P_5 Z, \phi Y) \). This proves the theorem.

**Theorem 7.4.7:** Let \( M \) be a pseudo-slant lightlike submanifold of an indefinite Sasakian manifold \( \bar{M} \) with structure vector field tangent to \( M \). Then \( D_1 \) defines a totally geodesic foliation if and only if

(i) \( \bar{g}(\nabla^s_X F Z, \phi Y) = -\bar{g}(h^s(X, f Z), \phi Y) \),

(ii) \( h^s(X, \phi N) \) and \( D^s(X, \phi W) \) have no components in \( \phi(D_1) \), for all \( X, Y \in \Gamma(D_1), Z \in \Gamma(D_2), N \in \Gamma(ltr(TM)) \) and \( W \in \Gamma(\phi ltr(TM)) \).

**Proof.** Let \( M \) be a pseudo-slant lightlike submanifold of an indefinite Sasakian manifold \( \bar{M} \). The distribution \( D_1 \) defines a totally geodesic foliation if and only if \( \nabla_X Y \in \Gamma(D_1) \), for all \( X, Y \in \Gamma(D_1) \). Since \( \nabla \)
is a metric connection, using (1.3.7), (1.5.2) and (1.5.6), for any \(X, Y \in \Gamma(D_1)\) and \(Z \in \Gamma(D_2)\), we get \(g(\nabla_X Y, Z) = -g(\nabla_X \phi Z, \phi Y)\), which gives \(g(\nabla_X Y, Z) = g(\nabla_X FZ + h^s(X, fZ), \phi Y)\). In view of (1.3.7), (1.5.2) and (1.5.6), for any \(X, Y \in \Gamma(D_1)\) and \(Z \in \Gamma(ltr(TM))\), we obtain \(g(\nabla_X Y, N) = -g(\phi Y, \nabla_X \phi N)\), which implies \(g(\nabla_X Y, N) = -g(\phi Y, h^s(X, \phi N))\). Now, from (1.3.7), (1.5.2) and (1.5.6), for any \(X, Y \in \Gamma(D_1)\) and \(W \in \Gamma(\phi ltr(TM))\), we have \(g(\nabla_X Y, W) = -g(\phi Y, \nabla_X \phi W)\), which gives \(g(\nabla_X Y, W) = g(\phi Y, D^s(X, \phi W))\). Thus, we obtain the required results.

Theorem 7.4.8: Let \(M\) be a pseudo-slant lightlike submanifold of an indefinite Sasakian manifold \(\overline{M}\) with structure vector field tangent to \(M\). Then \(D_2 \oplus \{V\}\) defines a totally geodesic foliation if and only if

\[
\begin{align*}
(i) \quad g(A_{\phi Z} X, fY) &= g(\nabla_X \phi Z, fY), \\
(ii) \quad g(fY, \nabla_X \phi N) &= -g(FY, h^s(X, \phi N)), \\
(iii) \quad g(fY, A_{\phi W} X) &= g(FY, D^s(X, \phi W)),
\end{align*}
\]

for all \(X, Y, Z \in \Gamma(D_2 \oplus \{V\})\), \(N \in \Gamma(ltr(TM))\) and \(W \in \Gamma(\phi ltr(TM))\).

Proof. Let \(M\) be a pseudo-slant lightlike submanifold of an indefinite Sasakian manifold \(\overline{M}\). The distribution \(D_2 \oplus \{V\}\) defines a totally geodesic foliation if and only if \(\nabla_X Y \in \Gamma(D_2 \oplus \{V\})\), for all \(X, Y \in \Gamma(D_2 \oplus \{V\})\). Since \(\overline{\nabla}\) is a metric connection, using (1.3.7), (1.5.2) and (1.5.6), for any \(X, Y \in \Gamma(D_2 \oplus \{V\})\) and \(Z \in \Gamma(D_1)\), we have \(g(\nabla_X Y, Z) = -g(\nabla_X \phi Z, \phi Y)\), which gives \(g(\nabla_X Y, Z) = g(A_{\phi Z} X, fY) - g(\nabla_X \phi Z, FY)\). In view of (1.3.7), (1.5.2) and (1.5.6), for any \(X, Y \in \Gamma(D_2 \oplus \{V\})\) and \(N \in \Gamma(ltr(TM))\), we obtain \(g(\nabla_X Y, N) = -g(\phi Y, \nabla_X \phi N)\), which implies \(g(\nabla_X Y, N) = -g(fY, \nabla_X \phi N) - g(FY, h^s(X, \phi N))\). Now, from (1.3.7), (1.5.2) and (1.5.6), for any \(X, Y \in \Gamma(D_2 \oplus \{V\})\) and \(W \in \Gamma(\phi ltr(TM))\), we get \(g(\nabla_X Y, W) = -g(\phi Y, \nabla_X \phi W)\), which gives \(g(\nabla_X Y, W) = g(FY, A_{\phi W} X) - g(FY, D^s(X, \phi W))\). This concludes the theorem.

Theorem 7.4.9: Let \(M\) be a mixed geodesic pseudo-slant lightlike submanifold of an indefinite Sasakian manifold \(\overline{M}\) with structure vector field tangent to \(M\). Then \(D_1\) defines a totally geodesic foliation if and only if \(\nabla_X FZ, h^s(X, \phi N)\) and \(D^s(X, \phi W)\) have no components in \(\phi(D_1)\), for all \(X \in \Gamma(D_1)\), \(Z \in \Gamma(D_2)\), \(N \in \Gamma(ltr(TM))\) and \(W \in \Gamma(\phi ltr(TM))\).

Proof. Since \(M\) is a mixed geodesic pseudo-slant lightlike submanifold of an indefinite Sasakian manifold \(\overline{M}\), we have \(h^s(X, Z) = 0\), for all \(X \in \Gamma(D_1)\) and \(Z \in \Gamma(D_2)\). Now the proof follows from theorem 7.4.7.