Chapter 1

Introduction

Geometric function theory is the branch of complex analysis which involves the study of geometric properties of analytic functions. Univalent functions play an important role in this study. The fundamentals of the univalent function theory emerged at the beginning of last century in the research papers of Koebe in 1907, Gronwall’s proof of area theorem in 1914, J. W. Alexander in 1915 and the second coefficient estimate of a normalized univalent function by Bieberbach in 1916. Although, the full development of the subject was possible only after 1945, as a result of the fundamental researches of Bernardi, S. D., Clunie, J., Duren, P. L., Goluzin, G. M., Libera, R. J., Littlewood, J. E., L"{O}ewner, K., MacGregor, T. H., Padmanabhan, K. S., Parvatham, R., Pinchuk, B., Ponnusamy, S., Robinson, R. M., Robertson, M. S., Rogosinski, W., Ruscheweyh, S., Schiffer, M., Schild, A., Shaffer, D. B., Shanmugam, T. N., Sheil-Small, T., Silverman, H., Singh, R., Singh, S., Gupta, S. and many others.

Their research works are reported in recently published books or monographs such as Nihari, Z. [79], Duren ([33], [34]), Goluzin [35], Gong [36], Goodman [37], Graham and Kohr [40], Hayman [42], Miller and Mocanu [74], Parvatham and Ponnusamy [92], Pommerenke [94] and Srivastava and Owa ([135], [136]), Bul-
Boaca, T. [25].

A function $f$ is said to be analytic at a point $z$ in a domain $\mathbb{D}$ in the complex plane $\mathbb{C}$ if it is differentiable not only at $z$ but also in some neighborhood of the point $z$. $f$ is said to be analytic in $\mathbb{D}$ if it is analytic at each point of $\mathbb{D}$. Further, we say that $f$ is univalent (schlicht or simple) in a domain $\mathbb{D}$ in the extended complex plane if and only if, it is analytic in $\mathbb{D}$ except for at most one simple pole and if for any two points $z_1, z_2 \in \mathbb{D}$, we have $f(z_1) = f(z_2)$ only if $z_1 = z_2$. In this case, the equation $f(z) = w$ has at most one root in $\mathbb{D}$ for any complex number $w$, further an analytic univalent function is also a conformal mapping. It immediately follows from the definition that for a function $f$ univalent in $\mathbb{D}$, $f'(z) \neq 0$ for all $z$ in $\mathbb{D}$. But the converse is not always true, for instance, for the function $f(z) = e^{2\pi z}$, $f'(z) \neq 0$ in $|z| < 1$, but it is only locally univalent.

As early as 1851, Riemann stated a remarkable result that every simply connected domain $\mathbb{D} \neq \mathbb{C}$ can be mapped conformally on to an open unit disk $\mathbb{E} = \{z : |z| < 1\}$. In view of this result, the study of univalent analytic functions on simply connected domains can be confined to the study of univalent analytic functions in the unit disk $\mathbb{E}$. Indeed, if $\mathbb{D} \neq \mathbb{C}$ is any simply connected domain then the properties of analytic univalent functions on $\mathbb{D}$ can be obtained through the Riemann mapping function which maps $\mathbb{E}$ onto $\mathbb{D}$. Therefore, we shall restrict our study to the univalent functions defined in the open unit disk $\mathbb{E}$.

Further, if $f$ is regular and univalent in $\mathbb{E}$, so is the function

$$
\frac{f(z) - f(0)}{f'(0)},
$$

since $f'(0) \neq 0$. Let $\mathcal{S}$ denote the class of all analytic univalent functions $f$ defined in the open unit disk $\mathbb{E}$ which are normalized by the conditions $f(0) = f'(0) - 1 = 0$. Thus $f \in \mathcal{S}$ has Taylor series expansion

$$
f(z) = z + a_2z^2 + a_3z^3 + \cdots.
$$
The theory of univalent functions began to take shape around the beginning of the twentieth century. In the year 1907, Koebe [54] proved the existence of an absolute constant $k \geq 0$ such that the disk $|w| < k$ is contained in the range of every function $f \in S$. A few years later, Bieberbach [11] determined the value of $k$ as 1/4. He also derived the wonderful result that $|a_2| \leq 2$ for every function $f$, $f(z) = z + a_2z^2 + \ldots$, in the class $S$. Since the equality in this result holds for $K(z) = z(1 - ze^{i\theta})^{-2}$, $\theta$ real, which is a member of the class $S$, it was natural to suspect that this function $K$ maximizes $|a_n|$ for every $n$. This led Bieberbach [11], in 1916, to propose a conjecture: “for every $f \in S$, $|a_n| \leq n$ for every $n$”. For many years, this conjecture, known as Bieberbach Conjecture, stood as a challenge to mathematicians and inspired the development of many new techniques in Complex Analysis. Finally, Branges [18] proved this conjecture to be true, in 1985, by making use of special functions.

Since the present work is mainly related to subordination between a pair of analytic functions, so we explain this concept, first.

### 1.1 Subordination

**Subordination:**

The principle of subordination was given by Lindelöf [60]. Later on, Littlewood ([61], [62]) and Rogosinski ([104], [105]) played an important role in the development of basic theory of subordination.

Let the functions $f$ and $g$ be analytic in $E_r = \{ z : |z| < r < 1 \}$. We say that $f$ is subordinate to $g$ written as $f \prec g$ in $E_r$, if there exists a Schwarz function $\phi$ in $E_r$ (i.e. $\phi$ is regular in $|z| < r$, $\phi(0) = 0$ and $|\phi(z)| \leq |z| < r$) such that

$$f(z) = g(\phi(z)) \text{ in } |z| < r.$$ 

Since $\phi(E_r) \subset \mathbb{E}_r$ and $\phi(0) = 0$, it follows from above that $f(E_r) \subset g(E_r)$ and
\( f(0) = g(0) \).

