Chapter 4

Starlike and $\phi-$like Functions in a Parabolic Region

4.1 Introduction

Recall the class $\mathcal{A}$ of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \ (z \in \mathbb{E})$$

which are analytic and univalent in the open unit disk $\mathbb{E}$. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ be two analytic functions, then convolution of $f$ and $g$, written as $f \ast g$ is defined by

$$(f \ast g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k.$$ 

A function $f \in \mathcal{A}$ is said to be starlike of order $\alpha$ ($0 \leq \alpha < 1$) in $\mathbb{E}$ if

$$\Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha, \ z \in \mathbb{E}.$$ 

Note: The contents of this chapter have appeared in paper 6 given in the list of publications.
Let $S^*(\alpha)$ denote the class of starlike functions of order $\alpha$. Write $S^*(0) = S^*$, the class of starlike functions.

A function $f \in \mathcal{A}$ is said to be parabolic starlike in $E$, if

$$\Re \left( \frac{zf'(z)}{f(z)} \right) > \left| \frac{zf'(z)}{f(z)} - 1 \right|, \quad z \in E. \quad (4.1)$$

The class of parabolic starlike functions is denoted by $S_P$.

A function $f \in \mathcal{A}$ is said to be strongly starlike of order $\alpha, 0 < \alpha \leq 1$, if

$$\left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\alpha \pi}{2}, \quad z \in E. \quad (4.2)$$

or, equivalently

$$\frac{zf'(z)}{f(z)} < \left( \frac{1+z}{1-z} \right)^\alpha, \quad z \in E.$$

Let $\tilde{S}(\alpha)$ denote the class of strongly starlike functions of order $\alpha$.

Let $\phi$ be analytic in a domain containing $f(E)$, $\phi(0) = 0$ and $\Re(\phi'(0)) > 0$. Then, the function $f \in \mathcal{A}$ is said to be $\phi-$like in $E$ if

$$\Re \left( \frac{zf'(z)}{\phi(f(z))} \right) > 0, \quad z \in E.$$

This concept was introduced by Brickman [20]. He proved that an analytic function $f \in \mathcal{A}$ is univalent if and only if $f$ is $\phi-$like for some $\phi$. Later, Ruscheweyh [109] investigated the following general class of $\phi-$like functions:

Let $\phi$ be analytic in a domain containing $f(E)$, where $\phi(0) = 0$, $\phi'(0) = 1$ and $\phi(w) \neq 0$ for some $w \in f(E) \setminus \{0\}$. Let $q$ be a fixed analytic function in $E$, $q(0) = 1$. Then the function $f \in \mathcal{A}$ is called $\phi-$like with respect to $q$, if

$$\frac{zf'(z)}{\phi(f(z))} < q(z), \quad z \in E.$$

A function $f \in \mathcal{A}$ is said to be parabolic $\phi-$like in $E$ if

$$\Re \left( \frac{zf'(z)}{\phi(f(z))} \right) > \left| \frac{zf'(z)}{\phi(f(z))} - 1 \right|, \quad z \in E. \quad (4.3)$$
Define the parabolic domain $\Omega$ as under:

$$\Omega = \left\{ u + iv : u > \sqrt{(u-1)^2 + v^2} \right\}.$$ 

Note that the conditions (4.1) and (4.3) are equivalent to the condition that $\frac{zf'(z)}{f(z)}$ and $\frac{zf'(z)}{\phi(f(z))}$ take values in the parabolic domain $\Omega$ respectively. Ronning [106] and Ma and Minda [65] showed that the function defined by

$$q(z) = 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2$$

maps the unit disk $\mathbb{D}$ onto the parabolic domain $\Omega$. Therefore, the condition (4.1) and (4.3) are equivalent to following conditions respectively:

$$\frac{zf'(z)}{f(z)} < q(z), \quad z \in \mathbb{D},$$

and

$$\frac{zf'(z)}{\phi(f(z))} < q(z), \quad z \in \mathbb{D},$$

where $q$ is given by (4.4).

In 2005, Ravichandran et al. [101] proved the following results for $\phi-$like functions.

**Theorem 4.1.1.** Let $\alpha \neq 0$ be a complex number and $q$ be convex univalent in $\mathbb{D}$.

Define $h(z) = \alpha q^2(z) + (1 - \alpha)q(z) + \alpha zq'(z)$. Further assume that

$$\Re \left\{ \frac{1 - \alpha}{\alpha} + 2q(z) + \left( 1 + \frac{zq''(z)}{q'(z)} \right) \right\} > 0, \quad z \in \mathbb{D}. If f \in \mathcal{A} satisfies

$$\frac{zf'(z)}{\phi(f(z))} \left[ 1 + \alpha \frac{zf''(z)}{f'(z)} + \alpha \frac{zf'(z) - \{\phi(f(z))\}'}{\phi(f(z))} \right] < h(z),$$

then

$$\frac{zf'(z)}{\phi(f(z))} < q(z),$$

and $q$ is the best dominant.
Theorem 4.1.2. Let $\alpha \neq 0$ be any complex number and $\beta = \max \left\{ 0, -\Re \frac{1}{\alpha} \right\}$. Let $q(z) \neq 0$ be analytic in $E$ and $Q(z) = z q'(z) q^{1-1}(z)$ be starlike of order $\beta$ in $E$. If $f \in A$ satisfies
\[
\frac{zf'(z)}{\Phi(f(z))} \left( 1 + \frac{zf''(z)}{f''(z)} - \frac{\Phi'(f(z))}{\Phi(f(z))} zf'(z) \right) ^\alpha < q(z) \left( 1 + \frac{zq'(z)}{q(z)} \right) ^\alpha,
\]
then
\[
\frac{zf'(z)}{\Phi(f(z))} < q(z), \quad z \in E
\]
and $q$ is the best dominant.

Later, Shanmugham et al. [116] obtained the following result.

Theorem 4.1.3. Let $q(z) \neq 0$ be analytic and univalent in $E$ with $q(0) = 1$ such that $\frac{zq'(z)}{q(z)}$ is starlike univalent in $E$. Let $q$ satisfy
\[
\Re \left[ 1 + \frac{zq'(z)}{\gamma} + \frac{zq''(z)}{q(z)} \right] \geq 0.
\]
Let for $f, g \in A$
\[
\Psi(\alpha, \gamma, g; z) := \alpha \left\{ \frac{z(f \ast g)'(z)}{\Phi(f \ast g)(z)} \right\} + \gamma \left\{ 1 + \frac{z(f \ast g)''(z)}{(f \ast g)'(z)} - \frac{z(\Phi(f \ast g)(z))'}{\Phi(f \ast g)(z)} \right\}.
\]
If $q$ satisfies
\[
\Psi(\alpha, \gamma, g; z) < \alpha q(z) + \frac{\gamma zq'(z)}{q(z)},
\]
then
\[
\frac{z(f \ast g)'(z)}{\Phi(f \ast g)(z)} < q(z), \quad z \in E,
\]
and $q$ is the best dominant.

Many other authors who also contributed to the study of $\phi$–like functions are Cho and Kim [26], Gupta et al. [41], Shanmugham et al. ([113], [114]), Ibrahim and Darus [43], Singh et al. [128] etc.
The aim of the present work is to study the similar type of results as mentioned above. We, here, find certain results as sufficient conditions for $\phi-$like functions with respect to a univalent function $q$. We have also derived a subordination theorem involving the convolution of analytic functions. As special cases of our main results, we obtain the sufficient conditions for analytic functions to be parabolic $\phi-$like, parabolic starlike, $\phi-$like and starlike.

