CHAPTER IV
RELATION BETWEEN DIFFERENT TYPES OF LABELING

4.1 INTRODUCTION

The relation between super-edge magic labeling with other classes of labeling was established by Figueroa-Centeno, Ichishima and Muntaner-Batle. Further Ali Ahmad, Muhammad Imran and Andrea Semanicova Fenovcikova showed the relation between mean labeling and \((a,d)\)-edge-antimagic vertex labeling. In this chapter, the relation between graceful labeling and other graceful labeling such as odd-even graceful labeling and even graceful labeling is discussed first. Then the relation between even graceful graph \(P_{n+k}^*\) and \((n+k)\)-equitable graph \(P_{n+k}^*\) are investigated.

Further the relation between graceful labeling and signed product cordial labeling is established. Even-even graceful labeling with other types of labeling such as E-cordial labeling, totally magic cordial labeling, multiplicative labeling, strongly multiplicative labeling and modular multiplicative divisor labeling are investigated. Special effort is made to introduce a new concept ‘complementary edge-odd graceful labeling’.

4.2 RELATION BETWEEN DIFFERENT TYPES OF GRACEFUL LABELING

Definition 4.2.1. An odd-even graceful labeling of a graph \(G\) with \(q\) edges is an injection \(f\) from \(V(G)\) to \(\{1,3,5,\ldots,2q+1\}\) such that, when each edge \(xy\) is assigned the label \(|f(u)−f(v)|\), the resulting edge labels are distinct. The graph \(G\) with an odd-even graceful labeling is called an odd-even graceful graph.

The following theorem brings out the simple relation that exists between an odd-even graceful labeling and graceful labeling.

Theorem 4.2.2. A graph \(G\) has an odd-even graceful labeling if and only if \(G\) has a graceful labeling.
Proof. Let $G$ be an odd-even graceful graph with $|V(G)| = p$ vertices and $|E(G)| = q$ edges. Let $f$ be an odd-even labeling of $G$. Denote the set of vertices and edges of the graph $G$ by $\{v_1, v_2, \ldots, v_p\}$ and $\{e_1, e_2, \ldots, e_q\}$ respectively. Then, there exists $f$ of $G$, $f: V(G) \rightarrow \{1, 3, 5, \ldots, 2q+1\}$ such that when each edge $uv$ is assigned the label $|f(u) - f(v)|$, the resulting edge labels are $2, 4, 6, \ldots, 2q$.

Define an injective function $\psi$ from $V(G)$ by

$$\psi(v) = \frac{f(v)-1}{2}, \ v \in V(G)$$

The above defined function $\psi$ provides distinct vertex labels and a graceful labeling for the graph $G$.

Obviously, $\psi : V(G) \rightarrow \{0, 1, 2, \ldots, q\}$

Then $\psi^*(uv) = |\psi(u) - \psi(v)|$

$$= \left| \left( \frac{f(u)-1}{2} \right) - \left( \frac{f(v)-1}{2} \right) \right|$$

$$= \left| \frac{f(u)-f(v)+1}{2} \right|$$

$$= \left| \frac{f(u)-f(v)}{2} \right|$$

This means $|\psi(u) - \psi(v)| = \left| \frac{f(u) - f(v)}{2} \right|$ gives distinct edge labels for the graph $G$.

Thus $\psi$ is a graceful labeling of the graph $G$ and hence the graph $G$ is graceful.

Conversely,

Let $G$ be a graceful graph. Then, there exists a function $\psi$,

$\psi : V(G) \rightarrow \{0, 1, 2, \ldots, q\}$ such that when each edge $uv$ is assigned the label $|\psi(u) - \psi(v)|$, the resulting edge labels are $1, 2, 3, \ldots, q$.

Define a new labeling, an injective function $f$ on $V(G)$ as follows:

For $v \in G, f(v) = 2\psi(v) + 1$

The above defined function $f$ provides distinct vertex labels and an odd-even graceful labeling of the graph $G$.

Further, for each edge $uv$ in $G,$ $|f(u) - f(v)| = \left| (2\psi(u)+1)-(2\psi(v)+1) \right|$
Then the induced edge labels are 2, 4, 6, ..., 2q.

Consequently $f$ is an odd-even graceful labeling of $G$.

Hence $G$ is an odd-even graceful graph if and only if $G$ is a graceful graph.

**Illustration 4.2.3.** The following figure shows an odd-even graceful labeling and the corresponding graceful labeling of the graph $P_3^{(3)}$.

![Figure 4.1: Graceful and odd-even graceful labeling of $P_3^{(3)}$](image)

**Illustration 4.2.4.** The Fig. 4.2 shows a graceful labeling and the corresponding odd-even graceful labeling of the double wheel graph $W_8$.

![Figure 4.2: Odd-even graceful and graceful labeling of double wheel graph $W_8$](image)
The following theorems establish the relation between graceful labeling and even graceful labeling.