In case the function \( g \) is univalent in \( |z| < r \), the subordination becomes more useful. In this case, \( f(z) \prec g(z) \) in \( |z| < r \) is equivalent to \( f(0) = g(0) \) and \( f(|z| < r) \subset g(|z| < r) \). Some of the important consequences of subordination \( f(z) \prec g(z) \) in \( \mathbb{E}_r \) are as follows:

(i) \( |f'(0)| \leq |g'(0)| \),

(ii) \( \max_{|z| \leq r} |f(z)| \leq \max_{|z| \leq r} |g(z)| \),

(iii) \( \Re((f(z))) \leq \max_{|z| \leq r} \Re(g(z)) \), and

(iv) \( \Re((f(z))) \geq \min_{|z| \leq r} \Re(g(z)) \),

\[ 1.1.1 \text{ Certain Subclasses of Analytic Functions} \]

Let us denote the class of analytic functions in unit disk \( \mathbb{E} \) by \( \mathcal{H} \). For \( a \in \mathbb{C} \) and \( n \in \mathbb{N} \), let \( \mathcal{H}[a,n] \) be a subclass of \( \mathcal{H} \) consisting of functions of the form

\[ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots \]

Let \( \mathcal{A} \) be the class of functions \( f \), analytic in the open unit disk \( \mathbb{E} = \{ z \in \mathbb{C} : |z| < 1 \} \) and normalized by the conditions \( f(0) = f'(0) - 1 = 0 \).

\[ \text{The Class of Convex Functions:} \]

A domain \( \mathbb{D} \) in \( \mathbb{C} \) is said to be convex if the line segment joining any two points of \( \mathbb{D} \) lies entirely in \( \mathbb{D} \), i.e. for all \( z_1, z_2 \in \mathbb{D}, \ 0 \leq \lambda \leq 1, \ \lambda z_1 + (1 - \lambda)z_2 \in \mathbb{D} \). A function \( f \in \mathcal{A} \) is said to be convex if it is univalent in \( \mathbb{E} \) and \( f(\mathbb{E}) \) is a convex domain. The class of functions \( f \in \mathcal{S} \) for which \( f(\mathbb{E}) \) is convex is denoted by \( \mathcal{K} \). A necessary and sufficient condition for a function \( f \in \mathcal{A} \) to be in \( \mathcal{K} \) is that

\[ \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0, \ z \in \mathbb{E}. \]
Geometrical meaning of above condition is that $f(re^{i\theta})$ maps the circle $|z| = r < 1$ onto a simple closed contour whose tangent rotates monotonically as $\theta$ increases in the counter clockwise direction. The class $\mathcal{K}(\alpha), 0 \leq \alpha < 1$, is the class of convex functions of order $\alpha$ and is given by:

$$
\mathcal{K}(\alpha) = \left\{ f \in \mathcal{A} : \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, \ 0 \leq \alpha < 1, \ z \in \mathbb{E} \right\}.
$$

Obviously, $f \in \mathcal{K}(\alpha) \iff zf'(z) \in S^*(\alpha)$. Also it is well known (see Marx [64] and Strohhäcker [139]) that if $f \in \mathcal{K}$, then $f \in S^*(1/2)$ i.e if $f$ is convex function then it is a starlike function of order $1/2$ and $\Re \left( \frac{f(z)}{z} \right) > \frac{1}{2}$ in $\mathbb{E}$.

The Class of Starlike Functions:

A domain $\mathbb{D}$ in $\mathbb{C}$ is said to be starlike (with respect to the origin) if $z = 0 \in \mathbb{D}$ and the linear segment joining 0 to any other point of $\mathbb{D}$ lies entirely in $\mathbb{D}$, i.e. for all $z \in \mathbb{D}$, $\lambda z \in \mathbb{D}$, $0 \leq \lambda \leq 1$. A function $f \in \mathcal{A}$ is said to be starlike in the open unit disk $\mathbb{E}$ if it is univalent in $\mathbb{E}$ and $f(\mathbb{E})$ is a starlike domain. A necessary and sufficient condition for the members of class $\mathcal{A}$ to be starlike is that

$$
\Re \left( \frac{zf'(z)}{f(z)} \right) > 0, \ z \in \mathbb{E}.
$$

(1.1)

The geometrical interpretation of the above condition is that for each fixed $r$, $0 < r < 1$, the functional $\arg f(re^{i\theta})$ strictly increases with $\theta$, $0 \leq \theta < 2\pi$. Let $S^*$ denote the subclass of $S$ consisting of all univalent starlike functions w.r.t. the origin. The class $S^*(\alpha), 0 \leq \alpha < 1$, is the class of starlike functions of order $\alpha$, and is given by

$$
S^*(\alpha) = \left\{ f \in \mathcal{A} : \Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha, \ z \in \mathbb{E} \right\}.
$$

Clearly $S^*(\alpha)$ is a subclass of $S^*$. For $\alpha < 0$, the functions in $S^*(\alpha)$ need not be univalent in $\mathbb{E}$. It follows from the above definition that $S^*(\beta) \subseteq S^*(\alpha)$ for $\alpha \leq \beta$. The concept of order of starlikeness was introduced by Robertson [103].
The Class of Strongly Starlike Functions of order \( \alpha \):

A function \( f \in \mathcal{A} \) is said to be strongly starlike of order \( \alpha \), \( 0 < \alpha \leq 1 \), if

\[
\left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\alpha \pi}{2}, \ z \in \mathbb{E},
\]

or equivalently

\[
\frac{zf'(z)}{f(z)} < \left( \frac{1+z}{1-z} \right)^\alpha, \ z \in \mathbb{E}.
\]

We denote the class of all such functions by \( \tilde{S}(\alpha) \). Note that by \( \tilde{S}(1) \equiv S^* \). For \( 0 < \alpha < 1 \), \( \tilde{S}(\alpha) \) consists only of bounded starlike functions and therefore, the inclusion \( \tilde{S}(\alpha) \subset S^* \) is proper. The class \( \tilde{S}(\alpha) \) was introduced and studied independently by Brannan and Kirwan [19] and Stankiewicz [138].

The Class of Close-to-Convex Functions:

The class of close-to-convex functions is the generalization of the class of starlike functions and was introduced by Kaplan [51].

