CHAPTER 2

SELF-SIMILAR GROUP OF KANNAN
CONTRACTION

2.1 Introduction

In order to implement the group structure on attractors generated from the Kannan iterated function system, this chapter introduces the definition of self-similar group and strong self-similar group of Kannan contraction and investigates their properties. Further, these groups are found as the attractor of KIFS. From this observation, if \( G \) is a self-similar group of Kannan contraction, then \( G \) is describable as the attractor of a KIFS and one of the Kannan contractions of KIFS is a group homomorphism. If \( G \) is a strong self-similar group of Kannan contraction, then \( G \) can be written as the union of finite copies of itself, and one of the Kannan mapping is isomorphism. The image of \( G \) under this Kannan contraction map is its proper subgroup \( H \) being homomorphic or isomorphic to \( G \) according to \( G \) is self-similar or strong self-similar group. Besides, this chapter reveals the necessary condition for strong self-similar group of Kannan contraction and the relation between profinite group and strong self-similar group of Kannan contraction.


2.2 Attractor of Kannan Iterated Function System

In 1969 [65], Kannan proposed a self mapping $f$ on a metric space $(X, d)$, which was a development over contraction mappings, known as Kannan contraction defined as follows:

A self mapping $f$ on a metric space $(X, d)$ is called **Kannan contraction**, if there exists $k \in (0, 1/2)$ such that

$$d(f(x), f(y)) \leq k[d(x, f(x)) + d(y, f(y))], \ \forall \ x, y \in X, \quad (2.1)$$

where $k$ is known as Kannan contraction ratio or K-contraction ratio. Further he proved that the existence and uniqueness of fixed point of the Kannan mapping by omitting the completeness of the space with the different sufficient conditions but obtained the same results as in Banach’s fixed point theorem. Those sufficient conditions together do not guarantee the completeness of the space.

Let $X = [0, 1]$ with Euclidean metric, define the self map $f$ on $X$ by

$$f(x) = \begin{cases} 
\frac{1}{2} & \text{if } 0 \leq x < 1, \\
\frac{1}{4} & \text{if } x = 1.
\end{cases}$$

Then $f$ is Kannan contraction but not Banach contraction. If $f$ is defined by $f(x) = \frac{x}{3}$ for all $x$ in $X$, then $f$ is Banach contraction but not a Kannan contraction. These examples show that the Banach contraction and the Kannan contraction are independent.

Besides that Kannan’s fixed point theorem is very important because Subrahmanyam [64] proved that Kannan’s mapping characterizes the completeness of the space. That is, a metric space $X$ is complete if and only if every Kannan mapping on $X$ has a fixed point. Hence, on the basis of iterated function system given by Barnsley [5]; Sahu et al. [63] introduced the Kannan iterated function system (KIFS) for constructing the fractal sets.

This section briefly describes the fractal constructed by Kannan iterated function system, more rigorous treatment as given in [63].
Definition 2.1. [63] Let \((X,d)\) be a complete metric space and \(f_n : X \rightarrow X, \ n = 1,2,3,\ldots,N \ (N \in \mathbb{N})\) be \(N\) - Kannan contraction mappings with the corresponding contraction ratios \(k_n, \ n = 1,2,3,\ldots,N\). Then the system \(\{X; f_n, \ n = 1,2,3,\ldots,N\}\) is called a Kannan iterated function system.

Lemma 2.1. [63] Let \(f : X \rightarrow X\) be a continuous Kannan mapping on the metric space \((X,d)\) with \(K\)-contraction ratio \(k\). Then \(F : \mathcal{K}(X) \rightarrow \mathcal{K}(X)\) defined by

\[
F(B) = \{f(x) : x \in B\}, \text{ for every } B \in \mathcal{K}(X),
\]

is a Kannan mapping on \((\mathcal{K}(X), H_d)\) with contraction ratio \(k\).

Lemma 2.2. [63] Let \((X,d)\) be a metric space. Let \(f_n, \ n = 1,2,3,\ldots,N\) be continuous Kannan mappings on \((\mathcal{K}(X), H_d)\). Let \(K\)-contraction ratio for \(f_n\) be denoted by \(k_n, \ n = 1,2,3,\ldots,N\). Define \(F : \mathcal{K}(X) \rightarrow \mathcal{K}(X)\) by

\[
F(B) = \bigcup_{n=1}^{N} f_n(B), \text{ for every } B \in \mathcal{K}(X).
\]

Then \(F\) is a Kannan mapping with \(K\)-contraction ratio \(k = \max_{n=1}^{N} k_n\).