**Theorem 4.2.5.** The graph $G$ has an even graceful labeling if it has a graceful labeling.

**Proof.** Suppose $G$ has a graceful labeling $f$ with the set of vertex labels $\{0, 1, 2, \ldots, q\}$. Then $\{f(u) - f(v) : uv \in E(G)\} = \{1, 2, \ldots, q\}$.

Now define the new vertex labels by multiplying each of the original vertex labels by the integer 2.

Explicitly $\psi(v) = 2f(v)$ for each vertex $v$ of $G$.

Then $|\psi(u) - \psi(v)| = |2f(u) - 2f(v)|$

From the above construction $\{\psi(u) - \psi(v) : uv \in E(G)\} = \{2, 4, \ldots, 2q\}$

So, $\psi$ is an even graceful labeling of $G$.

Hence the graph $G$ is even graceful if it is a graceful graph.

**Illustration 4.2.6** The relation between a graceful labeling and an even graceful labeling of the Petersen graph is shown in the Fig. 4.3.

![Figure 4.3: Graceful and even graceful labeling of Petersen graph](image)

**Theorem 4.2.7.** If $G$ has an even graceful labeling with all the vertex labels even, then $G$ is a graceful graph.
Proof. Let $G$ be an even graceful graph. Thus it has an even graceful labeling. Let $f : V(G) \to \{0, 1, 2, \ldots, 2q\}$ be the even graceful labeling of $G$ and the vertex labels are all even.

Then $\{ | f(u) - f(v) | : uv \in E(G) \} = \{2, 4, \ldots, 2q\}$.

Define $\psi : V(G) \to \{0, 1, 2, \ldots, q\}$ to be a new injective function such that $\psi(v) = \frac{f(v)}{2}$ for each vertex $v$ of $G$. $\psi$ is well-defined as $f(v)$ is even.

Then $|\psi(u) - \psi(v)| = \left| \frac{f(u)}{2} - \frac{f(v)}{2} \right| = \left| \frac{f(u) - f(v)}{2} \right| = \left| \frac{f(u) - f(v)}{2} \right|$

Thus, $\{ |\psi(u) - \psi(v)| : uv \in E(G) \} = \{1, 2, \ldots, q\}$

The resulting labeling is a graceful labeling of $G$.

Hence the graph $G$ is graceful if it is an even graceful graph whose vertex labels are all even.

Illustration 4.2.8. A graceful labeling and the corresponding even graceful labeling of the shadow graph $C_4(K_{1,2})$ is shown in Fig. 4.4.

![Figure 4.4: Even graceful and graceful labeling of $C_4(K_{1,2})$](image)

4.3 RELATION BETWEEN GRACEFUL LABELING AND SIGNED PRODUCT CORDIAL LABELING

Definition 4.3.1. A vertex labeling of graph $G$ with induced edge labeling defined by $f : V(G) \to \{-1, +1\}$ with induced edge labeling $f^*: E(G) \to \{-1, +1\}$ where $f^*(uv) = f(u)f(v)$ is called a signed product cordial labeling if $|v_f(+1) - v_f(-1)| \leq 1$

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and \( |e_f(+1) - e_f(-1)| \leq 1\), where \( v_f(+1) \) is the number of vertices labeled with 
(+1), \( v_f(-1) \) is the number of vertices labeled with 
(−1), \( e_f(+1) \) is the number of edges 
labeled with (1) and \( e_f(-1) \) is the number of edges labeled with 
(1).

A graph \( G \) is called signed product cordial if it admits a signed product cordial labeling.

The relation between graceful labeling and signed product cordial labeling is 
established in the following theorem.

**Theorem 4.3.2.** Let \( G \) be a tree. If \( G \) is graceful then it admits signed product 
cordial labeling.

**Proof.** Let \( G \) be a tree. Hence the graph \( G \) has \( q+1 \) vertices and \( q \) edges.

Assume that \( G \) has a graceful labeling. Then, there exists a vertex labeling \( f \) of \( G \),
\( f : V(G) \to \{0, 1, 2, \ldots, q\} \) with induced edge labels \( f^*(uv) = |f(u) - f(v)| \).

The resulting edge labels are \( 1, 2, 3, \ldots, q \).

Now, define \( \psi : V(G) \to \{-1, +1\} \) by \( \psi(v) = \begin{cases} 
-1 & \text{if } f(v) \text{ is odd} \\
+1 & \text{if } f(v) \text{ is even} 
\end{cases} \).

Then \( \psi^*(uv) = \psi(u) \psi(v) \)

\[
= (+1) (+1) \text{ if } f(u) \text{ and } f(v) \text{ are even} \\
= +1.
\]

\( \psi^*(uv) = \psi(u) \psi(v) \)

\[
= (-1) (-1) \text{ if } f(u) \text{ and } f(v) \text{ are odd} \\
= +1.
\]

\( \psi^*(uv) = \psi(u) \psi(v) \)

\[
= (+1) (-1) \text{ if } f(u) \text{ is even and } f(v) \text{ is odd} \\
= -1.
\]

\( \psi^*(uv) = \psi(u) \psi(v) \)

\[
= (-1) (+1) \text{ if } f(u) \text{ is odd and } f(v) \text{ is even} \\
= -1.
\]

The labeling pattern defined above satisfies the following conditions.

