CHAPTER 6
AN INTEGRATED INVENTORY MODEL WITH SETUP COST REDUCTION AND QUALITY IMPROVEMENT FOR NON-DEFECTIVE ITEMS AND DEFECTIVE ITEMS

6.1 INTRODUCTION

To survive in today’s highly competitive world, manufacturers need to find ways to reduce production time and costs in order to improve operating performance and product quality. Nowadays, targets of an increased productivity, operational availability and better overall efficiency of the production line are the most important goals for almost all manufacturing companies. Research on setup cost reduction in inventory models has received increasing attention and hot topic in recent years. Setup cost is the main part in the inventory system.

Manufacturing is the backbone of any industrialized nation. A high service level with regard to the product supply of customers is one of the most important objectives for manufacturing industries. Today this aim is achieved with the help of sufficient inventories and adequate delivery capacities within the supply chain. However, supply chain and/or inventory costs become a key factor in the manufacturing industries due to a changing environment. There are number of costs that make up the total inventory costs for manufacturing company. One of those costs is the setup cost which is the cost incurred to get equipment ready to process a different batch of goods. Hence, setup cost is regarded as a batch-level cost in activity based costing. A setup cost include the costs of changing the tools or dies on the equipment, moving materials or
components, and testing the initial output to be certain it meets the specifications. In addition to the out-of-pocket costs, such as the labor cost of setting up the equipment, there is a much greater cost. The greater cost of setup is the lost opportunity of manufacturing profitable output while the machine is idled during the setup time. Setup cost is viewed as a non-value-added cost that should be minimized. Companies have radically changed the way they act, nowadays, one of the main objectives while developing the productive process and work process is the cost reduction. Resources are being optimized as much as they can, to try to improve the efficiency in the productive process so this can make possible a significant cost reduction to make companies more competitive in this difficult economical moment that we are facing nowadays. After the recent recessions many companies are looking for ways to boost sales and increase profits; the most efficient way of funding the growth profitability is to free up costs and capital and then re-invest these funds in the most promising growth opportunities. One of the best examples of a company which has achieved this is Wrigley, the chewing gum manufacturer. Since the mid 1990’s they have significantly improved gross margins and overall operating efficiency, and then redirected much of the savings towards increased marketing, trade spend and innovation to drive growth. As a result they have been able to outperform competitors and provide good returns for shareholders. The success of the Japanese in the employment of Just-In-Time (JIT) production has received a great deal of attention in the past two decades. The critical aim of JIT from the production/inventory management standpoint is to produce high quality products with minimum cost. Investing capital in reducing setup/or ordering cost regarded as the most effective means of achieving this target. Today the
equipment cost, purchase cost, and other cost of a technology product become more and more significant in the market. Investing in equipment that makes the manufacturing process faster can actually lower the production costs in the long run. Likewise, machinery that uses less material can also reduce costs. For instance, in India, small scale industries spend an extra added cost by means of investment and it is expected to have a result to reduce the expected total cost of the supply chain. It is imperative to thoroughly research potential capital investment benefits versus costs required before purchasing new equipment. Hence this setup cost reduction inventory model involving probabilistic defective items at a vendor is more matched to real life supply chains.

Many manufacturers face cost reduction and efficiency challenges in their manufacturing operations. It includes the cost of tooling up, administrative cost, record keeping, placing large orders etc. Setup cost can be reduced by starting a business where raw materials, skilled laborers and purchasing ability are plenty. Aligarh locks are cheaper than company products because skilled laborers are available at cheaper wages. Further, in drought areas the government encourage the people to start (suitable) manufacturing units by giving several concessions like tax exemption, power tariff reduction etc.; consequently setup cost is reduced. For example, faster changeovers have been associated with lower inventory, faster throughput, shorter lead time, improved quality, and lower unit cost. Quick setups are also considered an important element for successfully implementing JIT production or time-based competition. Therefore, for attaining production system efficiency, reduced lot sizes alone are not sufficient, unless accompanied by corresponding setup cost reduction and quality improvement. Therefore,
considerable attention has been paid to the optimal lot sizing and investments in setup cost reduction and quality improvement. So, considerable attention has been paid to the optimal lot sizing and investments in setup cost reduction and quality improvement.