Theorem 2.1. [63] If \(\{X : f_1, f_2, \ldots, f_N\}\) is a KIFS with \(K\)-contraction ratio \(k\), then \(F : \mathcal{K}(X) \rightarrow \mathcal{K}(X)\) defined by

\[
F(B) = \bigcup_{n=1}^{N} f_n(B), \text{ for every } B \in \mathcal{K}(X),
\]

is a continuous Kannan mapping on the complete metric space \((\mathcal{K}(X), H_d)\) with contraction ratio \(k\). If \(A \in \mathcal{K}(X)\) is a unique fixed point of \(F\), which is also called an attractor, then \(A\) obeys

\[
F(A) = A = \bigcup_{n=1}^{N} f_n(A),
\]

and is given by \(A = \lim_{n \rightarrow \infty} F^n(B)\) for any \(B \in \mathcal{K}(X)\). \(F^n\) denotes the \(n\)-fold composition of \(F\).

As a consequence of above results the following theorem becomes easy, which is used to prove the disconnectedness of self-similar group in the next section.

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Theorem 2.2. Let \( \{X; f_0, f_1, \ldots, f_n\} \) be a KIFS with attractor \( A \). If the Kannan contraction mappings \( f_0, f_1, \ldots, f_n \) are one-to-one on \( A \) and \( f_i(A) \cap f_j(A) = \emptyset \) for all \( i, j \in \{0, 1, 2, \ldots, n\} \) with \( i \neq j \), then \( A \) is totally disconnected set.

2.3 Self-similar in the Sense of KIFS

This section introduces the definition and properties of self-similar group and strong self-similar group of Kannan contraction. Besides, the relation between strong self-similar and profinite group is investigated.

Definition 2.2. A compact topological group \((G, \ast, d)\) with a translation-invariant metric \(d\) is called a self-similar group of Kannan contraction, if there exists a proper subgroup \(H\) of finite index and a surjective homomorphism \(\phi: G \rightarrow H\), which is a Kannan contraction with respect to \(d\).

Definition 2.3. A compact topological group \((G, \ast, d)\) with a translation-invariant metric \(d\) is called a strong self-similar group of Kannan contraction, if there exists a proper subgroup \(H\) of finite index and a group isomorphism \(\phi: G \rightarrow H\), which is a Kannan contraction with respect to \(d\).

Proposition 2.1. A strong self-similar group of Kannan contraction is the attractor of KIFS.

Proof. Let \(G\) be a strong self-similar group of Kannan contraction. So there is a proper subgroup \(H\) of \(G\) with finite index such that the mapping \(\phi_0: G \rightarrow H\) is a group isomorphism and is a Kannan contraction. Let \(|G: H| = n\) and let \(x_0 = e\) be the identity element of \(G\). For all \(i, j \in \{0, 1, 2, \ldots, n - 1\}\) and \(i \neq j\), there are cosets of \(H\) in \(G\) such that \((H \ast x_i) \cap (H \ast x_j) = \emptyset\) and \(G = H \cup (H \ast x_1) \cup (H \ast x_2) \cup \cdots \cup (H \ast x_{n-1})\).

Define \(\phi_i: G \rightarrow G\) by \(\phi_i(g) = \phi_0(g) \ast x_i, i = 1, 2, 3, \ldots, n - 1\). Now, we show that
\( \phi_i(g) \) is a Kannan contraction for each \( i \). It is obvious that \( \phi_i(G) = H \ast x_i \), because of \( \phi_0 \) is surjective. Since \( \phi_0 \) is a Kannan contraction mapping with contraction ratio \( k \) and \( d \) is a translation invariant metric, we obtain that

\[
\begin{align*}
    d(\phi_i(g), \phi_i(h)) &= d(\phi_0(g) \ast x_i, \phi_0(h) \ast x_i) \\
    &= d(\phi_0(g), \phi_0(h)) \\
    &\leq \alpha [d(g, \phi_0(g)) + d(h, \phi_0(g))],
\end{align*}
\]

for all \( g, h \in G \). Therefore \( \phi_i \) is a Kannan contraction mapping with contraction ratio \( k \) for \( i = 1, 2, 3, \ldots, n - 1 \) and

\[
G = H \cup (H \ast x_1) \cup (H \ast x_2) \cup \cdots \cup (H \ast x_{n-1})
\]

\[
= \phi_0(G) \cup \phi_1(G) \cup \phi_2(G) \cup \cdots \cup \phi_{n-1}(G)
\]

\[
G = \bigcup_{i=0}^{n-1} \phi_i(G).
\]