A function \( f \in \mathcal{A} \) is said to be close-to-convex in \( \mathbb{E} \) if there exists a real number \( \alpha \), \( -\frac{\pi}{2} < \alpha < \frac{\pi}{2} \) and a convex function \( g \) (not necessarily normalized) such that

\[
\Re \left( e^{i\alpha} \frac{f''(z)}{g'(z)} \right) > 0, \ z \in \mathbb{E}.
\]  

(1.2)

In addition, if \( g \) is normalized by the conditions \( g(0) = 0 = g'(0) - 1 \), then the class of close-to-convex functions is denoted by \( C \). It is well-known that every close-to-convex function is univalent. Therefore, we have \( \mathcal{K} \subset S^* \subset C \subset S \). Kaplan [51] and Sakaguchi[110], independently proved that \( f \in C \) even if

\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > -\frac{1}{2}, \ z \in \mathbb{E}.
\]

Taking \( \alpha = 0 \) and \( g(z) = z \) in (1.2), it immediately follows that \( f \in C \) if \( \Re(f'(z)) > 0 \) in \( \mathbb{E} \). This simple and beautiful result was independently proved by Noshiro [82].
and Warchawski [149] in 1934/35.

The Class of $\phi-$ like Functions:

Another generalized class of starlike functions is the class of $\phi-$ like functions. Let $\phi$ be analytic in a domain containing $f(\mathbb{E})$, $\phi(0) = 0$ and $\Re(\phi'(0)) > 0$. Then, the function $f \in \mathcal{A}$ is said to be $\phi-$like in $\mathbb{E}$ if

$$\Re \left( \frac{zf'(z)}{\phi(f(z))} \right) > 0, \quad z \in \mathbb{E}.$$ 

This concept was introduced by Brickman [20]. He proved that an analytic function $f \in \mathcal{A}$ is univalent if and only if $f$ is $\phi-$like for some analytic function $\phi$. Later, Ruscheweyh [109] investigated the following general class of $\phi-$like functions:

Let $\phi$ be analytic in a domain containing $f(\mathbb{E})$, where $\phi(0) = 0$, $\phi'(0) = 1$ and $\phi(w) \neq 0$ for some $w \in f(\mathbb{E}) \setminus \{0\}$. Let $q$ be a fixed analytic function in $\mathbb{E}$, $q(0) = 1$. Then the function $f \in \mathcal{A}$ is called $\phi-$like with respect to $q$, if

$$\frac{zf'(z)}{\phi(f(z))} < q(z), \quad z \in \mathbb{E}.$$ 

The Classes of Uniformly Convex and Uniformly Starlike Functions:

The concept of uniformly convex and starlike functions was introduced by Goodman ([38], [39]) in 1991. A function $f \in \mathcal{A}$ is said to be uniformly convex if $f$ is convex and has the property that for every circular arc $\gamma$ contained in $\mathbb{E}$, with center $\zeta \in \mathbb{E}$, the arc $f(\gamma)$ is convex. Similarly the function $f \in \mathcal{A}$ is uniformly starlike if $f$ is starlike and has the property that for every circular arc $\gamma$ contained in $\mathbb{E}$, with center $\zeta \in \mathbb{E}$, the arc $f(\gamma)$ is a starlike arc. The classes of functions consisting of uniformly convex and uniformly starlike functions are denoted by UCV and UST respectively.

The following analytic characterization of UCV and UST are obtained by Goodman ([38], [39]). The class UCV of uniformly convex functions consists of functions

$$\frac{zf'(z)}{\phi(f(z))} < q(z), \quad z \in \mathbb{E}.$$
\( f \in \mathcal{A} \) satisfying the condition

\[
\Re \left\{ 1 - (z - \zeta) \frac{f''(z)}{f'(z)} \right\} \geq 0, \text{ for } z, \zeta \in \mathbb{E}.
\]

The class UST of uniformly starlike functions consists of functions \( f \in \mathcal{A} \) satisfying the condition

\[
\Re \left\{ \frac{(z - \zeta) f'(z)}{f(z) - f(\zeta)} \right\} \geq 0, \text{ for } z, \zeta \in \mathbb{E}.
\]

Goodman stated that the class UCV is preserved under the transformation \( e^{-i\alpha} f(e^{i\alpha} z) \) and no other transformation seems to be available.

Ronning [106] and Ma and Minda [65], studied the class of uniformly convex functions and obtained an interesting criterion in one variable for \( f \in \mathcal{A} \) to be uniformly convex in \( \mathbb{E} \). They proved that a function \( f \in \mathcal{A} \), is uniformly convex if and only if

\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \frac{|zf''(z)|}{|f'(z)|}, \text{ } z \in \mathbb{E}.
\]

The Class of Parabolic Starlike Functions:

The class of functions \( f \in \mathcal{A} \) such that \( f(z) = zF'(z) \) where \( F \) is uniformly convex, is denoted by \( S_P \), known as the class of parabolic starlike functions and is given as follows:

A function \( f \in \mathcal{A} \) is called parabolic starlike if

\[
\Re \left( \frac{zf'(z)}{f(z)} \right) > \frac{|zf'(z)|}{|f(z)|} - 1, \text{ } z \in \mathbb{E}.
\]

The Class of Uniformly Close-to-Convex Functions:

A function \( f \in \mathcal{A} \) is said to be uniformly close-to-convex in \( \mathbb{E} \), if

\[
\Re \left( \frac{zf'(z)}{g(z)} \right) > \frac{|zf'(z)|}{|g(z)|} - 1, \text{ } z \in \mathbb{E},
\]

for some \( g \in S_P \). Let \( UCC \) denote the class of all such functions.
The Class of Parabolic $\phi-$ like Functions:

A function $f \in \mathcal{A}$ is said to be parabolic $\phi-$like in $\mathbb{E}$, if

$$\Re \left( \frac{zf''(z)}{\phi(f(z))} \right) > \frac{|zf'(z)|}{\phi(f(z))} - 1, \quad z \in \mathbb{E}. \quad (1.3)$$

Multivalent Functions:

Let $f$ be a function analytic in the open unit disk $\mathbb{E}$. If the equation $f(z) = w$ does not have more than $p$-solutions in $\mathbb{E}$ and there exists some $w$ for which this equation has exactly $p$ solutions, then $f$ is said to be $p$-valent in $\mathbb{E}$.