### 4.2 A Subordination Theorem

In what follows, all the powers taken are principal ones.

**Theorem 4.2.1.** Let $\alpha$, $\beta$ and $\gamma$ be complex numbers such that $\alpha$, $\gamma \neq 0$. Let $q(z) \neq 0$, be a univalent function in $\mathbb{E}$ such that

\[(i) \Re\left[1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\beta}{\gamma} - 1\right)\frac{zq'(z)}{q(z)}\right] > 0\]

\[(ii) \Re\left[1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\beta}{\gamma} - 1\right)\frac{zq'(z)}{q(z)} + \left(1 + \frac{\beta}{\gamma}\right)\left(\frac{1 - \alpha}{\alpha}\right)q(z)\right] > 0.\]

If $f \in \mathcal{A}$ satisfies

\[
\left[\frac{zf'(z)}{\phi(f(z))}\right]^\beta \left[(1 - \alpha)\frac{zf'(z)}{\phi(f(z))} + \alpha \left\{1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))}\right\}\right]^{\gamma} \\
\prec (q(z))^\beta \left((1 - \alpha)q(z) + \alpha \frac{zq'(z)}{q(z)}\right)^\gamma,
\]

then $\frac{zf'(z)}{\phi(f(z))} \prec q(z)$, $z \in \mathbb{E}$, and $q$ is the best dominant.

**Proof.** Write $p(z) = \frac{zf'(z)}{\phi(f(z))}$, $z \in \mathbb{E}$. Then the function $p$ is analytic in $\mathbb{E}$ and $p(0) = 1$. Therefore, using this substitution in (4.5), we obtain:

\[
(p(z))^\beta \left((1 - \alpha)p(z) + \alpha \frac{zp'(z)}{p(z)}\right)^\gamma \prec (q(z))^\beta \left((1 - \alpha)q(z) + \alpha \frac{zq'(z)}{q(z)}\right)^\gamma.
\]
Equivalently,
\[
(p(z))^\beta \left((1 - \alpha)p(z) + \alpha \frac{zp'(z)}{p(z)}\right) \prec (q(z))^\beta \left((1 - \alpha)q(z) + \alpha \frac{zq'(z)}{q(z)}\right).
\]

Let us define the functions \( \Theta \) and \( \Phi \) as follows:
\[
\Theta(w) = (1 - \alpha)w^\beta + 1 \quad \text{and} \quad \Phi(w) = \alpha w^\beta - 1.
\]

Let \( \mathbb{D} = \mathbb{C} \setminus \{0\} \). Obviously, the functions \( \Theta \) and \( \Phi \) are analytic in domain \( \mathbb{D} \) and \( \Phi(w) \neq 0, \ w \in \mathbb{D} \). Therefore,
\[
Q(z) = \alpha q(z)^{\beta - 1} zq'(z)
\]
and
\[
h(z) = (q(z))^\beta \left((1 - \alpha)q(z) + \alpha \frac{zq'(z)}{q(z)}\right).
\]

On differentiating, we obtain
\[
\frac{zQ'(z)}{Q(z)} = 1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\beta}{\gamma} - 1\right) \frac{zq'(z)}{q(z)}
\]
and
\[
\frac{zh'(z)}{Q(z)} = 1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\beta}{\gamma} - 1\right) \frac{zq'(z)}{q(z)} + \left(1 + \frac{\beta}{\gamma}\right) \left(1 - \alpha\right) q(z).
\]

In view of the given conditions, we see that \( Q \) is starlike and \( \Re\left(\frac{zh'(z)}{Q(z)}\right) > 0 \).

Therefore, the proof, now follows from Lemma 2.2.1.

\[\square\]

### 4.2.1 Deductions

Selecting \( \alpha = \beta = 1 \) in Theorem 4.2.1, we get the following result which represents the correct version of Theorem 4.1.2.

**Theorem 4.2.2.** Let \( \gamma \neq 0 \) be any complex number. Let \( q(z) \neq 0 \) be univalent in \( \mathbb{E} \) and \( Q(z) = zq'(z)q^{\frac{1}{\gamma} - 1}(z) \) be starlike in \( \mathbb{E} \). If \( f \in \mathcal{A} \) satisfies
\[
\frac{zf'(z)}{\phi(f(z))} \left\{1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))}\right\} \gamma < q(z) \left(\frac{zq'(z)}{q(z)}\right)^\gamma,
\]

68
then
\[
\frac{zf'(z)}{\phi(f(z))} \prec q(z), \quad z \in \mathbb{E},
\]
and \(q\) is the best dominant.

Selecting \(\phi(w) = w\) in Theorem 4.2.1, we get:

**Theorem 4.2.3.** Let \(\alpha, \beta\) and \(\gamma\) be complex numbers such that \(\alpha, \gamma \neq 0\). Let \(q(z) \neq 0\) be a univalent function in \(\mathbb{E}\) such that
\[
(i) \quad \Re \left[ 1 + \frac{zq''(z)}{q'(z)} + \left( \frac{\beta}{\gamma} - 1 \right) \frac{zq'(z)}{q(z)} \right] > 0 \quad \text{and} \quad
(ii) \quad \Re \left[ 1 + \frac{zq''(z)}{q'(z)} + \left( \frac{\beta}{\gamma} - 1 \right) \frac{zq'(z)}{q(z)} + \left( 1 + \frac{\beta}{\gamma} \right) \left( \frac{1 - \alpha}{\alpha} \right) q(z) \right] > 0.
\]
If \(f \in \mathcal{A}\) satisfies
\[
\left[ \frac{zf'(z)}{f(z)} \right]^\beta \left[ (1 - 2\alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] \prec (q(z))^\beta \left( (1 - \alpha)q(z) + \alpha \frac{zq'(z)}{q(z)} \right)^\gamma,
\]
then
\[
\frac{zf'(z)}{f(z)} \prec q(z), \quad z \in \mathbb{E},
\]
and \(q\) is the best dominant.

### 4.2.2 Applications

**Remark 4.2.1.** When we select the dominant \(q(z) = 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2\) in Theorem 4.2.1, a little calculation yields that
\[
1 + \frac{zq''(z)}{q'(z)} + \left( \frac{\beta}{\gamma} - 1 \right) \frac{zq'(z)}{q(z)} = \frac{1 + z}{2(1 - z)} + \frac{\sqrt{z}}{(1 - z) \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)}
\]
\[
+ \left( \frac{\beta}{\gamma} - 1 \right) \frac{4\sqrt{z}}{\pi^2(1 - z)} \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \frac{1}{z} \quad \text{and}
\]
\[
1 + \frac{zq''(z)}{q'(z)} + \left( \frac{\beta}{\gamma} - 1 \right) \frac{zq'(z)}{q(z)} + \left( 1 + \frac{\beta}{\gamma} \right) \left( \frac{1 - \alpha}{\alpha} \right) q(z)
\]
\[
\frac{1+z}{2(1-z)} + \frac{\sqrt{z}}{(1-z)\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)} + \left(\frac{\beta}{\gamma} - 1\right) \frac{4\sqrt{z}}{\pi^2(1-z)} \log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right) + (1 + \frac{\beta}{\gamma}) \left(\frac{1-\alpha}{\alpha}\right) \left[1 + \frac{2}{\pi^2} \left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right].
\]

Thus for \(\alpha, \beta, \gamma \in \mathbb{R}\) such that \(-\frac{3}{4} < \frac{\beta}{\gamma} < \frac{3}{2}\) and \(0 < \alpha \leq 1\), we notice that \(q\) satisfies the conditions (i) and (ii) of Theorem 4.2.1 and consequently, conditions of Theorem 4.2.2 and Theorem 4.2.3 and therefore, we get the next three results respectively from Theorem 4.2.1, Theorem 4.2.2 and Theorem 4.2.3.