Quality improvement should improve an organization’s ability to achieve its business goals. Quality improvement processes using the data are essential to improve quality of care. It is also essential that quality monitoring processes are included to ensure that the desired results are being achieved and maintained. However, use of existing data, routinely collected primarily for clinical care to individual clients, offers an important opportunity for more efficient and rapid quality improvement interventions, as they do not require large additional investments in data collection.

Quality reduction has been extensively studied in accordance with Just-In-Time. The relation between the quality and inventory reduction is critical for both practitioners and academics because numerous modern production systems advocate reduction in inventory and improvement in quality. In recent years, the successful Japanese experience of employing JIT production has triggered considerable attention. Quality improvement and setup cost reduction played a very significant role in improving efficiency of businesses and organizations from every sphere by reducing redundancy in costs and enhancing productivity thereby accounting for the flexibility of today’s diverse business environment.

To the best of our knowledge, the author has developed integrated inventory model with setup cost reduction and quality improvement for non-defective items and defective items. The purpose of this chapter is to find out an optimal inventory strategy that can minimize the value of
the single vendor and the single buyer integrated total cost in a supply chain system.

The remainder of this chapter includes the following aspects: Problem description is given in section 6.2. Section 6.3, expresses the assumption and notations used throughout this chapter. In section 6.4, model formulation is developed for non-defective items. Solution procedure for non-defective items with setup cost reduction is given in section 6.5. An efficient algorithm is developed to obtain the optimal solution for non-defective items in section 6.6. In section 6.7, model formulation is developed for defective items with setup cost reduction under investment for quality improvement. Solution procedure for defective items with setup cost reduction under investment for quality improvement is given in section 6.8. An efficient algorithm is developed to obtain the optimal solution for defective items with setup cost reduction under investment for quality improvement in section 6.9. Numerical examples are provided for both defective and non-defective items in section 6.10 to illustrate the results. Finally, we give conclusions of the chapter in section 6.11.

6.2 PROBLEM DESCRIPTION

For quality improvement purposes often times, a manufacturing unit has to change certain parts of equipment. Any such changes in the assembly line manufacturing system or production process involves a cost known as the setup cost. Minimizing the setup cost and improving the product quality is of prime importance in today’s competitive business area. We have formulated the lead time and the setup cost reduction with quality improvement in a single vendor and a single buyer integrated inventory model. This chapter develops the effects of
setup cost reduction under investment for quality improvement in a supply chain system.

This chapter is focused on controllable lead time with setup cost reduction and quality improvement for non-defective items and defective items. In this chapter the concept of capital investment as logarithmic function is defined to reduce the process quality and setup cost.

Our objective is to develop an algorithm to determine the optimal order quantity, setup cost, process quality and lead time simultaneously, so that the integrated total cost incurred can be minimized. A solution procedure is developed to find the optimal solution and numerical examples are provided to illustrate the results of the defective items as well non-defective items.

6.3 ASSUMPTIONS AND NOTATIONS

To establish the mathematical model, the following assumptions and notations are adopted.

6.3.1 Assumptions

The assumptions made in this chapter are as follows.

1. The product is manufactured with a finite production rate \( P \), and \( P > D \).

2. This model deals with a single vendor and a single buyer for a single commodity.

3. Inventory is continuously reviewed. The purchaser places an order when the inventory position reaches the reorder point \( R \). The reorder point \( R = \) expected demand during lead time+ safety stock \( SS \), and \( SS = k \times (\text{standard deviation of lead time demand}) \), that is , \( R = DL + k\sigma\sqrt{L} \) where \( k \) is a safety factor.
4. The demand $X$ during lead time $L$ follows a normal distribution with mean $\mu L$ and standard deviation $\sigma \sqrt{L}$.

5. The lead time $L$ consists of $n$ mutually independent components. The $i^{th}$ component has a normal duration $b_i$, minimum duration $a_i$ and crashing cost per unit time $c_i$. For convenience, we rearrange $c_i$ such that $c_1 < c_2 < c_3 < \ldots < c_n$. The components of lead time are crashed one at a time starting from the first component because it has the minimum unit crashing cost and then the second component and so on.