The class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad (p \in \mathbb{N} \setminus \{1\}; \ z \in \mathbb{E}),$$

is denoted by $\mathcal{A}_p$. Clearly, the members of class $\mathcal{A}_p$ are analytic and $p$-valent in the open unit disk $\mathbb{E}$. For detailed study of these functions, we refer to the book by Hayman [42].

Some useful characterizations for the members of class $\mathcal{A}_p$ to be $p$-valent starlike of order $\alpha$, $p$-valent convex of order $\alpha$, $p$-valent close-to-convex, uniformly $p$-valent convex, parabolic $p$-valent starlike and uniformly $p$-valent close-to-convex are given below:

A function $f \in \mathcal{A}_p$, is said to be $p$-valent starlike of order $\alpha$ ($0 \leq \alpha < p$) in $\mathbb{E}$, if it satisfies the condition

$$\Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha, \quad z \in \mathbb{E},$$

and the class of such functions is denoted by $S^*_p(\alpha)$. We write $S^*_p(0) = S^*_p$ — the class of $p$-valent starlike functions.

A function $f \in \mathcal{A}_p$, is said to be $p$-valent convex of order $\alpha$ ($0 \leq \alpha < p$) in $\mathbb{E}$, if it satisfies the condition

$$\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, \quad z \in \mathbb{E},$$
and the class of such functions is denoted as $K_p(\alpha)$. In particular, $K_p(0) = K_p$ — the class of p-valent convex functions.

A function $f \in \mathcal{A}_p$ is said to be p-valent close-to-convex in $E$, if it satisfies

$$\mathbb{R}\left(\frac{zf'(z)}{g(z)}\right) > 0, \ z \in \mathbb{E}, \quad (1.4)$$

for some $g \in S_p^*$. Let the class of such functions be denoted by $C_p$. Note that $C_1 = C$.

Selecting $g(z) \equiv z^p \in S_p^*$. Therefore, condition (1.4), becomes

$$\mathbb{R}\left(\frac{f'(z)}{z^p-1}\right) > 0, \ z \in \mathbb{E}. \quad (1.5)$$

Hence, a function $f \in \mathcal{A}_p$ is said to be p-valent close-to-convex in $E$, if it satisfies condition (1.5).

A function $f \in \mathcal{A}_p$ is said to be uniformly p-valent convex in $E$, if it satisfies the condition

$$\mathbb{R}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)} - (p - 1)\right|, \ z \in \mathbb{E},$$

and denote the class of such functions by $UCV_p$.

A function $f \in \mathcal{A}_p$ is said to be parabolic p-valent starlike in $E$, if it satisfies the condition

$$\mathbb{R}\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - p\right|, \ z \in \mathbb{E},$$

and the class of such functions is denoted by $S_p^p$.

A function $f \in \mathcal{A}_p$ is said to be uniformly p-valent close-to-convex in $E$, if it satisfies the condition

$$\mathbb{R}\left(\frac{zf'(z)}{g(z)}\right) > \left|\frac{zf'(z)}{g(z)} - p\right|, \ z \in \mathbb{E},$$

for some $g \in S_p^p$ and the class of such functions is denoted by $UCC_p$. 
1.2 Methodology

In the theory of geometric functions, the introduction of the concept of differential subordination has made a very significant change. The old techniques used in geometric function theory to prove various results are lengthy and cumbersome, whereas the differential subordination technique provides very simple proofs for the results which were proved by using some difficult technique. The technique of differential subordination came into existence with the wonderful article “Differential Subordination and Univalent Functions” by Miller and Mocanu [69] in 1981. Since then this area has been expanded tremendously. Hundreds of papers in the field of Geometric Function Theory have appeared where the technique of differential subordination has been used successfully. The applications of differential subordination have been developed in various other fields like differential equations, partial differential equations, harmonic functions, meromorphic functions, Banach spaces, integral operators and functions of several complex variables.

Differential Subordination:

Let $\Phi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be an analytic function, $p$ an analytic function in $\mathbb{E}$ with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$ and $h$ be univalent in $\mathbb{E}$. Then the function $p$ is said to satisfy first order differential subordination if

$$\Phi(p(z), zp'(z); z) \prec h(z), \Phi(p(0), 0; 0) = h(0). \quad (1.6)$$

A univalent function $q$ is called a dominant of the differential subordination $(1.6)$ if $p(0) = q(0)$ and $p \prec q$ for all $p$ satisfying $(1.6)$. A dominant $\bar{q}$ that satisfies $\bar{q} \prec q$ for all dominants $q$ of $(1.6)$, is said to be the best dominant of $(1.6)$. The best dominant is unique up to a rotation of $\mathbb{E}$.

The study of differential subordination can be classified as under:
(i) Given univalent functions \( h \) and \( q \), find conditions on \( \Phi \) such that (1.6) implies that \( p \preceq q \).

(ii) Given the differential subordination in (1.6), find a dominant \( q \) or a best dominant \( \tilde{q} \).

(iii) Given \( \Phi \) and dominant \( q \), find the largest class of univalent functions \( h \) such that (1.6) implies that \( p \preceq q \).

For the thorough study of differential subordination, we refer to the book entitled “Differential Subordinations - Theory and Applications”, authored by Miller and Mocanu [74].

Following result will be used in the course of our study involving differential subordination.

**Lemma 1.2.1.** (Miller and Mocanu [74, p.132]) Let \( q \) be univalent in \( \mathbb{E} \) and let \( \Theta \) and \( \Phi \) be analytic in a domain \( \mathbb{D} \) containing \( q(\mathbb{E}) \), with \( \Phi(w) \neq 0 \), when \( w \in q(\mathbb{E}) \).

Set \( Q(z) = zq'(z)\Phi[q(z)], h(z) = \Theta[q(z)] + Q(z) \) and suppose that either

(i) \( h \) is convex, or

(ii) \( Q \) is starlike.

In addition, assume that

(iii) \( \Re\left(\frac{zh'(z)}{Q(z)}\right) > 0 \) for all \( z \) in \( \mathbb{E} \).

If \( p \) is analytic in \( \mathbb{E} \), with \( p(0) = q(0) \), \( p(\mathbb{E}) \subset \mathbb{D} \) and

\[
\Theta[p(z)] +zp'(z)\Phi[p(z)] \prec \Theta[q(z)] + zq'(z)\Phi[q(z)], \ z \in \mathbb{E},
\]

then \( p(z) \preceq q(z) \) and \( q \) is the best dominant.