**Theorem 4.2.4.** Assume \(\alpha, \beta, \gamma\) are real numbers such that \(-\frac{3}{4} < \frac{\beta}{\gamma} < \frac{3}{2}\), and \(0 < \alpha \leq 1\). If \(f \in \mathcal{A}\) satisfies

\[
\left[\frac{zf'(z)}{\phi(f(z))}\right]^\beta \left[(1-\alpha) \frac{zf'(z)}{\phi(f(z))} + \alpha \left\{1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))}\right\}\right]^\gamma
\]

\[
< \left\{1 + \frac{2}{\pi^2} \left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2\right)^\beta \right\}^{\gamma}
\]

\[
\left\{(1-\alpha) \left(1 + \frac{2}{\pi^2} \left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2\right) + \alpha \frac{4\sqrt{z}}{\pi^2(1-z)} \log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)\right\}^\gamma
\]

then

\[
\frac{zf'(z)}{\phi(f(z))} < 1 + \frac{2}{\pi^2} \left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2\right), \quad z \in \mathbb{E},
\]

i.e. \(f\) is parabolic \(\phi\)-like in \(\mathbb{E}\).

**Theorem 4.2.5.** Let \(\gamma\) be real number such that \(-\frac{3}{4} < \frac{1}{\gamma} < \frac{3}{2}\). If \(f \in \mathcal{A}\) satisfies

\[
\left[\frac{zf'(z)}{\phi(f(z))}\right] \left\{1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))}\right\}^\gamma < \left(1 + \frac{2}{\pi^2} \left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2\right)\right)^\gamma
\]

\[
\left\{\frac{4\sqrt{z}}{\pi^2(1-z)} \log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right\}^\gamma
\]

\[
\left\{1 + \frac{2}{\pi^2} \left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2\right)\right\}^\gamma,
\]
then
\[
\frac{zf''(z)}{f(z)} < 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2, \quad z \in \mathbb{E},
\]
i.e. \( f \) is parabolic \( \phi \)--like in \( \mathbb{E} \).

**Theorem 4.2.6.** Let \( \alpha, \beta \) and \( \gamma \) be real numbers such that \(-\frac{3}{4} < \frac{\beta}{\gamma} < \frac{3}{2}, \) and \( 0 < \alpha \leq 1 \). If \( f \in \mathcal{A} \) satisfies
\[
\left[ \frac{zf''(z)}{f(z)} \right]^\beta \left[ (1 - 2\alpha) \frac{zf''(z)}{f(z)} + \alpha \left\{ 1 + \frac{zf''(z)}{f(z)} \right\} \right]^\gamma < \left\{ 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2 \right\}^\beta \frac{4\sqrt{z}}{\pi^2(1 - z)} \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^2,
\]
then
\[
\frac{zf''(z)}{f(z)} < 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2, \quad z \in \mathbb{E},
\]
i.e. \( f \) is parabolic starlike in \( \mathbb{E} \).

**Remark 4.2.2.** When we select the dominant \( q(z) = \frac{1 + z}{1 - z} \) in Theorem 4.2.1, a little calculation gives
\[
1 + \frac{2z}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2
\]
and
\[
1 + \frac{2z}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2
\]
Clearly, for \( 0 < \alpha \leq 1 \) and \( \gamma = \beta = 1, \) \( q \) satisfies conditions (i) and (ii) of Theorem 4.2.1 and consequently, conditions of Theorem 4.2.2 and Theorem 4.2.3 and we obtain the following three results from Theorem 4.2.1, Theorem 4.2.2 and Theorem 4.2.3 respectively.
Theorem 4.2.7. Let $0 < \alpha \leq 1$. If $f \in \mathcal{A}$ satisfies
\[
\frac{zf'(z)}{\phi(f(z))} \left[ (1 - \alpha) \frac{zf''(z)}{\phi(f(z))} + \alpha \left\{ 1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))} \right\} \right]
< (1 - \alpha) \left( \frac{1+z}{1-z} \right)^2 + \frac{2\alpha z}{(1-z)^2},
\]
then
\[
\frac{zf'(z)}{\phi(f(z))} < \frac{1+z}{1-z}, \quad z \in \mathbb{E},
\]
i.e. $f$ is $\phi$–like in $\mathbb{E}$.

Theorem 4.2.8. If $f \in \mathcal{A}$ satisfies
\[
\frac{zf'(z)}{\phi(f(z))} \left\{ 1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))} \right\} < \frac{2z}{(1-z)^2},
\]
then
\[
\frac{zf'(z)}{\phi(f(z))} < \frac{1+z}{1-z}, \quad z \in \mathbb{E},
\]
i.e. $f$ is $\phi$–like in $\mathbb{E}$.

Theorem 4.2.9. Let $0 < \alpha \leq 1$. If $f \in \mathcal{A}$ satisfies
\[
\frac{zf'(z)}{f(z)} \left[ (1 - 2\alpha) \frac{zf''(z)}{f(z)} + \alpha \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \right] < (1 - \alpha) \left( \frac{1+z}{1-z} \right)^2 + \frac{2\alpha z}{(1-z)^2},
\]
then
\[
\frac{zf'(z)}{f(z)} < \frac{1+z}{1-z}, \quad z \in \mathbb{E},
\]
i.e. $f$ is starlike in $\mathbb{E}$.

Remark 4.2.3. When we select the dominant $q(z) = e^{\frac{3}{2}z}$ in Theorem 4.2.1, after a little calculation, we obtain
\[
1 + \frac{zq''(z)}{q'(z)} + \left( \frac{\beta}{\gamma} - 1 \right) \frac{zq'(z)}{q(z)} = 1 + \frac{3\beta}{2\gamma} z \quad \text{and}
\]
\[
1 + \frac{zq''(z)}{q'(z)} + \left( \frac{\beta}{\gamma} - 1 \right) \frac{zq'(z)}{q(z)} + \left( 1 + \frac{\beta}{\gamma} \right) \frac{1 - \alpha}{\alpha} \frac{q(z)}{q(z)} = 1 + \frac{3\beta}{2\gamma} z.
\]
Thus for $\beta, \gamma, \alpha \in \mathbb{R}$ such that $0 \leq \frac{\beta}{\gamma} < \frac{2}{3}$ and $0 < \alpha \leq 1$, we notice that $q$ satisfies the conditions (i) and (ii) of Theorem 4.2.1 and consequently, conditions of Theorem 4.2.2 and Theorem 4.2.3 are also satisfied. Therefore, we, respectively, derive the next three results from Theorem 4.2.1, Theorem 4.2.2 and Theorem 4.2.3.