6. Let $L_o = \sum_{i=1}^{n} b_i$ and $L_i$ be the length of lead time with components 1, 2, 3,..., $i$ crashed to their minimum duration and then $L_i$ can be expressed as $L_i = L_o - \sum_{j=1}^{i} (b_j - a_j), i = 1, 2, \ldots, n$ and the lead time crashing cost per cycle $R(L) = c_i (L_{i-1} - L) + \sum_{j=1}^{i-1} c_j (b_j - a_j), L \in [L_{i-1}, L_i]$. In addition, the length of lead time is equal for all shipping cycles and the lead time crashing costs occur in each shipping cycle. (Liao and Shyu [98]).

7. To make a product more perfect, the relationship between lot size and product quality is considered. The production process is unspecified to be at an in-control circumstance at the beginning of production. During the production process, the process may transfer to an out-of-control circumstance, and then it starts to produce defective units and continues awaiting the entire lot is produced (see, for instance Sarkar et al. [150]).

8. Two investments are considered to reduce setup cost and to improve the quality of products.
9. The basic inventory model is regularly based on the assumption of fixed setup cost, process quality. By using investment; we can decrease the setup cost and process quality of this chapter. The opening investment may be high but total cost will be concentrated in each step by using the opening investment function we assume a logarithmic investment function for this purpose (refer to Porteus [132]).

10. The extra costs incurred by the vendor will be fully transferred to the buyer if shortened lead time is requested. (Yang and Pan [191]).

11. At the start of each production cycle, the production process is always in an in-control state and perfect items are produced. During a production run, the production process may shift from an in-control state to an out-of-control state.

12. Once the production process shifts to an out-of-control state, the shift cannot be detected until the end of the production cycle and the process continues production and a fixed proportion of the produced items are defective.

13. All defective items produced are detected after the production cycle is over, and rework cost for defective items will be incurred. (Hou [67]).

6.3.2 Notations
To develop the model, the following notations are used.

$I(S)$ Capital investment required to achieve setup cost $S$, $0 < S \leq S_0$

$I(\theta)$ Capital investment required to reduce process quality $\theta$, $0 < \theta \leq \theta_0$

$\alpha$ Annual fractional opportunity cost of capital per unit time. (e.g. interest rate).

$s$ Safety stock
\( k \) Safety factor

\( b \) Percentage decrease in setup cost per dollar increase in the investment to reduce the setup cost.

\( q \) Percentage decrease in out of control probability per dollar increase in the investment to reduce the out of control probability

\( TVB \) Total cost of the vendor and the buyer.

### 6.4 MODEL FORMULATION FOR NON-DEFECTIVE ITEMS

The buyer places an order after every \( Q \) units, therefore, for average cycle time of \( \frac{Q}{D} \), the expected ordering and lead time crashing costs per unit can be given by \( \frac{AD}{Q} \) and \( \frac{DR(L)}{Q} \) respectively.

The expected net inventory level just before arrival of a procurement is the safety stock \( s = R - DL \). The expected net inventory level immediately after arrival of procurement is \( Q + s \). Hence the average inventory over the cycle can be approximated by \( \left( \frac{Q}{2} \right) + s \), i.e. \( \left( \frac{Q}{2} \right) + k\sigma \sqrt{L} \) (Assumption 3) so, the buyer’s holding cost per unit time is \( rC_b \left( \frac{Q}{2} + k\sigma \sqrt{L} \right) \). Since \( S \) is the vendor’s setup cost per setup, and the production quantity for the vendor in a lot is \( mQ \), the expected set up cost per year is given by \( \left( \frac{D}{mQ} \right) S \).

The integrated inventory model is designed for a vendor’s production situations in which once an order is placed, the production begins and a constant number of units are added to the inventory each day until the production run has been completed. The vendor produces
the item in the quantity of $mQ$ and the buyer would receive it in $m$ lots, with which each having a quantity of $Q$. The inventory pattern for this model is shown in figure (6.1).