\( (i) \left| v_\psi(+1) - v_\psi(-1) \right| \leq 1 \)
Case 1: $p$ is even
\[ v_\psi (+1) = \frac{p}{2} \quad \text{and} \quad v_\psi (-1) = \frac{p}{2} \]

Case 2: $p$ is odd
\[ v_\psi (+1) = \frac{p + 1}{2} \quad \text{and} \quad v_\psi (-1) = \frac{p - 1}{2} \]

Case 3: $q$ is even
\[ e_\psi (+1) = \frac{q}{2} \quad \text{and} \quad e_\psi (-1) = \frac{q}{2} \]

Case 4: $q$ is odd
\[ e_\psi (+1) = \frac{q - 1}{2} \quad \text{and} \quad e_\psi (-1) = \frac{q + 1}{2} \]

Therefore, $\psi$ is a signed product cordial labeling of graph $G$.

Hence $G$ is a signed product cordial graph if it is graceful.

**Illustration 4.3.3.** Signed product cordial labeling and the corresponding graceful labeling of the tree is shown in the following figure.

![Graceful labeling and signed product cordial labeling of tree](image)

Figure 4.5: Graceful labeling and signed product cordial labeling of tree

### 4.4 RELATION BETWEEN EVEN GRACEFUL LABELING AND $(n + k)$-EQUITABLE LABELING OF $P_{n+k}^*$

Even graceful labeling for the graph $P_{n+k}^*$ is discussed in Chapter 3. In this section initially the $(n + k)$-equitable labeling for the graph $P_{n+k}^*$ is established and later the
relation between an even graceful labeling and \((n+k)\)-equitable labeling of \(P^*_n\) is discussed.

**Theorem 4.4.1.** The graph \(P^*_n\) is \((n+k)\)-equitable for all positive integers \(n\) and \(k\).

**Proof.** Define the \((n+k)\)-equitable labeling of \(P^*_n\) using the algorithm given below:

**Input:** Let \(G\) be the graph \(P^*_n\) with \(p=(n+k)+2\) vertices and \(q=2n+k\) edges.

**Step 1:** Represent \((n+k)+2\) vertices as \(\{v_0, v_1, v_2, \ldots, v_n, v_{n+1}, v_{n+2}, \ldots, v_{n+k}\}\)

**Step 2:** Represent \(2n+k\) edges as \(E(G) = \{e_0, e_1, e_2, \ldots, e_n, e'_0, e'_1, e'_2, \ldots, e'_{n+1}, e'_{n+2}, \ldots, e'_{n+k}\}\)

**Step 3:** Define \(f : V \rightarrow \{0, 1, 2, \ldots, (n+k)-1\}\) such that
\[
\begin{align*}
&f(v_0) = f(v_n) = 0, \\
&f(v) = n, \\
&f(v_i) = n-i; 1 \leq i \leq n-1 \text{ and} \\
&f(v_{n+j}) = n+j-1; 1 \leq j \leq k.
\end{align*}
\]

**Step 4:** Construct the induced edge labeling \(f^* : E \rightarrow \{0, 1, 2, \ldots, (n+k)-1\}\) as follows:
\[
\begin{align*}
E_1 &= \{ | f(v_i) - f(v_0) | : 1 \leq i \leq n-1 \} \\
&= \{ | f(v_1) - f(v_0) |, | f(v_2) - f(v_0) |, \ldots, | f(v_{n-1}) - f(v_0) | \} \\
&= \{ (n-1) - 0, (n-2) - 0, \ldots, (n-(n-1)) - 0 \} \\
&= \{ (n-1), (n-2), \ldots, 1 \} \\
E_1 &= \{ n-i : 1 \leq i \leq n-1 \} \\
E_2 &= \{ | f(v_i) - f(v) | : 1 \leq i \leq n-1 \} \\
&= \{ | f(v_1) - f(v) |, | f(v_2) - f(v) |, \ldots, | f(v_{n-1}) - f(v) | \} \\
&= \{ | (n-1) - n |, | (n-2) - n |, \ldots, | (n-(n-1)) - n | \} \\
&= \{ n-1-n, n-2-n, \ldots, n-n+1-n \} \\
&= \{ -1, -2, \ldots, 1-n \} \\
&= \{ 1, 2, \ldots, n-1 \}
\end{align*}
\]
That is, \(E_2 = \{ i : 1 \leq i \leq n-1 \}\)
Step 5: Consider the following two conditions:

(i) \( |v_f(s) - v_f(t)| \leq 1 \) and (ii) \( |e_{f'}(s) - e_{f'}(t)| \leq 1; 0 \leq s, t \leq n + k - 1 \)

Step 6: Consider the 6 cases.