**Theorem 4.2.10.** Let $\alpha, \beta, \gamma$ be real numbers such that $0 \leq \frac{\beta}{\gamma} < \frac{2}{3}$ and $0 < \alpha \leq 1$. If $f \in \mathcal{A}$ satisfies

$$\left[ \frac{zf'(z)}{\phi(f(z))} \right]^\beta \left[ (1 - \alpha) \frac{zf''(z)}{\phi(f(z))} + \alpha \left\{ 1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))} \right\} \right]^\gamma \prec e^{\frac{3}{2} \beta z} \left( (1 - \alpha)e^\frac{3}{2}z + \frac{3\alpha}{2} \frac{e^{\frac{3}{2}z}}{z} \right)^\gamma,$$

then $\frac{zf'(z)}{\phi(f(z))} \prec e^{\frac{3}{2}z}$, $z \in \mathbb{E}$, i.e. $f$ is $\phi$-like in $\mathbb{E}$.

**Theorem 4.2.11.** Let $\gamma$ be real number such that $0 \leq \frac{1}{\gamma} < \frac{2}{3}$. If $f \in \mathcal{A}$ satisfies

$$\frac{zf'(z)}{\phi(f(z))} \left\{ 1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))} \right\} \gamma \prec e^{\frac{3}{2}z} \left( \frac{3}{2} \right)^\gamma,$$

then $\frac{zf'(z)}{\phi(f(z))} \prec e^{\frac{3}{2}z}$, $z \in \mathbb{E}$, i.e. $f$ is $\phi$-like in $\mathbb{E}$.

**Theorem 4.2.12.** Let $\alpha, \beta, \gamma$ be real numbers such that $0 \leq \frac{\beta}{\gamma} < \frac{2}{3}$ and $0 < \alpha \leq 1$. If $f \in \mathcal{A}$ satisfies

$$\left\{ \frac{zf'(z)}{f(z)} \right\}^\beta \left[ (1 - 2\alpha) \frac{zf'(z)}{f(z)} + \alpha \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \right]^\gamma \prec e^{\frac{3\beta}{2} z} \left( 1 - \alpha \right)e^\frac{3}{2}z + \frac{3\alpha}{2} \frac{e^{\frac{3}{2}z}}{z} \right)^\gamma,$$

then $\frac{zf'(z)}{f(z)} \prec e^{\frac{3}{2}z}$, $z \in \mathbb{E}$, i.e. $f$ is starlike in $\mathbb{E}$.

We illustrate the results of $\phi$-like and starlike functions by selecting $\alpha = \gamma = 1$ and $\beta = 0$ in Theorem 4.2.10 and Theorem 4.2.12 respectively.
Example 4.2.1. If \( f \in \mathcal{A} \) satisfies
\[
1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))} \preceq \frac{3}{2} z = h(z),
\]
then \( \frac{zf'(z)}{\phi(f(z))} \preceq e^{\frac{3}{2} z} = q(z), \ z \in \mathbb{E}, \ i.e. \ f \) is \( \phi \)-like in \( \mathbb{E} \).

Example 4.2.2. If \( f \in \mathcal{A} \) satisfies
\[
1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \preceq \frac{3}{2} z = h(z),
\]
then \( \frac{zf'(z)}{f(z)} \preceq e^{\frac{3}{2} z} = q(z), \ z \in \mathbb{E}, \ i.e. \ f \) is starlike in \( \mathbb{E} \).

Remark 4.2.4. For illustration, we plot the images of unit disk under the functions \( h(z) = \frac{3}{2} z \) and \( q(z) = e^{\frac{3}{2} z} \), which are given by Figure 4.1 and Figure 4.2 respectively.

![Figure 4.1](image-url)

In the light of Example 4.2.1 when the differential operator \( 1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))} \) takes values in the shaded region of complex plane as given in Figure 4.1, then \( \frac{zf'(z)}{\phi(f(z))} \) takes values in the shaded portion shown in Figure 4.2. Thus \( f(z) \) is
\( \phi \)-like. Similarly, according to Example 4.2.2, when the differential operator \( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \) takes values in the disk centered at origin with radius \( \frac{3}{2} \), shown in the shaded region of Figure 4.1, then \( \frac{zf'(z)}{f(z)} \) takes values in the shaded portion of the complex plane given in Figure 4.2. Thus \( f(z) \) is starlike.

![Figure 4.2](image)

### 4.3 A Subordination Theorem Involving Convolution

In what follows, all the powers taken are principal ones.

**Theorem 4.3.1.** Let \( \alpha, \beta, \gamma \) be complex numbers such that \( \alpha \neq 0 \). Let \( q(z) \neq 0 \), be a univalent function in \( \mathbb{E} \) such that

\[
\begin{align*}
(i) \quad & \Re \left[ 1 + \frac{zq''(z)}{q'(z)} + (\gamma - 1) \frac{zq'(z)}{q(z)} \right] > 0 \quad \text{and} \\
(ii) \quad & \Re \left[ 1 + \frac{zq''(z)}{q'(z)} + (\gamma - 1) \frac{zq'(z)}{q(z)} + \frac{\beta(1 - \alpha)}{\alpha} (q(z))^{\beta - \gamma} + \gamma \right] > 0.
\end{align*}
\]

If \( f \) and \( g \) \( \in \mathcal{A} \) satisfy

\[
(1 - \alpha) \left[ \left( \frac{z(f * g)'(z)}{\phi(f * g)(z)} \right)^\beta + \alpha \left[ \left( \frac{z(f * g)'(z)}{\phi(f * g)(z)} \right)^\gamma 2 + \left( \frac{z(f * g)''(z)}{(f * g)'(z)} - \frac{z(\phi((f * g)(z)))'}{\phi((f * g)(z))} \right) \right] \right] < (1 - \alpha)(q(z))^\beta + \alpha(q(z))^\gamma \left( 1 + \frac{zq'(z)}{q(z)} \right), \quad (4.6)
\]
then \( \frac{z(f * g)'(z)}{\Phi((f * g)(z))} \prec q(z), \ z \in \mathbb{E} \) and \( q \) is the best dominant.

**Proof.** Define the function \( p \) by

\[
p(z) = \frac{z(f * g)'(z)}{\Phi((f * g)(z))}, \ z \in \mathbb{E}.
\]

Then the function \( p \) is analytic in \( \mathbb{E} \) and \( p(0) = 1 \). Therefore, from equation (4.6) we obtain:

\[
(1 - \alpha)(p(z))^{\beta} + \alpha(p(z))^{\gamma} \left( 1 + \frac{zp'(z)}{p(z)} \right) \prec (1 - \alpha)(q(z))^{\beta} + \alpha(q(z))^{\gamma} \left( 1 + \frac{zq'(z)}{q(z)} \right),
\]

Let us define the functions \( \Theta \) and \( \Phi \) as follows:

\[
\Theta(w) = (1 - \alpha)w^{\beta} + \alpha w^{\gamma} \quad \text{and} \quad \Phi(w) = \alpha w^{\gamma - 1}.
\]

Obviously, both the functions \( \Theta \) and \( \Phi \) are analytic in \( D = \mathbb{C} \setminus \{0\} \) and \( \Phi(w) \neq 0 \) in \( D \). Therefore,

\[
Q(z) = \Phi(q(z))zq'(z) = \alpha zq'(z)(q(z))^{\gamma - 1}
\]

and

\[
h(z) = \Theta(q(z)) + Q(z) = (1 - \alpha)(q(z))^{\beta} + \alpha(q(z))^{\gamma} \left( 1 + \frac{zq'(z)}{q(z)} \right).
\]