In figure (6.2), for the vendor, the average inventory can be evaluated as follows: Vendor’s average inventory is evaluated as the difference of the vendor’s accumulated inventory and the buyer’s accumulated inventory and hence the vendor’s holding total cost is

$$\left[ mQ \left( \frac{Q}{P} + (m-1) \frac{Q}{D} \right) - \frac{m^2Q^2}{2P} \right] - \left[ \frac{Q^2}{D}(1+2+\ldots+(m-1)) \right] \frac{mQ}{D}$$

$$= \frac{Q}{2} \left[ m \left( 1 - \frac{D}{P} \right) - 1 + \frac{2D}{P} \right].$$

So the vendor’s holding cost per unit time is given by

$$\left( \frac{Q}{2} \left[ m \left( 1 - \frac{D}{P} \right) - 1 + \frac{2D}{P} \right] \right) r_v.$$

![Figure (6.1) Inventory pattern for the vendor](image-url)
Figure (6.2) Inventory pattern for the vendor and the buyer

If the buyer’s order quantity is $Q$ and the vendor’s lot size is $mQ$, then the integrated total cost is given by

$$ITVB(Q, L, m) = \frac{DA}{Q} + \frac{D}{mQ} R(L) + \left( \frac{D}{mQ} \right) S + \left( \frac{Q}{2} \left( m \left( 1 - \frac{D}{P} \right) - 1 + \frac{2D}{P} \right) \right) rC_v + rC_b$$

$$+ \left( \frac{Q}{2} + k\sigma \sqrt{L} \right) rC_b$$

(6.1)

Building upon equation (6.1), we desire to study the effects of investment on setup cost reduction. Now we take setup cost, which is given in equation (6.1) as decision variables. When setup cost $S$, is no longer considered to be one of the fixed parameters but one of the decision variables, the control of the setup cost level is accomplished by varying the capital investment allocated to reducing setup cost.

That is, by introducing capital investment, we can lower the setup cost from the original setup cost level. Several relationships between the amount of capital investment and the setup cost level have been reported by numerous researchers. Kim et al. [87] and Priyan and Uthayakumar [135], Priyan and Uthayakumar [136] presented several classes of setup reduction functions and described a general solution procedure. Several
relationships between the amount of capital investment and the setup cost level have been reported by numerous researchers.

This chapter assumes that the relationship between setup cost and process quality capital investment can be described by the logarithmic investment function. Logarithmic is closely related operations with vital important production applications.

The logarithmic function is one of many possible investment functions, and it may be an interesting research topic to consider a general investment function. Thus, in this study, a logarithmic investment function is assumed for investment in setup cost reduction and process quality. It is commonly used by bankers to determine the present value and future value of investments. Economists also make frequent use of exponents to describe cost and production functions, as well as demand curves.

Besides, natural logarithmic is important primarily because they allow you to solve complicated exponential expressions that use $e$ as the exponential base. It occurs naturally in many situations, including the continuous compounding of interest. Thus the logarithmic relationship of setup cost to investment discussed is not only an interesting special case but also a practical one. Therefore, we assume that the relationship between setup cost reduction and capital investment can be described by the continuous logarithmic investment function in order to accommodate more practical features of the real manufacturing inventory systems.

For the logarithmic investment function, the setup cost decreases exponentially as the investment amount increases. For example, if $S_0$ is 100, it may cost $150 to reduce the setup cost by 10% to 90, another $150 to reduce it to 81, and so on.
Accordingly, the relationship between setup cost $S$, and setup cost reduction investment $I$, is given by $S = S_0 \exp(-\alpha t)$, where $\alpha$ factor such that $\exp(-\alpha)$ is the fractional decrease in $S$ per dollar increase in $I$. Alternately, the relationship can be stated as

$$I(S) = b \ln \left( \frac{S_0}{S} \right) \text{ for } 0 < S \leq S_0$$

(6.2)

where for $b = \left( \frac{1}{\delta} \right)$, $I(S)$ is a convex and strictly decreasing function. This logarithmic investment function has been utilized by several authors for example see Porteus [131], Porteus [132] and others (e.g., [110], [115], [113], [114], [119], [120], [122], [134], [151], [152], [177] and [191]). They stated that this logarithmic investment function is more appropriate and we can obtain a significant savings using this logarithmic investment function.