### 1.3 Survey of Literature

In this course of study, we have studied various classes of analytic functions for finding the results on uniform convexity, parabolic starlikeness, uniform close-
to-convexity, parabolic $\phi$–likeness, convexity, starlikeness, close-to-convexity and $\phi$–likeness of analytic functions. The technique of differential subordination has played an important role in the course of our study. The major contributions in the technique of differential subordination are due to Miller and Mocanu. For their contributions, we refer to ([69], [70], [71], [72], [73], [74], [75], [76]).

In the present work, we have successfully used the technique of differential subordination to obtain the results on uniform convexity and parabolic starlikeness of analytic functions. The concept of uniform convexity and uniform starlikeness for the members of class $\mathcal{A}$ was introduced by Goodman in 1991. Later on, Ronning, Ma and Minda independently obtained the more analytic criteria for normalized analytic functions to be uniformly convex and starlike. For more contributions to the development of uniformly convex / parabolic starlike / uniformly close-to-convex / parabolic $\phi$–like / convex / starlike / close-to-convex / $\phi$–like functions, we refer to Goodman ([38], [39]), Ronning ([106], [107], [108]), Kanas et al. ([48], [49], [50]), Ma and Minda ([65], [66], [67]), Ali et al. ([4], [5], [6]), Kim ([53]), Shanmugam [115], Ravichandran et al. ([98], [100]), Nezhmetdinov ([80], [81]), Owa [88], Padmanabhan [90], Subramanian et al. ([140], [141]), Bharti et al. [10], Brown [21], Merkes, Salmassi [68], Xu, Yang [150], Kharsani, Hajiry ([7], [8]), Aghalary et al. ([2], [3]), Dorin [32], Sham et al. [112], Singh et al. ([118], [119], [120], [121], [122], [123], [124], [125], [126], [127], [128], [129], [130], [131], [132]) etc.

A few years back, the concept of differential superordination was also introduced by Miller and Mocanu in their papers ([74], [75]). Later on, Bulboaca contributed significantly for the development of the concept of differential superordination. For his contributions, we refer to ([22], [23], [24], [25]).
## 1.4 Synopsis of the Work Done

The present work is devoted to the study of certain subclasses of univalent as well as multivalent functions defined on the unit disk \( \mathbb{E} = \{ z; |z| < 1 \} \). An attempt has been made to explore certain classes of analytic functions for obtaining the results regarding uniform convexity, parabolic starlikeness, uniform close-to-convexity, parabolic \( \phi \)-likeness, convexity, starlikeness, close-to-convexity and \( \phi \)-likeness.

In the course of this study, new results on above mentioned geometric characterization have been obtained and certain existing results pertaining to some well-known subclasses of univalent and multivalent functions have been improved. A brief chapter-wise description of the work done is given below:

**Chapter 2:**

In the literature of univalent function theory, the operators \( f'(z), \frac{zf''(z)}{f'(z)} \) and \( 1 + \frac{zf''(z)}{f'(z)} \) play an important role to find sufficient conditions for starlikeness, convexity and close-to-convexity. Various classes involving the combinations of above differential operators have been introduced and studied in literature by different authors. The main objective of this chapter is to study them for parabolic starlikeness, uniform convexity and uniform close-to-convexity.

For \( f \in \mathcal{A} \), define differential operator \( J(\alpha; f) \) as follows:

\[
J(\alpha; f)(z) = (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right), \quad \alpha \in \mathbb{R}.
\]

In 1973, Miller et al. [77] studied the class \( \mathcal{M}_\alpha \) (known as the class of \( \alpha \)-convex functions) defined as follows:

\[
\mathcal{M}_\alpha = \left\{ f \in \mathcal{A} : \Re[J(\alpha; f)(z)] > 0, \; z \in \mathbb{E} \right\}.
\]

They proved that if \( f \in \mathcal{M}_\alpha \), then \( f \) is starlike in \( \mathbb{E} \). In 1976, Lewandowski et al.
[57] studied the class consisting of functions $f \in \mathcal{A}$ which satisfy

$$\Re \left\{ \frac{zf''(z)}{f'(z)} \left(1 + \frac{zf''(z)}{f'(z)}\right) \right\} > 0, \quad z \in \mathbb{E}. $$

Further, Silverman [117] defined the class $\mathcal{G}_b$ by taking quotient of operators $1 + \frac{zf''(z)}{f'(z)}$ and $\frac{zf'(z)}{f(z)}$:

$$\mathcal{G}_b = \left\{ f \in \mathcal{A} : \left| 1 + \frac{zf''(z)/f'(z)}{zf'(z)/f(z)} - 1 \right| < b, \quad z \in \mathbb{E} \right\}. $$

The class $\mathcal{G}_b$ had been studied by Tuneski ([143], [144], [145], [146], [147]).

All above mentioned differential operators have been studied in literature for obtaining the starlikeness of analytic functions. In this chapter, we study them for parabolic starlikeness and uniform close-to-convexity of analytic functions. Here, in each section, we present subordination theorem and its applications to univalent and multivalent functions. The improved version of the result of Billing et al. [17] shall follow as a special case of our main result given in the closing section of the chapter. Using Mathematica 7.0, we shall illustrate our claims pictorially. Besides many other results, we, in particular, obtain the following results:

1. Suppose $\alpha$ be a positive real number and if $f \in \mathcal{A}_p$ satisfies

$$\frac{zf'(z)}{f(z)} \left(1 + \alpha \frac{zf''(z)}{f'(z)}\right) < (1 - \alpha)p \left\{ 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2 \right\}$$

$$+ \alpha p^2 \left\{ 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2 \right\}^2 + \alpha p \frac{4\sqrt{z}}{\pi^2(1-z)} \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right), \quad z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{pf(z)} < 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2$$

i.e. $f \in S_p^p$.