Case 1: \( s = n \) and \( t = n - i; 1 \leq i \leq n - 1 \)

Case 2: \( s = 0 \) and \( t = n - i; 1 \leq i \leq n - 1 \)

Case 3: \( s = n \) and \( t = 0 \)

Case 4: \( s = n \) and \( t = n + j - 1; 1 \leq j \leq k \)

Case 5: \( s = 0 \) and \( t = n + j - 1; 1 \leq j \leq k \)

Case 6: \( s = n - j; 1 \leq j \leq n - 1 \) and \( t = n + j - 1; 1 \leq j \leq k \)

Step 7: Consider each case in step 6 for the conditions in step 5.

Output: \( (n+k) \)-equitable labeling of the graph \( P_{n+k}^* \).

Illustration 4.4.2. The Fig. 4.6 given below shows the \((5+5)\)-equitable labeling of the graph \( P_{5+5}^* \).
Corollary 4.4.3. The graph $P_{n+k}$ is even graceful if and only if it is $(n+k)$-equitable.

Proof. Let the graph $P_{n+k}$ be even graceful with $n + k + 2$ vertices and $2n + k$ edges. Therefore, there exists a vertex labeling $f$ of $P_{n+k}$, $f : V(P_{n+k}) \rightarrow \{0, 1, 2, \ldots, 2q\}$ and the induced function $f^* : E(P_{n+k}) \rightarrow \{2, 4, \ldots, 2q\}$ defined by

$$f^*(e = uu') = |f(u) - f(u')| : u, u' \in V$$

form an edge labeling.

Now, let us consider a vertex function $\psi$ of the graph $P_{n+k}$ defined as:

$$\psi(v_0) = f(v_0), \quad \psi(v_i) = \frac{f(v_i)}{2} - (n+1); \quad 1 \leq i \leq n,$$

$$\psi(v) = \frac{f(v)}{2} \quad \text{and} \quad \psi(v_{n+j}) = \frac{f(v_{n+j})}{2} - (n+1); \quad 1 \leq j \leq k.$$
Further for each edge in the graph $P_{n+k}^+$,

\[\psi'(v_0v_i) = |\psi(v_0) - \psi(v_i)| = \left| 0 - \left[ \frac{2n-(i-1)}{2} - (n+1) \right] \right| \quad 1 \leq i \leq n\]

\[= \left| 0 - [2n-(i-1)] + n + 1 \right| \quad 1 \leq i \leq n\]

\[= \left| -2n+i-1+n+1 \right| \quad 1 \leq i \leq n\]

\[= \left| -n+i \right| \quad 1 \leq i \leq n\]

\[= n-i \quad 1 \leq i \leq n\]

\[\psi'(v_0v_1) = \{ n-1, n-2, \ldots, 0 \}\]

\[\psi'(v_0v_{n+j}) = \left| \psi(v_0) - \psi(v_{n+j}) \right| \quad 1 \leq j \leq k\]

\[= \left| 0 - \left[ \frac{2n+j}{2} - (n+1) \right] \right| \quad 1 \leq j \leq k\]

\[= \left| -2n-j+n+1 \right| \quad 1 \leq j \leq k\]

\[= \left| -n-j+1 \right| \quad 1 \leq j \leq k\]

\[= n+j-1 \quad 1 \leq j \leq k\]

\[= \{ n, n+1, \ldots, n+k-1 \}\]

\[\psi'(v_i) = \left| \psi(v) - \psi(v_i) \right| = \left| n-(n-i) \right| \quad 1 \leq i \leq n\]

\[= n-i \quad 1 \leq i \leq n\]

\[= \{ 1, 2, \ldots, n \}\]

Therefore the induced function $\psi^* : E(P_{n+k}^+) \to \{ 0, 1, 2, \ldots, n+k-1 \}$ defined as

\[\psi^*(e = uu') = \left| \psi(u) - \psi(u') \right| \quad u, u' \in V \text{ form an edge labeling.}\]

Obviously, $v_\psi(0) = 2, v_\psi(n-i) = 1; 1 \leq i \leq n-1$,

$v_\psi(n) = 2$ and $v_\psi(n+j-1) = 1; 2 \leq j \leq k$.

Also, $e_\psi(0) = 1, e_\psi(n-i) = 1; 1 \leq i \leq n-1, e_\psi(i) = 1; 1 \leq i \leq n-1$

$e_\psi(n) = 2$ and $e_\psi(n+j-1) = 1; 2 \leq j \leq k$.

But it is noted that $n-i$ are same as $i$, when $1 \leq i \leq n-1$

Therefore, $e_\psi(i) = e_\psi(n-i) = 2; 1 \leq i \leq n-1$.