On differentiating, we obtain

\[
\frac{zQ'(z)}{Q(z)} = 1 + \frac{zq''(z)}{q'(z)} + (\gamma - 1)\frac{zq'(z)}{q(z)}
\]

and

\[
\frac{zh'(z)}{Q(z)} = 1 + \frac{zq''(z)}{q'(z)} + (\gamma - 1)\frac{zq'(z)}{q(z)} + \frac{\beta(1 - \alpha)}{\alpha}(q(z))^{\beta - \gamma + \gamma}.
\]

In view of the given conditions, we see that \( Q \) is starlike and \( \Re \left( \frac{zh'(z)}{Q(z)} \right) > 0 \). Therefore, the proof, now follows from Lemma 2.2.1. \( \square \)
4.3.1 Deductions

Selecting \( g(z) = \frac{z}{1 - z} \) in Theorem 4.3.1, we get the following result:

**Theorem 4.3.2.** Let \( \alpha, \beta, \gamma \) be complex numbers such that \( \alpha \neq 0 \). Let \( q(z) \neq 0 \), be a univalent function in \( E \) such that

(i) \( \Re \left[ 1 + \frac{zq''(z)}{q'(z)} + (\gamma - 1) \frac{zq'(z)}{q(z)} \right] > 0 \) and

(ii) \( \Re \left[ 1 + \frac{zq''(z)}{q'(z)} + (\gamma - 1) \frac{zq'(z)}{q(z)} + \frac{\beta(1 - \alpha)}{\alpha} (q(z))^{\beta - \gamma} + \gamma \right] > 0 \).

If \( f \in \mathcal{A} \) satisfy

\[
(1 - \alpha) \left[ \frac{zf'(z)}{\phi(f(z))} \right]^\beta + \alpha \left[ \frac{zf'(z)}{\phi(f(z))} \right]^{\gamma} \left[ 2 + \frac{zf'''(z)}{f''(z)} - \frac{zf'(z)}{f'(z)} \right]
\]

\[
< (1 - \alpha)(q(z))^\beta + \alpha(q(z))^{\gamma} \left( 1 + \frac{zq'(z)}{q(z)} \right),
\]

then \( \frac{zf'(z)}{\phi(f(z))} < q(z) \), \( z \in E \) and \( q \) is the best dominant.

Selecting \( \phi(w) = w \) in Theorem 4.3.2, we get:

**Theorem 4.3.3.** Let \( \alpha, \beta, \gamma \) be complex numbers such that \( \alpha \neq 0 \). Let \( q(z) \neq 0 \), be a univalent function in \( E \) such that

(i) \( \Re \left[ 1 + \frac{zq''(z)}{q'(z)} + (\gamma - 1) \frac{zq'(z)}{q(z)} \right] > 0 \) and

(ii) \( \Re \left[ 1 + \frac{zq''(z)}{q'(z)} + (\gamma - 1) \frac{zq'(z)}{q(z)} + \frac{\beta(1 - \alpha)}{\alpha} (q(z))^{\beta - \gamma} + \gamma \right] > 0 \).

If \( f \in \mathcal{A} \) satisfy

\[
(1 - \alpha) \left[ \frac{zf'(z)}{f(z)} \right]^\beta + \alpha \left[ \frac{zf'(z)}{f(z)} \right]^{\gamma} \left[ 2 + \frac{zf'''(z)}{f''(z)} - \frac{zf'(z)}{f'(z)} \right]
\]

\[
< (1 - \alpha)(q(z))^\beta + \alpha(q(z))^{\gamma} \left( 1 + \frac{zq'(z)}{q(z)} \right),
\]

then \( \frac{zf'(z)}{f(z)} < q(z) \), \( z \in E \) and \( q \) is the best dominant.
4.3.2 Applications

Remark 4.3.1. When we select $\beta = 1, \gamma = 0$ and $q(z) = 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2$ in Theorem 4.3.2 and Theorem 4.3.3, a little calculation yields that

$$1 + \frac{zq''(z)}{q(z)} - \frac{zq'(z)}{q(z)} = \frac{1 + z}{2(1 - z)} + \frac{\sqrt{z}}{(1 - z) \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)} - \frac{4\sqrt{z}}{\pi^2(1 - z)} \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^2$$

and

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q'(z)} + \left( \frac{1 - \alpha}{\alpha} \right) q(z) = \frac{1 + z}{2(1 - z)} + \frac{\sqrt{z}}{(1 - z) \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)}$$

$$- \frac{4\sqrt{z}}{\pi^2(1 - z)} \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^2 + \left( 1 - \alpha \right) \left( 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2 \right).$$

Thus for real number $\alpha$ such that $0 < \alpha \leq 1$, we notice that $q$ satisfies the condition (i) and (ii) in Theorem 4.3.2 and Theorem 4.3.3. Therefore, we derive next two results from Theorem 4.3.2 and Theorem 4.3.3 respectively.

Theorem 4.3.4. For a real number $\alpha$, $0 < \alpha \leq 1$. If $f \in \mathcal{A}$ satisfies

$$(1 - \alpha) \frac{zf'(z)}{\phi(f(z))} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z))')}{\phi(f(z))} \right)$$

$$\prec (1 - \alpha) \left[ 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2 \right] + \frac{4\alpha\sqrt{z}}{\pi^2(1 - z)} \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^2,$$

then

$$\frac{zf'(z)}{\phi(f(z))} \prec 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2, \quad z \in \mathbb{E},$$

i.e. $f$ is parabolic $\phi-$like in $\mathbb{E}$.

Theorem 4.3.5. Suppose $\alpha$ is a real number such that $0 < \alpha \leq 1$. If $f \in \mathcal{A}$ satisfies

$$(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf''(z)}{f(z)} \right)$$

78
\[
< (1 - \alpha) \left[ 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2 \right] + \frac{4\alpha e^z}{\pi^2 (1 - z)} \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^2,
\]
then
\[
\frac{zf'(z)}{f(z)} < 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2, \ \ z \in \mathbb{E},
\]
i.e. \( f \) is parabolic starlike in \( \mathbb{E} \).

**Remark 4.3.2.** When we select \( \beta = 1, \gamma = 0 \) and \( q(z) = e^z \) in Theorem 4.3.2 and Theorem 4.3.3, a little calculation yields that
\[
1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} = 1
\]
and
\[
1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \left( \frac{1 - \alpha}{\alpha} \right) q(z) = 1 + \left( \frac{1 - \alpha}{\alpha} \right) e^z.
\]
Thus for positive real number \( \alpha \) such that \( 0 < \alpha \leq 1 \), we notice that \( q \) satisfies the condition (i) and (ii) in Theorem 4.3.2 and Theorem 4.3.3. Therefore, we get the following results from Theorem 4.3.2 and Theorem 4.3.3 respectively.

**Theorem 4.3.6.** Let \( \alpha \) be a real number such that \( 0 < \alpha \leq 1 \). If \( f \in A \) satisfies
\[
(1 - \alpha) \frac{zf'(z)}{\phi(f(z))} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z))')}{\phi(f(z))} \right) < (1 - \alpha)e^z + \alpha z,
\]
then
\[
\frac{zf'(z)}{\phi(f(z))} < e^z, \ \ z \in \mathbb{E},
\]
i.e. \( f \) is \( \phi \)-like in \( \mathbb{E} \).