2. Suppose $\alpha \in \mathbb{R} \setminus [0, 1)$ and if $f \in \mathcal{A}_p$ satisfies

$$1 + \alpha \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} < (1 - \alpha) \left\{ 1 - p - \frac{2p}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2 \right\}$$
\[
\frac{zf'(z)}{pf(z)} \prec 1 + \frac{2}{\pi z} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2, \quad z \in E, \text{ i.e. } f \in \mathcal{S}_p.
\]

3. Let \( \alpha \) be a positive real number. If \( f \in \mathcal{A} \) satisfies

\[
(1 - \alpha)zf'(z) + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \prec 1 + \frac{2}{\pi z} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2
\]

\[
+ \alpha \left\{ \frac{4}{\pi^2} \frac{\sqrt{z}}{1 - z} \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right\}, \quad z \in E,
\]

then

\[
\frac{zf'(z)}{f(z)} \prec 1 + \frac{2}{\pi z} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2, \quad z \in E, \text{ i.e. } f \in \mathcal{S}_P.
\]

4. If \( f \in \mathcal{A} \), satisfies the differential subordination

\[
(1 - \alpha)f'(z) + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \prec (1 - \alpha) \left\{ 1 + \frac{2}{\pi z} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2 \right\}
\]

\[
+ \alpha \left\{ 1 - \frac{4}{\pi^2} \frac{\sqrt{z}}{1 - z} \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right\}, \quad z \in E,
\]

for \( 0 < \alpha \leq 1 \), then

\[
f'(z) \prec 1 + \frac{2}{\pi z} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2, \quad z \in E \text{ i.e. } f \in \mathcal{UCC}.
\]

Chapter 3:

The work of this chapter is motivated by the work of Irmak et al. [44]. In 2003, Irmak et al. [44] studied the class \( T_\lambda(\alpha) \) consisting of functions \( f \in \mathcal{A} \) satisfying the following condition

\[
\frac{zf'(z) + \lambda z^2 f''(z)}{(1 - \lambda)f(z) + \lambda z f'(z)} \prec 1 + (1 - \alpha)z, \quad 0 \leq \alpha < 1, \quad z \in \mathbb{E},
\]

16
and obtained certain conditions for \( f \in \mathcal{A} \) to be a member of class \( T_{\lambda}(\alpha) \) and consequently, they get some sufficient conditions for starlike and convex functions.

Let \( S(\lambda) \) denote the class of functions \( f \in \mathcal{A} \) for which

\[
\Re \left( \frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} \right) > \frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} - 1, \quad 0 \leq \lambda \leq 1, \quad z \in \mathbb{E}.
\]

Clearly, \( S(0) \) and \( S(1) \) are usual classes \( S_P \) and UCV respectively.

In this chapter, we find three sufficient conditions for \( f \in \mathcal{A} \) to be a member of the class \( S(\lambda) \). As a consequence, we obtain certain criteria for uniform convexity and parabolic starlikeness. In particular, we get following results:

1. Let \( \beta \) be a positive real number. If \( f \in \mathcal{A} \) satisfies

\[
(1-\beta) \left[ \frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} \right] + \beta \left[ \frac{zf'(z) + (1+2\lambda)z f''(z) + \lambda z^2 f'''(z)}{f'(z) + \lambda z f''(z)} \right] < 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1+\sqrt{z}}{1-\sqrt{z}} \right) \right)^2 + \beta \left[ \frac{4\sqrt{z}}{\pi^2(1-z)} \log \left( \frac{1+\sqrt{z}}{1-\sqrt{z}} \right) \right]^2, \quad z \in \mathbb{E},
\]

then \( f \in S(\lambda) \), \( 0 \leq \lambda \leq 1 \).

2. If \( f \in \mathcal{A} \), satisfies the differential subordination

\[
\alpha \left[ \frac{(1+2\lambda)z f''(z) + f'(z) + \lambda z^2 f'''(z)}{f'(z) + \lambda z f''(z)} \right] - \frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} < \beta \left[ 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1+\sqrt{z}}{1-\sqrt{z}} \right) \right)^2 \right] + \gamma \left[ 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1+\sqrt{z}}{1-\sqrt{z}} \right) \right)^2 \right] + \frac{4\alpha \sqrt{z}}{\pi^2(1-z)} \log \left( \frac{1+\sqrt{z}}{1-\sqrt{z}} \right),
\]

where \( \alpha, \beta, \gamma, \lambda \in \mathbb{R} \) such that \( \alpha \neq 0, \frac{\beta}{\alpha} > 0, \frac{\gamma}{\alpha} > 0, 0 \leq \lambda \leq 1 \) then

\[
\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} < 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1+\sqrt{z}}{1-\sqrt{z}} \right) \right)^2, \quad z \in \mathbb{E}.
\]
3. If $f \in \mathcal{A}$, satisfies the differential subordination

$$\beta \left( \frac{zf'(z) + \lambda z^2 f''(z)}{(1 - \lambda) f(z) + \lambda z f'(z)} \right) + \alpha \left\{ \frac{(1 + 2\lambda)zf''(z) + f'(z) + \lambda z^2 f'''(z)}{f'(z) + \lambda z f''(z)} \right\} - \alpha \left\{ \frac{zf'(z) + \lambda z^2 f''(z)}{(1 - \lambda) f(z) + \lambda z f'(z)} \right\} \prec \beta \left\{ 1 + 2 \frac{\log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^2}{\pi^2} + \frac{4q \sqrt{z}}{\pi^2} \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right\} \lambda \phi(z),$$

where $\alpha, \beta, \lambda \in \mathbb{R}$ such that $\alpha \neq 0, \frac{\beta}{\alpha} > 0, 0 \leq \lambda \leq 1$ then

$$\frac{zf'(z) + \lambda z^2 f''(z)}{(1 - \lambda) f(z) + \lambda z f'(z)} \prec 1 + 2 \frac{\log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^2}{\pi^2}, \quad z \in \mathbb{E}.$$

Chapter 4:

The concept of $\phi$–like functions was introduced by Brickman [20]. The concept of $\phi$–likeness is a generalization of the concept of starlikeness. Ruscheweyh [109] also contributed to study of $\phi$–like functions. In 2005, Ravichandran et al. [101] proved the following results for $\phi$–like functions.