Consider the following two conditions:
(i) $|v_\psi(s) - v_\psi(t)| \leq 1$ and (ii) $|e_\psi^*(s) - e_\psi^*(t)| \leq 1; \ 0 \leq s, t \leq n + k - 1.$

**Case 1:** $s = 0$ and $t = n - i$ (or) $i; \ 1 \leq i \leq n - 1$

(i) $|v_\psi(s) - v_\psi(t)| = |v_\psi(0) - v_\psi(i)| = |2 - 1| = 1$

(ii) $|e_\psi^*(s) - e_\psi^*(t)| = |e_\psi^*(0) - e_\psi^*(i)| = |1 - 2| = 1$

**Case 2:** $s = n$ and $t = n - i$ (or) $i; \ 1 \leq i \leq n - 1$

(i) $|v_\psi(s) - v_\psi(t)| = |v_\psi(n) - v_\psi(i)| = |2 - 1| = 1$

(ii) $|e_\psi^*(s) - e_\psi^*(t)| = |e_\psi^*(n) - e_\psi^*(i)| = |2 - 2| = 0$

**Case 3:** $s = 0$ and $t = n + j - 1; \ 2 \leq j \leq k$

(i) $|v_\varepsilon(s) - v_\varepsilon(t)| = |v_\varepsilon(0) - v_\varepsilon(n + j - 1)| = |2 - 1| = 1$

(ii) $|e_\varepsilon^*(s) - e_\varepsilon^*(t)| = |e_\varepsilon^*(0) - e_\varepsilon^*(n + j - 1)| = |1 - 1| = 0$

**Case 4:** $s = n$ and $t = n + j - 1; \ 2 \leq j \leq k$

(i) $|v_\varepsilon(s) - v_\varepsilon(t)| = |v_\varepsilon(n) - v_\varepsilon(n + j - 1)| = |2 - 1| = 1$

(ii) $|e_\varepsilon^*(s) - e_\varepsilon^*(t)| = |e_\varepsilon^*(n) - e_\varepsilon^*(n + j - 1)| = |2 - 1| = 1$

Therefore, it satisfies all the conditions.

So from the above construction, it follows that the graph $P^*_n$ is $(n + k)$-equitable.

Conversely, suppose that the graph $P^*_n$ is $(n + k)$-equitable.

To prove: The graph $P^*_n$ is even graceful.

Since the graph $P^*_n$ is $(n + k)$-equitable, for the vertex labeling

$$\psi: V(P^*_n) \to \{0, 1, 2, \ldots, n + k - 1\},$$

the induced function

$$\psi': E(P^*_n) \to \{0, 1, 2, \ldots, n + k - 1\}$$

defined as

$$\psi'(e = uu') = |\psi(u) - \psi(u')|: u, u' \in V$$

satisfies the following conditions:

(i) $|v_\psi(s) - v_\psi(t)| \leq 1$ and (ii) $|e_\psi^*(s) - e_\psi^*(t)| \leq 1; \ 0 \leq s, t \leq n + k - 1.$

Now, define a new function $f$ such that

$$f(v_i) = \psi(v_i), \ f(v_j) = 2[(\psi(v_j)) + (n + 1)]; \ 1 \leq i \leq n,$$

$$f(v) = 2\psi(v)$$

and

$$f(v_{n + j}) = 2[\psi(v_{n + j}) + (n + 1)]; \ 1 \leq j \leq k.$$

According to proof of theorem 4.4.1, it is clear that $f$ is an injective mapping from
V(P^*_n \rightarrow \{0, 1, 2, \ldots, 2(2n + k)\}) and the induced function

\( f^* : E(P^*_n) \rightarrow \{0, 1, 2, \ldots, 2(2n + k)\} \) defined by

\[ f^*(e = uu') = |f(u) - f(u')| \] \( u, u' \in V \) is a one-one and onto mapping.

Accordingly from the above construction, the graph \( P^*_n \) is even graceful.

Hence the graph \( P^*_n \) is even graceful if and only if it is \( (n + k) \)-equitable.

**Illustration 4.4.4.** The following figure illustrates that \( P^*_{5+3} \) is even graceful if and only if it is \( (5+3) \)-equitable.

![Figure 4.7: Even graceful and (5+3)-equitable labeling of \( P^*_5 \)](image)

**4.5 RELATION BETWEEN EVEN-EVEN GRACEFUL LABELING AND OTHER LABELINGS**

In this section, the relation between even-even graceful labeling and other labelings such as \( E \)-cordial labeling, totally magic cordial labeling, multiplicative labeling, strongly multiplicative labeling and modular multiplicative divisor labeling are investigated.

**Definition 4.5.1.** Let \( G = (V(G), E(G)) \) with \( p \) vertices and \( q \) edges and \( f : E(G) \rightarrow \{0, 1\} \). Define \( f^* \) on by \( f^*(v) = \left( \sum f(uv) \right) \mod 2 \), sum taken over all edges incident to \( v \). The function \( f \) is called an \( E \)-cordial labeling of \( G \) if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1
and the number of edges labeled 0 and the number of edges labeled 1 differ by at most 1. A graph that admits $E$-cordial labeling is called $E$-cordial.

**Definition 4.5.2.** A graph $G$ is said to have totally magic cordial labeling with constant $C$ if there exists a mapping $f : V(G) \cup E(G) \to \{0, 1\}$ such that $f(u) + f(v) + f(uv) \equiv C \pmod{2}$ for all $uv \in E(G)$ and $|n_f(0) - n_f(1)| \leq 1$ where $n_f(i)(i = 0, 1)$ is the sum of the number of vertices and edges with label $i$.