**Theorem 4.3.7.** Let \( \alpha \) be a real number such that \( 0 < \alpha \leq 1 \). If \( f \in A \) satisfies
\[
(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) < (1 - \alpha)e^z + \alpha z,
\]
then
\[
\frac{zf'(z)}{f(z)} < e^z, \ \ z \in \mathbb{E},
\]
i.e. \( f \) is starlike in \( \mathbb{E} \).
Remark 4.3.3. When we select $\beta = 1, \gamma = 0$ and $q(z) = \frac{1 + (1 - 2\delta)z}{1 - z}; 0 \leq \delta < 1$ in Theorem 4.3.2 and Theorem 4.3.3, after a little calculation, we have

$$1 + \frac{z q''(z)}{q'(z)} - \frac{z q'(z)}{q(z)} = \frac{1 + (1 - 2\delta)z^2}{(1 - z)(1 + (1 - 2\delta)z)}$$

and

$$1 + \frac{z q''(z)}{q'(z)} - \frac{z q'(z)}{q(z)} + \left(\frac{1 - \alpha}{\alpha}\right) q(z) = \frac{1 + (1 - 2\delta)z^2}{(1 - z)(1 + (1 - 2\delta)z)} + \left(\frac{1 - \alpha}{\alpha}\right) \left[1 + (1 - 2\delta)z\right].$$

Thus for real number $\alpha$ such that $0 < \alpha \leq 1$, we notice that $q$ satisfies the condition (i) and (ii) in Theorem 4.3.2 and Theorem 4.3.3. Therefore, we arrive at the following results from Theorem 4.3.2 and Theorem 4.3.3 respectively.

**Theorem 4.3.8.** Let $\alpha$ be a real number such that $0 < \alpha \leq 1$. If $f \in \mathcal{A}$ satisfies

$$(1 - \alpha) \frac{zf'(z)}{\phi(f(z))} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z))'}{\phi(f(z))}\right) \prec (1 - \alpha) \left[1 + (1 - 2\delta)z\right] + \frac{2\alpha z(1 - \delta)}{(1 - z)[1 + (1 - 2\delta)z]}, z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{\phi(f(z))} \prec 1 + (1 - 2\delta)z \frac{1 - z}{1 - z}, 0 \leq \delta < 1,$$

i.e. $f$ is $\phi$-like in $\mathbb{E}$.

**Theorem 4.3.9.** Let $\alpha$ be a real number such that $0 < \alpha \leq 1$. If $f \in \mathcal{A}$ satisfies

$$(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec (1 - \alpha) \left[1 + (1 - 2\delta)z\right] \frac{1 - z}{1 - z} + \frac{2\alpha z(1 - \delta)}{(1 - z)[1 + (1 - 2\delta)z]}, z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)} \prec 1 + (1 - 2\delta)z \frac{1 - z}{1 - z},$$

i.e. $f \in S^*(\delta), 0 \leq \delta < 1$.
Remark 4.3.4. When we select $\beta = 1$, $\gamma = 0$ and $q(z) = 1 + az$, $0 \leq a < 1$ in Theorem 4.3.2 and Theorem 4.3.3, a little calculation yields that

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} = \frac{1}{1 + az}.$$

and

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \left(1 - \frac{\alpha}{\alpha}\right)q(z) = \frac{1}{1 + az} + \left(1 - \frac{\alpha}{\alpha}\right)(1 + az).$$

Thus for real number $\alpha$ such that $0 < \alpha \leq 1$, we notice that $q$ satisfies the condition (i) and (ii) in Theorem 4.3.2 and Theorem 4.3.3. Therefore, we obtain the following results from Theorem 4.3.2 and Theorem 4.3.3 respectively.

**Theorem 4.3.10.** Let $\alpha$ be a real number such that $0 < \alpha \leq 1$. If $f \in A$ satisfies

$$(1 - \alpha)\frac{zf'(z)}{\phi(f(z))} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} - \frac{z\phi'(f(z))}{\phi(f(z))}\right) < (1 - \alpha)(1 + az) + \frac{\alpha az}{1 + az},$$

then

$$\frac{zf'(z)}{\phi(f(z))} < 1 + az, \; 0 \leq a < 1, \; z \in \mathbb{E}.$$

i.e. $f$ is $\phi$-like in $\mathbb{E}$.

**Theorem 4.3.11.** Let $\alpha$ be a real number such that $0 < \alpha \leq 1$. If $f \in A$ satisfies

$$(1 - \alpha)\frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) < (1 - \alpha)(1 + az) + \frac{\alpha az}{1 + az}, \; z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)} < 1 + az, \; 0 \leq a < 1, \; z \in \mathbb{E}.$$

i.e. $f$ is starlike in $\mathbb{E}$.

**Remark 4.3.5.** When we select $\beta = 1, \gamma = 0$ and $q(z) = \left(\frac{1 + z}{1 - z}\right)^{\eta}$, $0 < \eta \leq 1$ in Theorem 4.3.2 and Theorem 4.3.3, a little calculation yields that

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} = \frac{1 + z^2}{1 - z^2}.$$
and

\[ 1 + \frac{z q''(z)}{q'(z)} - \frac{z q'(z)}{q(z)} + \left( \frac{1 - \alpha}{\alpha} \right) q(z) = \frac{1 + z^2}{1 - z^2} + \left( \frac{1 - \alpha}{\alpha} \right) \left( \frac{1 + z}{1 - z} \right)^\eta. \]

Thus for real number \( \alpha \) such that \( 0 < \alpha \leq 1 \), we notice that \( q \) satisfies the condition (i) and (ii) in Theorem 4.3.2 and Theorem 4.3.3. Therefore, we have the following results from Theorem 4.3.2 and Theorem 4.3.3 respectively.

**Theorem 4.3.12.** Let \( \alpha \) be a real number such that \( 0 < \alpha \leq 1 \). If \( f \in \mathcal{A} \) satisfies

\[ (1 - \alpha) \frac{zf'(z)}{\phi(f(z))} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))} \right) \prec (1 - \alpha) \left( \frac{1 + z}{1 - z} \right)^\eta + 2\eta \alpha z \]

then

\[ \frac{zf'(z)}{\phi(f(z))} \prec \left( \frac{1 + z}{1 - z} \right)^\eta, \quad 0 < \eta \leq 1, \quad z \in \mathbb{E}. \]

**Theorem 4.3.13.** Let \( \alpha \) be a real number such that \( 0 < \alpha \leq 1 \). If \( f \in \mathcal{A} \) satisfies

\[ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) \prec (1 - \alpha) \left( \frac{1 + z}{1 - z} \right)^\eta + 2\eta \alpha z \]

then

\[ \frac{zf'(z)}{f(z)} \prec \left( \frac{1 + z}{1 - z} \right)^\eta, \quad z \in \mathbb{E}, \]

i.e. \( f \in \mathcal{S}(\eta), \quad 0 < \eta \leq 1 \).