Thus, \( G \) is the attractor of the KIFS \( \{G; \phi_0, \phi_1, \ldots, \phi_{n-1}\} \). \( \square \)

Proposition 2.1 elucidates that the group \( G \) is a fractal generated by the KIFS \( \{G; \phi_0, \phi_1, \ldots, \phi_{n-1}\} \), where \( \phi_0 \) defines the isomorphism between \( H \) and \( G \). According to the self-similar property of the fractals, the group \( G \) is called as a self-similar group of Kannan contraction. Proposition 2.1 explains the reason that why the groups defined in Definition 2.2 and Definition 2.3 are named as self-similar groups.

**Proposition 2.2.** Let \( (G, \ast, d) \) and \( (G', \ast', d') \) be compact topological groups. If \( G \) is a strong self-similar group of Kannan contraction and \( f : G \rightarrow G' \) is an isometry map as well as a group isomorphism, then \( G' \) is also a strong self-similar group of Kannan contraction.

**Proof.** Since \( f \) is surjective and isometry, so there exist \( x, y, z \in G \) such that \( f(x) = x', f(y) = y' \) and \( f(z) = z' \) for all \( x', y', z' \in G' \). By \( d \) is a translation-invariant metric,
we compute that

\[ d'(x' \ast' z', y' \ast' z') = d'(f(x) \ast' f(z), f(y) \ast' f(z)) = d'(f(x \ast z), f(y \ast z)) = d(x \ast z, y \ast z) = d(x, y) = d'(f(x), f(y)) = d'(x', y'). \]

As \( G \) is a strong self-similar group of Kannan contraction, there exists a subgroup \( H \) of finite index and a group isomorphism \( \phi : G \to H \). Let \( f(H) = H' \). As \( f \) is a group isomorphism, so it is obvious that \( H' \) is a subgroup of \( G' \) with finite index.

Define \( f|_H : H \to H' \) by \( f|_H(x) = f(x) \) for all \( x \in H \subseteq G \). Now we prove that \( \phi' = f|_H \circ \phi \circ f^{-1} : G' \to H' \) is both a Kannan contraction mapping and a group isomorphism. Here, \( f, f|_H \) and \( \phi \) are group isomorphisms, so it is clear that \( \phi' \) is also a group isomorphism. As \( \phi \) is a Kannan contraction mapping with contraction ratio \( k \) and \( f, f|_H \) are isometries, we get

\[ d'((\phi'|g'), \phi'|h')) = d'(f|_H \circ \phi \circ f^{-1}(g'), f|_H \circ \phi \circ f^{-1}(h')) = d'(f|_H(\phi \circ f^{-1}(g')), f|_H(\phi \circ f^{-1})(h')) = d(\phi \circ f^{-1}(g'), \phi \circ f^{-1}(h')) \leq k[d'(f^{-1}(g'), \phi(f^{-1}(g'))) + d'(f^{-1}(h'), \phi(f^{-1}(h')))] = k[d'(f|_H \circ f^{-1}(g'), f|_H \circ \phi(f^{-1}(g'))) + d'(f|_H \circ f^{-1}(h'), f|_H \circ \phi(f^{-1}(h')))] = k[d'(g', \phi'(g')) + d'(h', \phi'(h'))], \]

for all \( g', h' \in G' \). It gives that \( \phi' \) is Kannan contraction mapping on \( G' \). \( \square \)
Proposition 2.3. If $G_1, G_2, \ldots, G_n$ are strong self-similar groups of Kannan contraction, so is $G_1 \times G_2 \times \cdots \times G_n$.