**Definition 4.5.3.** A graph $G = (V(G), E(G))$ with $p$ vertices is said to be multiplicative if the vertices of $G$ can be labeled with distinct positive integers such that labels induced on the edges by the product of labels of end vertices are all distinct.

**Definition 4.5.4.** A graph $G$ with $p$ vertices is said to be strongly multiplicative if the vertices of $G$ can be labeled with $p$ distinct integers $1, 2, 3, \ldots, p$ such that the labels induced on the edges by the product of labels of the end vertices are all distinct.

**Definition 4.5.5.** A graph $G = (V(G), E(G))$ with $|V| = p$ is said to have a modular multiplicative divisor labeling if there exists a bijection $f : V(G) \to \{1, 2, \ldots, p\}$ and the induced function $f^* : E(G) \to \{0, 1, 2, \ldots, p - 1\}$ where $f^*(uv) = (f(u)f(v)) \pmod{p}$ such that $p$ divides the sum of all edge labels of $G$.

The following theorems bring out the relation between even-even graceful labeling and other labelings.

**Theorem 4.5.6.** Let $G$ be a perfect $m$-ary tree. If $G$ has an even-even graceful labeling then it is $E$-cordial.

**Proof:** Let $f : E(G) \to \{2, 4, 6, \ldots, 2q\}$ is even-even graceful edge labeling of the graph $G$. Let $\{0, 2, 4, 6, \ldots, 2k - 2\}$ be the set of vertices under the labeling of $f$, where $k = p = q + 1$

i.e. the set of vertex labels of a perfect $m$-ary tree is $\{0, 2, 4, 6, \ldots, 2p - 2\}$
Now, construct a new function $\psi : E(G) \rightarrow \{0, 1\}$ as follows:

$$
\psi(uv) \equiv \left\lceil \frac{f(uv)}{2} \right\rceil \pmod{2}.
$$

Clearly, $\psi^*(v) \equiv \left(\sum \psi(uv)\right) \pmod{2}$.

The labeling pattern defined above satisfies the following conditions.

(i) $\left| v_\psi^*(1) - v_\psi^*(0) \right| \leq 1$ and (ii) $\left| e_\psi^*(1) - e_\psi^*(0) \right| \leq 1$

Obviously, $e_\psi^*(1) = e_\psi^*(0) = \frac{q}{2}$

$$
v_\psi^*(1) = \frac{p+1}{2} \text{ if } \frac{p}{2} \equiv 1 \pmod{2} \text{ and } v_\psi^*(1) = \frac{p-1}{2} \text{ if } \frac{p}{2} \equiv 0 \pmod{2}.
$$

$$
v_\psi^*(0) = \frac{p-1}{2} \text{ if } \frac{p}{2} \equiv 1 \pmod{2} \text{ and } v_\psi^*(0) = \frac{p+1}{2} \text{ if } \frac{p}{2} \equiv 0 \pmod{2}.
$$

This implies that $\psi$ is an $E$-cordial labeling of the graph $G$.

Hence the even-even graceful graph $G$ has an $E$-cordial labeling.

**Illustration 4.5.7.** The Fig. 4.8 shows that the perfect 4-ary tree is $E$-cordial if it is even-even graceful.

![Figure 4.8: Even-even graceful and $E$-cordial labeling of perfect 4-ary tree](image)

**Theorem 4.5.8.** If a tree $G$ is even-even graceful then it is totally magic cordial.

**Proof.** If $G$ is even-even graceful with edge labeling $f : E(G) \rightarrow \{2, 4, 6, \ldots, 2q\}$ defined by $f^*(v) \equiv \left(\sum f(uv)\right) \pmod{2k}$.

The resulting vertex labels are $0, 2, 4, 6, \ldots, 2k - 2$, where $k = p$. 

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Hence the set of vertices of a tree are \( \{0, 2, 4, 6, \ldots, 2p - 2\} \)

**Case 1:**
Define bijective function \( \psi : V(G) \cup E(G) \to \{0, 1\} \) by
\[
\psi(v) \equiv [f^*(v) + 1] \pmod{2} \text{ for all } v \in V(G)
\]
\[
\Rightarrow \psi(v) = 1
\]
\[
\psi(uv) \equiv f(uv) \pmod{2} \text{ for all } uv \in E(G)
\]
\[
\Rightarrow \psi(uv) = 0
\]
\[
\psi(u) + \psi(uv) + \psi(v) \equiv [f^*(u) + 1 + f(uv) + f^*(v) + 1] \pmod{2}
\]
Then \( \psi(u) + \psi(uv) + \psi(v) \equiv [f^*(u) + f(uv) + f^*(v) + 2] \pmod{2} = 0 \)

Thus, the constant \( c = 0 \)

Clearly, \( n_{\psi}(0) = n_{\psi}(1) - 1 \)

Thus, \( \psi \) is a total magic cordial labeling of a graph \( G \) with \( c = 0 \)