**Remark 4.3.6.** When we select \( \beta = 1, \gamma = 0 \) and \( q(z) = \frac{\alpha'(1 - z)}{\alpha' - z} \), \( \alpha' > 1 \) in Theorems 4.3.2 and 4.3.3, a little calculation yields that

\[ 1 + \frac{z q''(z)}{q'(z)} - \frac{z q'(z)}{q(z)} = \frac{1}{1 - z} + \frac{z}{\alpha' - z} \]

and

\[ 1 + \frac{z q''(z)}{q'(z)} - \frac{z q'(z)}{q(z)} + \left( \frac{1 - \alpha}{\alpha'} \right) q(z) = \frac{1}{1 - z} + \frac{z}{\alpha' - z} + \left( \frac{1 - \alpha}{\alpha'} \right) \left[ \alpha'(1 - z) \right]. \]

Thus for real number \( \alpha \) such that \( 0 < \alpha \leq 1 \), we notice that \( q \) satisfies the condition (i) and (ii) in Theorem 4.3.2 and Theorem 4.3.3. Therefore, we get the following results from Theorem 4.3.2 and Theorem 4.3.3 respectively.
Theorem 4.3.14. Let $\alpha$ be a real number such that $0 < \alpha \leq 1$. If $f \in \mathcal{A}$ satisfies
\[
(1 - \alpha) \frac{zf'(z)}{\phi(f(z))} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z))')}{\phi(f(z))}\right)< (1 - \alpha) \frac{\alpha'(1 - z)}{\alpha' - z} + \frac{(1 - \alpha')\alpha z}{(1 - z)(\alpha' - z)},
\]
then
\[
\frac{zf'(z)}{\phi(f(z))} < \frac{\alpha'(1 - z)}{\alpha' - z}, \quad \alpha' > 1, \quad z \in \mathbb{E},
\]
i.e. $f$ is $\phi$–like in $\mathbb{E}$.

Theorem 4.3.15. Let $\alpha$ be a real number such that $0 < \alpha \leq 1$. If $f \in \mathcal{A}$ satisfies
\[
(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)< (1 - \alpha) \frac{\alpha'(1 - z)}{\alpha' - z} + \frac{(1 - \alpha')\alpha z}{(1 - z)(\alpha' - z)},
\]
then
\[
\frac{zf'(z)}{f(z)} < \frac{\alpha'(1 - z)}{\alpha' - z}, \quad \alpha' > 1, \quad z \in \mathbb{E},
\]
i.e. $f$ is starlike in $\mathbb{E}$.

Remark 4.3.7. When we select $\beta = 1, \gamma = 1$ and $q(z) = 1 + \frac{2}{\pi^2} \left(\log \left(\frac{1 + \sqrt{z}}{1 - \sqrt{z}}\right)\right)^2$ in Theorem 4.3.2 and Theorem 4.3.3, after a little calculation, we obtain
\[
1 + \frac{zq''(z)}{q'(z)} = \frac{1 + z}{2(1 - z)} + \frac{\sqrt{z}}{(1 - z)\log \left(\frac{1 + \sqrt{z}}{1 - \sqrt{z}}\right)}
\]
and
\[
1 + \frac{zq''(z)}{q'(z)} + \frac{1}{\alpha} = \frac{1 + z}{2(1 - z)} + \frac{\sqrt{z}}{(1 - z)\log \left(\frac{1 + \sqrt{z}}{1 - \sqrt{z}}\right)} + \frac{1}{\alpha}.
\]
Thus for positive real number $\alpha$, we notice that $q$ satisfies the condition (i) and (ii) in Theorem 4.3.2 and Theorem 4.3.3. Therefore, we obtain the following results from Theorem 4.3.2 and Theorem 4.3.3 respectively.

Theorem 4.3.16. Let $\alpha$ be a positive real number. If $f \in \mathcal{A}$ satisfies
\[
\frac{zf'(z)}{\phi(f(z))} + \alpha \left(\frac{zf'(z)}{\phi(f(z))}\right)\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z))')}{\phi(f(z))}\right)
\]

83
\[
< 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2 + \frac{4\alpha\sqrt{z}}{\pi^2(1 - z)} \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right),
\]
then
\[
\frac{zf'(z)}{\phi(f(z))} < 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2, \quad z \in \mathbb{E},
\]
i.e. \( f \) is parabolic \( \phi \)-like in \( \mathbb{E} \).

**Theorem 4.3.17.** Let \( \alpha \) be a positive real number. If \( f \in \mathcal{A} \) satisfies
\[
\frac{zf'(z)}{f(z)} + \alpha \left( \frac{zf''(z)}{f(z)} \right) \left( 1 + \frac{zf''(z)}{f(z)} - \frac{zf'(z)}{f(z)} \right) < 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2 + \frac{4\alpha\sqrt{z}}{\pi^2(1 - z)} \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right),
\]
then
\[
\frac{zf'(z)}{f(z)} < 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2, \quad z \in \mathbb{E},
\]
i.e. \( f \) is parabolic starlike in \( \mathbb{E} \).

**Remark 4.3.8.** When we select \( \beta = 1, \gamma = 1 \) and \( q(z) = \frac{1 + (1 - 2\delta)z}{1 - z} \), for \( 0 \leq \delta < 1 \) in Theorem 4.3.2 and Theorem 4.3.3, a little calculation yields that
\[
1 + \frac{zq''(z)}{q'(z)} = \frac{1 + z}{1 - z}
\]
and
\[
1 + \frac{zq''(z)}{q'(z)} + \frac{1}{\alpha} = \frac{1 + z}{1 - z} + \frac{1}{\alpha}.
\]
Thus for positive real number \( \alpha \), we notice that \( q \) satisfies the condition (i) and (ii) in Theorem 4.3.2 and Theorem 4.3.3. Therefore, we arrive at the following results from Theorem 4.3.2 and Theorem 4.3.3 respectively.

**Theorem 4.3.18.** Let \( \alpha \) be a positive real number. If \( f \in \mathcal{A} \) satisfies
\[
\frac{zf'(z)}{\phi(f(z))} + \alpha \left( \frac{zf'(z)}{\phi(f(z))} \right) \left( 1 + \frac{zf''(z)}{\phi(f(z))} - \frac{z(\phi(f(z))')}{\phi(f(z))} \right) < \frac{1 + (1 - 2\delta)z}{1 - z} + \frac{2\alpha z(1 - \delta)}{(1 - z)^2},
\]

84
then
\[
\frac{zf'(z)}{\phi(f(z))} < \frac{1 + (1 - 2\delta)z}{1 - z}, \quad 0 \leq \delta < 1, \quad z \in \mathbb{E},
\]
i.e. \( f \) is \( \phi \)-like in \( \mathbb{E} \).

**Theorem 4.3.19.** Let \( \alpha \) be a positive real number. If \( f \in \mathcal{A} \) satisfies
\[
\frac{zf'(z)}{f(z)} + \alpha \left( \frac{zf'(z)}{f(z)} \right) \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) < \frac{1 + (1 - 2\delta)z}{1 - z} + \frac{2\alpha z(1 - \delta)}{(1 - z)^2},
\]
then
\[
\frac{zf'(z)}{f(z)} < \frac{1 + (1 - 2\delta)z}{1 - z}, \quad z \in \mathbb{E},
\]
i.e. \( f \in S^*(\delta), \quad 0 \leq \delta < 1 \).

**Remark 4.3.9.** When we select \( \beta = 1, \gamma = 1 \) and \( q(z) = e^z \), in Theorem 4.3.2 and
Theorem 4.3.3, a little calculation yields that
\[
1 + \frac{zq''(z)}{q'(z)} = 1 + z
\]
and
\[
1 + \frac{zq''(z)}{q'(z)} + \frac{1}{\alpha} = 1 + z + \frac{1}{\alpha}.
\]
Thus for positive real number \( \alpha \), we notice that \( q \) satisfies the condition (i) and (ii)
in Theorem 4.3.2 and Theorem 4.3.3. Therefore, we have the following results from
Theorem 4.3.2 and Theorem 4.3.3 respectively.