Proof. Since $(G_1, *_1, d_1), (G_2, *_2, d_2), \ldots, (G_n, *_n, d_n)$ are compact topological groups, $G_1 \times G_2 \times \cdots \times G_n$ is a compact topological group. Moreover, there are subgroups $H_1, H_2, \ldots, H_n$ of $G_1, G_2, \ldots, G_n$ respectively such that $[G_i : H_i] = m_i$ and the mappings $\phi_i : G_i \rightarrow H_i$ are Kannan contraction with corresponding contraction ratios $k_i$ and also group isomorphisms for $i = 1, 2, \ldots, n$, since these groups are strong self-similar group in the sense of Kannan contraction. Define the mapping $\phi : G_1 \times G_2 \times \cdots \times G_n \rightarrow H_1 \times H_2 \times \cdots \times H_n$ by

$$\phi(g_1, g_2, \ldots, g_n) = (\phi_1(g_1), \phi_2(g_2), \ldots, \phi_n(g_n)).$$

It is obvious that $H_1 \times H_2 \times \cdots \times H_n$ is a subgroup of $G_1 \times G_2 \times \cdots \times G_n$ and $[G_1 \times G_2 \times \cdots \times G_n : H_1 \times H_2 \times \cdots \times H_n] = m_1 m_2 \cdots m_n$. Since $\phi_1, \phi_2, \ldots, \phi_n$ are group homomorphisms, we compute that

$$\phi(g * h) = \phi((g_1, g_2, \ldots, g_n) * (h_1, h_2, \ldots, h_n))$$

$$= \phi((g_1 *_1 h_1, g_2 *_2 h_2, \ldots, g_n *_n h_n))$$

$$= (\phi_1(g_1) *_1 \phi_1(h_1), \phi_2(g_2) *_2 \phi_2(h_2), \ldots, \phi_n(g_n) *_n \phi_n(h_n))$$

$$= (\phi_1(g_1), \ldots, \phi_n(g_n)) * (\phi_1(h_1), \ldots, \phi_n(h_n))$$

$$= \phi((g_1, g_2, \ldots, g_n)) * \phi((h_1, h_2, \ldots, h_n))$$

$$= \phi(g) * \phi(h).$$

It is clear that $\phi$ is bijective due to the definitions of $\phi_1, \phi_2, \ldots, \phi_n$. Hence $\phi$ is a group homomorphism. Let $k = \max \{ k_1, k_2, \ldots, k_n \}$. Then, we obtain that

$$d(\phi(g), \phi(h)) = d(\phi(g_1, g_2, \ldots, g_n) * \phi(h_1, h_2, \ldots, h_n))$$
\[
\begin{align*}
&= \max \{ d_1(\phi_1(g_1), \phi_1(h_1)), \ldots, d_n(\phi_n(g_n), \phi_n(h_n)) \} \\
&\leq \max \{ k_1[d_1(g_1, \phi_1(g_1)) + d_1(h_1, \phi_1(h_1))), \ldots, k_n[d_n(g_n, \phi_n(g_n)) + d_n(h_n, \phi_n(h_n))] \} \\
&\leq \max \{ k[d_1(g_1, \phi_1(g_1)) + d_1(h_1, \phi_1(h_1))), \ldots, k[d_n(g_n, \phi_n(g_n)) + d_n(h_n, \phi_n(h_n))] \} \\
&= k \max \{ [d_1(g_1, \phi_1(g_1)) + d_1(h_1, \phi_1(h_1))), \ldots, [d_n(g_n, \phi_n(g_n)) + d_n(h_n, \phi_n(h_n))] \} \\
&= k \{ (d_1(g_1, \phi_1(g_1)), \ldots, d_n(g_n, \phi_n(g_n))) + (d_1(h_1, \phi_1(h_1)), \ldots, d_n(h_n, \phi_n(h_n))) \} \\
&= k[d(g, \phi(g) + d(h, \phi(h))].
\end{align*}
\]

Hence, \( \phi \) is a Kannan contraction with contraction ratio \( k \) consequently, \( G_1 \times G_2 \times \cdots \times G_n \) is a strong self-similar group of Kannan contraction.

Proposition 2.3 shows that, the finite product of strong self-similar groups of Kannan contraction is also a strong self-similar group of Kannan contraction. If \( G \) is self-similar group of Kannan contraction then the Propositions 2.1, 2.2 and 2.3 are true and the proofs are similarly done for \( G \).