**Case 2:**
Define a bijective function \( \psi : V(G) \cup E(G) \to \{0, 1\} \) by
\[
\psi(v) \equiv f^*(v) \pmod{2} \text{ for all } v \in V(G)
\]
\[
\Rightarrow \psi(v) = 0
\]
\[
\psi(uv) \equiv [f(uv) + 1] \pmod{2} \text{ for all } uv \in E(G)
\]
\[
\Rightarrow \psi(uv) = 1
\]
Then \( \psi(u) + \psi(uv) + \psi(v) \equiv [f^*(u) + f(uv) + 1 + f^*(v)] \pmod{2} \)
\[
\equiv [0+1+0] \pmod{2} = 1
\]
\[
\Rightarrow c = 1
\]

Clearly, \( n_{\psi}(0) = n_{\psi}(1) + 1 \)

Thus, \( \psi \) is a totally magic cordial labeling of a graph \( G \) with \( c = 1 \).

Hence a graph \( G \) has a totally magic labeling if \( G \) has an even-even graceful labeling.

**Illustration 4.5.9.** The relation between even-even graceful labeling and totally magic labeling of a tree graph is shown in the following figure.
Theorem 4.5.10. Let $G$ be an $m$-ary tree graph. If $G$ is even-even graceful then it admits multiplicative labeling.

**Proof.** Consider a graph $G$ which is an $m$-ary tree with $m$ an even integer. If $G$ is even-even graceful there exists a function $f : E(G) \to \{2, 4, 6, \ldots, 2q\}$ with induced vertex labeling defined by $f^*(v) \equiv \left(\sum f(uv)\right)(\text{mod } 2k)$. The vertex labels are $0, 2, 4, 6, \ldots, 2p - 2$.

Define $\psi$ from $V(G)$ to a set of any $p$ distinct positive integers by $\psi(v) = 2f^*(v) + 1$ for all $v \in V(G)$.

Subsequently, $\psi^*(uv) = \psi(u)\psi(v) = \left(2f^*(u) + 1\right)\left(2f^*(v) + 1\right)$

$\psi^*(uv) = 4f^*(u)f^*(v) + 2f^*(u) + 2f^*(v) + 1$

It is easy to conclude that the induced edge labels are distinct positive integers.

Hence even-even graceful $m$-ary tree is multiplicative graph.

**Illustration 4.5.11.** Fig. 4.10 shows that the 16-ary tree has multiplicative labeling if it is even-even graceful.
Theorem 4.5.12. Let $G$ be an $m$-ary tree. If $G$ has an even-even graceful labeling then it is strongly multiplicative.

Proof. Consider the $m$-ary tree $G$ with even-even graceful labeling $f : E(G) \rightarrow \{2, 4, 6, \ldots, 2q\}$ and an induced vertex labeling $f^*(v) \equiv (\sum f(uv)) \mod 2k$. The induced vertex labels are $0, 2, 4, 6, \ldots, 2p - 2$.

Consider the following labeling $\psi$ of the vertices of the graph $G$:

Consider the following labeling $\psi$ from $V$ of $G$ to $\{1, 2, 3, \ldots, p\}$ defined by $\psi(v) = \frac{f^*(v)}{2} + 1$ for all $v \in V(G)$.

Then $\psi^*(uv) = \psi(u) \psi(v) = \left[\frac{f^*(u)}{2} + 1\right]\left[\frac{f^*(v)}{2} + 1\right]$.

[since $f^*(u)$ and $f^*(v)$ are distinct and even]

$\Rightarrow \left[\frac{f^*(u)}{2} + 1\right]$ and $\left[\frac{f^*(v)}{2} + 1\right]$ are distinct positive integers.

This implies the graph $G$ admits a strongly multiplicative labeling whenever $G$ is even-even graceful.

Illustration 4.5.13. The following figure shows that the 6-ary tree admits strongly multiplicative labeling if it has even-even graceful labeling.
Figure 4.11: Even-even graceful and strongly multiplicative labeling of 6-ary tree

**Theorem 4.5.14.** Let $G$ be an $m$-ary tree. If $G$ has an even-even graceful labeling then, it admits modular multiplicative divisor labeling.

**Proof.** Let $G$ be an $m$-ary tree with $p = n$ vertices and $q = p - 1 = n - 1$ edges. Assume that $G$ has an even-even graceful labeling. Then, there exists an edge labeling $f$ of $G$, $f : E(G) \rightarrow \{2, 4, 6, \ldots, 2(p-1)\}$ with induced vertex labels $f^*(v) \equiv \left(\sum f(uv)\right) \pmod{2k}$. The resulting vertex labels are $0, 2, 4, 6, \ldots, 2p - 2$.