Let $\alpha \neq 0$ be a complex number and $q$ be convex univalent in $\mathbb{E}$. Define $h(z) = \alpha q^2(z) + (1 - \alpha) q(z) + \alpha z q'(z)$. Further assume that

$$\Re \left\{ \frac{1 - \alpha}{\alpha} + 2q(z) + \left( 1 + \frac{zq''(z)}{q'(z)} \right) \right\} > 0, \quad z \in \mathbb{E}.$$  

If $f \in \mathcal{A}$ satisfies

$$\frac{zf'(z)}{\phi(f(z))} \left[ 1 + \alpha z \frac{f''(z)}{f'(z)} + \alpha \frac{zf'(z) - \{\phi(f(z))\}'}{\phi(f(z))} \right] \prec h(z),$$

then

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z), \quad z \in \mathbb{E} \text{ and } q \text{ is the best dominant.}$$

Let $\alpha \neq 0$ be any complex number and $\beta = \max \left\{ 0, -\Re \frac{1}{\alpha} \right\}$. Let $q(z) \neq 0$ be analytic in $\mathbb{E}$ and $Q(z) = zq'(z)q^{\frac{1}{\alpha} - 1}(z)$ be starlike of order $\beta$ in $\mathbb{E}$. If $f \in \mathcal{A}$ satisfies

$$\frac{zf'(z)}{\phi(f(z))} \left( 1 + \frac{z f''(z)}{f'(z)} - \frac{\phi'(f(z))}{\phi(f(z))} z f'(z) \right)^{\alpha} \prec q(z) \left( 1 + \frac{zq'(z)}{q(z)} \right)^{\alpha},$$

18
then
\[ \frac{zf'(z)}{\phi(f(z))} < q(z), \; z \in \mathbb{E} \text{ and } q \text{ is the best dominant.} \]

Later, Shanmugham et al. [116] obtained the following result.

Let \( q(z) \neq 0 \) be analytic and univalent in \( \mathbb{E} \) with \( q(0) = 1 \) such that \( \frac{zq'(z)}{q(z)} \) is starlike univalent in \( \mathbb{E} \). Let \( q \) satisfy
\[
\Re \left[ 1 + \frac{zq(z)}{q(z)} - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q(z)} \right] \geq 0.
\]

Let for \( f, g \in \mathcal{A} \)
\[
\Psi(\alpha, \gamma, g; z) := \alpha \left\{ \frac{z(f * g)'(z)}{\Phi(f * g)(z)} \right\} + \gamma \left\{ 1 + \frac{z(f * g)''(z)}{(f * g)'(z)} - \frac{z(\Phi(f * g)(z))'}{\Phi(f * g)(z)} \right\}.
\]

If \( q \) satisfies
\[
\Psi(\alpha, \gamma, g; z) \prec \alpha q(z) + \gamma \frac{zq'(z)}{q(z)},
\]
then
\[
\frac{z(f * g)'(z)}{\Phi(f * g)(z)} < q(z), \; z \in \mathbb{E},
\]
and \( q \) is the best dominant.

Many other authors who also contributed to the study of \( \phi \)-like functions are Cho and Kim [26], Gupta et al. [41], Shanmugham et al. ([113], [114]), Ibrahim and Darus [43], Singh et al. [128] etc.

The primary objective of this chapter to study the similar type of results as mentioned above. We, here, find certain results as sufficient conditions for \( \phi \)-like functions with respect to a univalent function \( q \). We have also derived a subordination theorem involving the convolution of analytic functions. As special cases of our main results, we obtain the sufficient conditions for analytic functions to be parabolic \( \phi \)-like, parabolic starlike, \( \phi \)-like and starlike. In particular we derive following results:
1. Assume \( \alpha, \beta \) and \( \gamma \) are real numbers such that \( -\frac{3}{4} < \frac{\beta}{\gamma} < \frac{3}{2} \), and \( 0 < \alpha \leq 1 \). If \( f \in \mathcal{A} \) satisfies

\[
\left[ \frac{zf'(z)}{\phi(f(z))} \right]^\beta \left[ (1 - \alpha) \frac{zf''(z)}{\phi(f(z))} + \alpha \left\{ 1 + \frac{zf''(z)}{f'(z)} - z\left(\frac{\phi(f(z))'}{\phi(f(z))}\right)' \right\} \right]^\gamma
\]

\[
< \left\{ 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2 \right\}^\beta
\]

\[
\left\{ (1 - \alpha) \left[ 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2 \right] + \frac{4\sqrt{z}}{\pi^2(1-z)} \frac{\log \left( \frac{1+\sqrt{z}}{1-\sqrt{z}} \right)}{1 + \frac{2}{\pi^2} \left( \log \left( \frac{1+\sqrt{z}}{1-\sqrt{z}} \right) \right)^2} \right\}^\gamma,
\]

then

\[
\frac{zf'(z)}{\phi(f(z))} < 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2, \quad z \in \mathbb{E},
\]
i.e. \( f \) is parabolic \( \phi \)-like in \( \mathbb{E} \).

2. For a real number \( \alpha \), \( 0 < \alpha \leq 1 \). If \( f \in \mathcal{A} \) satisfies

\[
(1 - \alpha) \frac{zf'(z)}{\phi(f(z))} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} - z\left(\frac{\phi(f(z))'}{\phi(f(z))}\right)' \right)
\]

\[
< (1 - \alpha) \left[ 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2 \right] + \frac{4\sqrt{z}}{\pi^2(1-z)} \frac{\log \left( \frac{1+\sqrt{z}}{1-\sqrt{z}} \right)}{1 + \frac{2}{\pi^2} \left( \log \left( \frac{1+\sqrt{z}}{1-\sqrt{z}} \right) \right)^2},
\]

then

\[
\frac{zf'(z)}{\phi(f(z))} < 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2, \quad z \in \mathbb{E},
\]
i.e. \( f \) is parabolic \( \phi \)-like in \( \mathbb{E} \).

3. Suppose \( \alpha \) is a real number such that \( 0 < \alpha \leq 1 \). If \( f \in \mathcal{A} \) satisfies

\[
(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} - z\left(\frac{\phi(f(z))'}{\phi(f(z))}\right)' \right)
\]

\[
< (1 - \alpha) \left[ 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2 \right] + \frac{4\sqrt{z}}{\pi^2(1-z)} \frac{\log \left( \frac{1+\sqrt{z}}{1-\sqrt{z}} \right)}{1 + \frac{2}{\pi^2} \left( \log \left( \frac{1+\sqrt{z}}{1-\sqrt{z}} \right) \right)^2},
\]

\[
20
\]
then
\[
\frac{zf'(z)}{f(z)} < 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2, \quad z \in \mathbb{E},
\]
i.e. \( f \) is parabolic starlike in \( \mathbb{E} \).