**Proposition 2.4.** A self-similar group of continuous Kannan contraction is a disconnected set.

**Proof.** Let \( G \) be a self-similar group of Kannan contraction. Then \( G \) is a topological group. Proposition 2.1 shows that \( G \) is the attractor of the KIFS \( \{ \phi_0, \ldots, \phi_{n-1} \} \). For every \( i = 1, 2, \ldots, n-1 \), the mappings
\[
\phi_i : G \longrightarrow \phi_i(G),
\]
are Kannan contraction. Furthermore, we have
\[
G = \phi_0(G) \cup \phi_1(G) \cup \cdots \cup \phi_{n-1}(G)
\]
\[
\emptyset = \phi_i(G) \cap \phi_j(G)
\]
for all \( i, j \in \{ 0, 1, 2, \ldots, n-1 \} \) and \( i \neq j \). It is well known that the image of a compact set under a continuous map is compact and every compact subspace of a Hausdorff
space is closed. Therefore, \( \phi_i(G) \) is a closed set for \( i = 0, 1, 2, \ldots, n - 1 \). Due to the fact that

\[
G = \phi_0(G) \cup [\phi_1(G) \cup \cdots \cup \phi_{n-1}(G)]
\]

\[
\emptyset = \phi_0(G) \cap [\phi_1(G) \cup \cdots \cup \phi_{n-1}(G)],
\]

we obtain that \( \{\phi_0(G), [\phi_1(G) \cup \cdots \cup \phi_{n-1}(G)]\} \) is a closed separation of \( G \). Hence \( G \) is disconnected set.

Proposition 2.4 demonstrates that the disconnectedness of a topological group is necessary for self-similar group of Kannan contraction while the Proposition 2.5 shows that the totally disconnectedness of a topological group is necessary for strong self-similar group in the sense of KIFS.

**Proposition 2.5.** A strong self-similar group of continuous Kannan contraction is a totally disconnected set.

**Proof.** Let \( G \) be a strong self-similar group of continuous Kannan contraction. By Proposition 2.1, \( G \) is the attractor of KIFS \( \{\phi_0, \ldots, \phi_{n-1}\} \). Since \( \phi_0 : G \rightarrow H \) is one-to-one, we get

\[
\phi_i(g) = \phi_i(h)
\]

\[
\phi_0(g) \ast x_i = \phi_0(h) \ast x_i
\]

\[
\phi_0(g) = \phi_0(h)
\]

\[
g = h,
\]

for all \( g, h \in G \). This shows that \( \phi_i \) is one-to-one for \( i = 1, 2, \ldots, n - 1 \). In addition to that, \( \phi_i(G) \cap \phi_j(G) = \emptyset \), for all \( i, j \in \{0, 1, 2, \ldots, n - 1\} \) and \( i \neq j \). Consequently, \( G \) is a totally disconnected set on account of the Theorem 2.2.

The following theorem gives the relation between profinite group and strong self-similar group of Kannan contraction.
**Theorem 2.3.** A strong self-similar group of Kannan contraction is a profinite group.

**Proof.** Let $G$ be a strong self-similar group of Kannan contraction. By Definition 2.3, $G$ is a compact topological group and also $G$ is Hausdorff, since every metric space is Hausdorff. Proposition 2.5 shows that $G$ is totally disconnected set. Finally we get $G$ is compact, Hausdorff and totally disconnected. Thus, we have the properties which characterize the profinite group. This shows that a strong self-similar group of Kannan contraction is a profinite group.

The converse of the Theorem 2.3 is need not be true. That is, profinite is the necessary condition for the topological group being strong self-similar group of Kannan contraction. For example, consider the group $G = (\mathbb{Z}_n, \oplus)$ with discrete metric

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

It is clear that $G$ is compact topological group with respect to discrete topology generated by the discrete metric $d$. Besides, $G$ is Hausdorff and totally disconnected so $G$ is profinite. Let $H = \{0\}$, where 0 is the identity element of $G$. So, it is clear that $[G : H] = n$. Further, the mapping $\phi : G \to H$ defined by

$$\phi(g) = 0,$$

is a surjective group homomorphism and Kannan contraction. As per the Definition 2.2, $G$ is a self-similar group of Kannan contraction but not a strong self-similar group. Since, $H$ is not isomorphic to $G$. From this example, we conclude that the converse of the Theorem 2.3 is not true.
2.4 Concluding Remarks

In this chapter, we have introduced the self-similar group and strong self-similar group of Kannan contraction. The representation of the strong self-similar group of Kannan contraction has presented through the attractor of KIFS and one of the Kannan contractions of KIFS is a group isomorphism. Further, the necessary condition for the strong self-similar group has demonstrated. Finally, we have established the relation between profinite group and strong self-similar group of Kannan contraction.