**Theorem 4.3.20.** Let \( \alpha \) be a positive real number. If \( f \in \mathcal{A} \) satisfies
\[
\frac{zf'(z)}{\phi(f(z))} + \alpha \left( \frac{zf'(z)}{\phi(f(z))} \right) \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z))'}{\phi(f(z))} \right) < e^z(1 + \alpha z),
\]
then
\[
\frac{zf'(z)}{\phi(f(z))} < e^z, \quad z \in \mathbb{E},
\]
i.e. \( f \) is \( \phi \)-like in \( \mathbb{E} \).
Theorem 4.3.21. Let $\alpha$ be a positive real number. If $f \in A$ satisfies
\[
\frac{zf'(z)}{f(z)} + \alpha \left( \frac{zf'(z)}{f(z)} \right) \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) \prec e^{z}(1 + \alpha z),
\]
then
\[
\frac{zf'(z)}{f(z)} \prec e^{z}, \quad z \in \mathbb{E},
\]
i.e. $f$ is starlike in $\mathbb{E}$.

Remark 4.3.10. When we select $\beta = 1, \gamma = 1$ and $q(z) = \frac{\alpha'(1 - z)}{\alpha' - z}, 1 < \alpha' < \frac{3}{2}$ in Theorem 4.3.2 and Theorem 4.3.3, a little calculation yields that
\[
1 + \frac{zq''(z)}{q'(z)} = \frac{\alpha' + z}{\alpha' - z},
\]
\[
1 + \frac{zq''(z)}{q'(z)} + \frac{1}{\alpha} = \frac{\alpha' + z}{\alpha' - z} + \frac{1}{\alpha},
\]
For a positive real number $\alpha$, we see that $q$ satisfies the conditions (i) and (ii) of Theorem 4.3.2 and Theorem 4.3.3. Therefore, we get the following results from Theorem 4.3.2 and Theorem 4.3.3 respectively.

Theorem 4.3.22. Let $\alpha$ be a positive real number. If $f \in A$ satisfies
\[
\frac{zf'(z)}{\phi(f(z))} + \alpha \left( \frac{zf'(z)}{\phi(f(z))} \right) \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{\phi(f(z))} \right) \prec \frac{\alpha'(1 - z)}{\alpha' - z} + \alpha \frac{(\alpha' - \alpha^2)z}{(\alpha' - z)^2},
\]
then
\[
\frac{zf'(z)}{\phi(f(z))} \prec \frac{\alpha'(1 - z)}{\alpha' - z}, \quad 1 < \alpha' < \frac{3}{2}, \quad z \in \mathbb{E},
\]
i.e. $f$ is $\phi$–like in $\mathbb{E}$.

Theorem 4.3.23. Let $\alpha$ be a positive real number. If $f \in A$ satisfies
\[
\frac{zf'(z)}{f(z)} + \alpha \left( \frac{zf'(z)}{f(z)} \right) \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) \prec \frac{\alpha'(1 - z)}{\alpha' - z} + \alpha \frac{(\alpha' - \alpha^2)z}{(\alpha' - z)^2},
\]
then
\[
\frac{zf'(z)}{f(z)} \prec \frac{\alpha'(1 - z)}{\alpha' - z}, \quad 1 < \alpha' < \frac{3}{2}, \quad z \in \mathbb{E},
\]
i.e. $f$ is starlike in $\mathbb{E}$.
Remark 4.3.11. When we select $\beta = 1, \gamma = 1$ and $q(z) = 1 + az, 0 \leq a < 1$ in Theorem 4.3.2 and Theorem 4.3.3, a little calculation gives

\[
1 + \frac{zq''(z)}{q'(z)} = 1,
\]

\[
1 + \frac{zq''(z)}{q'(z)} + \frac{1}{\alpha} = 1 + \frac{1}{\alpha}.
\]

For a positive real number $\alpha$, we see that $q$ satisfies the conditions (i) and (ii) of Theorem 4.3.2 and Theorem 4.3.3. Therefore, we derive next two results from Theorem 4.3.2 and Theorem 4.3.3 respectively.

Theorem 4.3.24. Let $\alpha$ be a positive real number. If $f \in \mathcal{A}$ satisfies

\[
\frac{zf'(z)}{\phi(f(z))} + \alpha \left( \frac{zf'(z)}{\phi(f(z))} \right) \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{\phi(f(z))} \right) < 1 + az + \alpha az,
\]

then

\[
\frac{zf'(z)}{\phi(f(z))} < 1 + az, \quad 0 \leq a < 1, \quad z \in \mathbb{E},
\]

i.e. $f$ is $\phi$-like in $\mathbb{E}$.

Theorem 4.3.25. Let $\alpha$ be a positive real number. If $f \in \mathcal{A}$ satisfies

\[
\frac{zf'(z)}{f(z)} + \alpha \left( \frac{zf'(z)}{f(z)} \right) \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) < 1 + az + \alpha az,
\]

then

\[
\frac{zf'(z)}{f(z)} < 1 + az, \quad 0 \leq a < 1, \quad z \in \mathbb{E},
\]

i.e. $f$ is starlike in $\mathbb{E}$.

Remark 4.3.12. When we select $\beta = 1, \gamma = 1$ and $q(z) = \left(\frac{1 + z}{1 - z}\right)^{\eta}, 0 < \eta \leq 1$ in Theorem 4.3.2 and Theorem 4.3.3, a little calculation yields that

\[
1 + \frac{zq''(z)}{q'(z)} = \frac{1 + z^2 + 2\eta z}{1 - z^2},
\]

\[
1 + \frac{zq''(z)}{q'(z)} + \frac{1}{\alpha} = \frac{1 + z^2 + 2\eta z}{1 - z^2} + \frac{1}{\alpha}.
\]
For a positive real number $\alpha$, we see that $q$ satisfies the conditions (i) and (ii) of Theorem 4.3.2 and Theorem 4.3.3. Therefore, we obtain the following results from Theorem 4.3.2 and Theorem 4.3.3 respectively.

**Theorem 4.3.26.** Let $\alpha$ be a positive real number. If $f \in \mathcal{A}$ satisfies

$$\frac{zf'(z)}{\phi(f(z))} + \alpha \left( \frac{zf'(z)}{\phi(f(z))} \right) \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{z\phi(f(z))'}{\phi(f(z))} \right) < \left( 1 + \frac{1 + \eta}{1 - \eta z} \right)^\eta \left[ 1 + \frac{2\alpha \eta z}{1 - z^2} \right],$$

then

$$\frac{zf'(z)}{\phi(f(z))} < \left( 1 + \frac{1 + \eta}{1 - \eta z} \right)^\eta, \ 0 < \eta \leq 1, \ z \in \mathcal{E}.$$  

**Theorem 4.3.27.** Let $\alpha$ be a positive real number. If $f \in \mathcal{A}$ satisfies

$$\frac{zf'(z)}{f(z)} + \alpha \left( \frac{zf'(z)}{f(z)} \right) \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf''(z)}{f(z)} \right) < \left( 1 + \frac{1 + \eta}{1 - \eta z} \right)^\eta \left[ 1 + \frac{2\alpha \eta z}{1 - z^2} \right],$$

then

$$\frac{zf'(z)}{f(z)} < \left( 1 + \frac{1 + \eta}{1 - \eta z} \right)^\eta, \ z \in \mathcal{E},$$

i.e. $f \in \mathcal{S}(\eta), \ 0 < \eta \leq 1.$

88