Chapter 5:
For \( f \in \mathcal{A}_p \), we define the multiplier transformation \( I_p(n, \lambda) \) as
\[
I_p(n, \lambda)[f](z) = z^n + \sum_{k=p+1}^{\infty} \left( \frac{k + \lambda}{p + \lambda} \right)^n a_k z^k, \text{ where } \lambda \geq 0, \ n \in \mathbb{Z}.
\]
The operator \( I_1(n, 0) \) is the well-known Sălăgean [111] derivative operator \( D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \) and \( f \in \mathcal{A} \). In 1992, Uralegaddi and Somanatha [148] investigated the operator \( I_1(n, 1) \). In 1993, Jung et al. [47] studied the transformation
\[
I_1(-1, \lambda)[f](z) = z + \sum_{k=2}^{\infty} \left( \frac{1 + \lambda}{k + \lambda} \right) a_k z^k, \text{ where } \lambda > -1, \ f \in \mathcal{A}.
\]
In 2003, Cho and Srivastava [30] and Cho and Kim [27] investigated the operator \( I_1(n, \lambda) \). In 2005, Aghalary et al. [1] studied the operator \( I_p(n, \lambda) \) and obtained certain sufficient conditions for starlike and convex functions. For \( f \in \mathcal{A}_p \), we define a class \( S_n(p, \lambda) \) consisting of functions which satisfy
\[
\Re \left( \frac{I_p(n+1, \lambda)[f](z)}{I_p(n, \lambda)[f](z)} \right) > \left| \frac{I_p(n+1, \lambda)[f](z)}{I_p(n, \lambda)[f](z)} - p \right|, \quad z \in \mathbb{E},
\]
where \( \lambda \geq 0, \ n \in \mathbb{N}_0 \). Note that \( S_0(p, 0) = S_p^p \) and \( S_1(p, 0) = UCV_p \).

In Section 5.2 of this chapter, we obtain sufficient condition for a function \( f \in \mathcal{A}_p \) to be a member of class \( S_n(p, \lambda) \). As consequences of this result, we obtain sufficient conditions for parabolic starlikeness, starlikeness, uniform convexity, convexity, uniform close-to-convexity and close-to-convexity of multivalent/univalent analytic functions. In the closing section 5.5, we present a subordination theorem.
that gives sufficient condition for uniform close-to-convexity of multivalent analytic functions. In particular, we obtain the following results:

1. Let \( \gamma \) and \( \beta \) be real numbers such that \( \frac{3}{4} < \frac{\gamma}{\beta} < \frac{3}{2} \). If \( f \in \mathcal{A}_p \) satisfies

\[
\left( \frac{I_p(n+1, \lambda)[f](z)}{I_p(n, \lambda)[f](z)} \right)^\gamma \left( \frac{I_p(n+2, \lambda)[f](z)}{I_p(n+1, \lambda)[f](z)} \right)^\beta < \left\{ 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2 \right\}^\gamma
\]

\[
1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2 + \frac{4\sqrt{z}}{\pi^2(1-z)} \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^2
\]

\[, \quad z \in \mathbb{E}, \]

then \( f \in S_n(p, \lambda) \), where \( \lambda \geq 0, n \in \mathbb{N}_0 \).

2. Let \( \alpha \) be a positive real number. If \( f \in \mathcal{A}_p \) satisfies

\[
(1 - \alpha) \left( \frac{I_p(n+1, \lambda)[f](z)}{I_p(n, \lambda)[f](z)} \right) + \alpha \left( \frac{I_p(n+2, \lambda)[f](z)}{I_p(n+1, \lambda)[f](z)} \right) < 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2
\]

\[
+ \frac{\alpha}{p + \lambda} \left\{ \frac{4\sqrt{z}}{\pi^2(1-z)} \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right\}, \quad z \in \mathbb{E},
\]

then

\[
\frac{I_p(n+1, \lambda)[f](z)}{I_p(n, \lambda)[f](z)} < 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2, \quad z \in \mathbb{E},
\]

where \( \lambda \geq 0, n \in \mathbb{N}_0 \).

3. Let \( \alpha \) be a positive real number. If \( f \in \mathcal{A}_p \) satisfies

\[
(1 - \alpha) \left( 1 + \frac{z^f''(z)}{f'(z)} \right) + \alpha \left( 1 + \frac{2zf''(z) + z^2f'''(z)}{f'(z) + zf''(z)} \right)
\]

\[
< p + \frac{2p}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2 + \frac{4\alpha \sqrt{z}}{\pi^2(1-z)} \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^2, \quad z \in \mathbb{E},
\]

then \( f \in UCV_p \).

4. Let \( \alpha \) be a positive real number. If \( f \in \mathcal{A}_p \) satisfies

\[
(1 - \alpha) \left( \frac{zf'(z)}{f(z)} \right) + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) < p + \frac{2p}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2
\]
\[
\frac{4\alpha\sqrt{z}}{\pi^2(1-z)} \log \left( \frac{1+\sqrt{z}}{1-\sqrt{z}} \right), \quad z \in \mathbb{E},
\]
then \( f \in S^p_p \).

5. Let \( \alpha \) be a positive real number. If \( f \in \mathcal{A}_p \) satisfies

\[
(1 - \alpha) \frac{f'(z)}{pz^{p-1}} + \frac{\alpha}{p^2} \left[ \frac{f'(z)}{z^{p-1}} + \frac{f''(z)}{z^{p-2}} \right] < 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2
\]

\[
+ \frac{4\alpha\sqrt{z}}{p\pi^2(1-z)} \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right), \quad z \in \mathbb{E},
\]
then \( f \in UCC^p \).

Most of the work presented in this thesis has been published in the research papers (see list of publications).

The thesis is appended with a bibliography consisting of 150 references although we don’t claim it be an exhaustive one.