Now, define a bijective function $\psi : V(G) \rightarrow \{1, 2, 3, \ldots, p\}$ by

$$\psi(v) = \left\lfloor \frac{f^*(v)}{2} \right\rfloor \text{ for all } v \in V(G) - \{v_0\}, \text{ where } v_0 \text{ is the root vertex.}$$

$\psi(v_0) = p$

Then $\psi^*(uv) = \left[\psi(u) \psi(v)\right] \pmod{p}$

$$= \left\lfloor \frac{f^*(u)}{2} \right\rfloor \left\lfloor \frac{f^*(v)}{2} \right\rfloor \pmod{p}$$

$$= \left\lfloor \frac{f^*(u)f^*(v)}{4} \right\rfloor \pmod{p}$$

$\psi^*(uv_0) = \left[\psi(u) \psi(v_0)\right] \pmod{p}$

$$= \left\lfloor \frac{f^*(u)}{2} \right\rfloor \left[p\right] \pmod{p}$$

$$= \left\lfloor \frac{f^*(u)(p)}{2} \right\rfloor \pmod{p} \text{ [since } f^*(u) \text{ is even, } \left\lfloor \frac{f^*(u)(p)}{2} \right\rfloor \text{ is an integer].}$$

Therefore, $p$ divides sum of all edge labels of $G$. 

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Thus, the graph $G$ has modular multiplicative divisor labeling.
Accordingly, $G$ is modular multiplicative divisor graph if it is even-even graceful.

Illustration 4.5.15. The relation between even-even graceful labeling and modular multiplicative divisor labeling of 6-ary tree is shown in the Fig. 4.12

Figure 4.12: Even-even graceful and modular multiplicative divisor labeling of 6-ary tree.

4.6 COMPLEMENTARY EDGE-ODD GRACEFUL LABELING

Kotzig and Rosa [40] defined the complementary labeling of an edge-magic labeling. Later Kourosh Eshghi [41] has defined and discussed the concept of complementary labeling of a graceful labeling. The concept of complementary edge-odd graceful labeling is introduced in this section.

Definition 4.6.1. A $(p, q)$-graph $G$ is said to be edge-odd graceful if there is a bijection $f$ from $E(G)$ to the set $\{1, 3, \ldots, 2q - 1\}$ such that the induced mapping $f^*$ from $V(G)$ to the set $\{0, 1, 2, \ldots, (2k - 1)\}$ defined by $f^*(v) = (\sum f(uv)) \mod 2k$, sum taken over all edges incident to $v$, where $k = \max\{p, q\}$ makes all edge labels distinct.

Definition 4.6.2. If $f$ is an edge-odd graceful labeling of a graph $G = (V(G), E(G))$ with $q$ edges then, the labeling $\psi$ of $G$ defined by $\psi(x) = 2q - f(x)$ for all edges $x \in E(G)$ is said to be complementary to $f$. 
Theorem 4.6.3. If \( f \) is an edge-odd graceful labeling of a graph \( G = (V(G), E(G)) \) with \( q \) edges, then the labeling \( \psi \) of \( G \) defined by \( \psi(x) = 2q - f(x) \) for all edges \( x \in G \) is again an edge-odd graceful labeling of \( G \).

Proof. Consider the graph \( G(V,E) \) with number of vertices \( p \) and number of edges \( q \). Let \( f \) be an edge-odd graceful labeling of \( G \). Then, \( f : E(G) \rightarrow \{ 1, 3, 5, ..., 2q-1 \} \) and the induced vertex labeling \( f^*: V(G) \rightarrow \{ 0, 1, 2, ..., (2k-1) \} \) is defined by \( f^*(v) = (\sum f(uv)) \mod (2k) \), where \( k = \max \{ p, q \} \).

Now, let us consider the edge labeling \( \psi \) of the graph \( G \) defined as:
\[
\psi(e = uv) = 2q - f(e).
\]
Clearly \( \psi \) is an injective mapping from the edge set of \( G \) to \( \{ 1, 3, 5, ..., 2q-1 \} \).

Consider the vertex labeling \( \psi^* \) of the graph \( G \) defined as:
\[
\psi^*(v) = 2q - f^*(v).
\]
It is easy to verify that the labeling \( \psi^* \) is from the set \( V(G) \) to distinct positive integers.

Therefore, the function \( \psi \) provides an edge-odd graceful labeling for the graph \( G \).

Illustration 4.6.3. Edge-odd graceful labeling and complementary edge-odd graceful labeling are illustrated in the Fig. 4.13.

![Figure 4.13: Edge-odd graceful and complementary edge-odd graceful graphs](image)
4.7 CONCLUSION

This chapter demonstrated the relationship between different types of graceful labeling with suitable examples. The study established that even graceful graphs are graceful if they have an even graceful labeling whose vertex labels are all even. Also the theorems proved in this chapter established the relation between graceful labeling and other types of labeling. More specifically relationship of even-even graceful labeling with other types of labeling such as $E$-cordial labeling, totally magic cordial labeling, multiplicative labeling, strongly multiplicative labeling and multiplicative divisor labeling have been found. In addition, a new concept, complementary edge-odd graceful labeling has been introduced. In the next chapter, a more general edge graceful labeling namely, $k$-even even edge graceful labeling and its applications are discussed.