Here, investment $I(S)$ (expressed in equation (6.2)) is the one-time investment cost whose benefits will extend indefinitely into the future. Thus, the annual cost of such an investment is $\alpha d(s)$, where $\alpha$ is the annual fractional cost of capital investment (e.g., interest rate).

Then our objective is to minimize the integrated total cost per unit time denoted by $ITC(Q,L,m,S)$ namely, the sum of the capital investment cost for reducing setup cost $S$ as expressed in equation (6.2) and the inventory integrated total cost as expressed in equation (6.1) by optimizing $Q$, $S$ and $L$ constrained on $0 < S \leq S_0$.

$$ITC(Q,L,m,S) = \frac{DA}{Q} + \frac{D}{Q} R(L) + \left( \frac{D}{mQ} \right) S + \left( \frac{Q}{2} \left[ \frac{D}{m} \left( \frac{1}{m} - \frac{D}{P} \right) - 1 + \frac{2D}{P} \right] rC_v + rC_b \right)$$

$$+ \left( \frac{Q}{2} + k \sigma \sqrt{L} \right) rC_b + ab \ln \left( \frac{S_0}{S} \right)$$

(6.3)
Subject to the constraint, \(0 < S \leq S_0\), where \(\alpha\) is the annual fractional cost of capital investment (e.g., interest rate).

### 6.5 Solution Procedure for Non-Defective Items with Setup Cost Reduction

To solve the above non-linear programming problem, temporarily ignore the constraint \(0 < S \leq S_0\) and relax the integer requirement on \(m\) (the number of deliveries from the vendor to the buyer during one production cycle). For a fixed \(Q, S\) and \(L \in [L_s, L_{-1}]\), \(ITC(Q, L, m, S)\) can be proved to be a convex function of \(m\). Consequently, the search for the optimal delivery \(m^*\) is reduced to find a local minimum.

**Property 1.** For a fixed \(Q, S\) and \(L \in [L_s, L_{-1}]\), \(ITC(Q, L, m, S)\), is convex in \(m\). Taking the first and second order partial derivatives of \(ITC(Q, L, m, S)\) with respect to \(m\), we have

\[
\frac{\partial ITC(Q, L, m, S)}{\partial m} = \frac{-DA}{Qm^2} + \frac{Q}{2} \left[ rC_v \left(1 - \frac{D}{p}\right)\right],
\]

and

\[
\frac{\partial^2 ITC(Q, L, m, S)}{\partial m^2} = \frac{2DA}{Qm^3} > 0.
\]

Therefore, \(ITC(Q, L, m, S)\) is convex in \(m\), for a fixed \(Q, S\) and \(L \in [L_s, L_{-1}]\).

Next, the first order partial derivatives of \(ITC(Q, L, m, S)\) with respect to \(Q, S\) and \(L \in [L_s, L_{-1}]\) respectively are taken for a fixed \(m\). This process yields

\[
\frac{\partial ITC(Q, L, m, S)}{\partial Q} = -\frac{D}{Q^2} \left[ A + \frac{S}{m} + R(L) \right] + \frac{r}{2} \left( m \left(1 - \frac{D}{P}\right) - 1 + \frac{2D}{P} \right) C_v + C_b, \quad (6.6)
\]

\[
\frac{\partial ITC(Q, L, m, S)}{\partial S} = \frac{D}{Qm} - \frac{\alpha b}{S} \quad (6.7)
\]
\[
\frac{\partial ITC(Q, L, m, S)}{\partial L} = \frac{Dc_i}{Q} + \frac{rC_vk\sigma L^{\frac{1}{2}}}{2}
\]  
\hspace{1cm} (6.8)

Furthermore, for a fixed \((Q, S, m)\), \(ITC(Q, L, m, S)\) is noted to be a concave function in \(L \in [L_i, L_{r-1}]\), because

\[
\frac{\partial ITC(Q, L, S, m)}{\partial L^2} = -\frac{rC_vk\sigma L^{\frac{3}{2}}}{4} < 0
\]  
\hspace{1cm} (6.9)

Hence, for a fixed \((Q, S, m)\) the minimum integrated total cost per unit time occurs at the end points of the interval \(L \in [L_i, L_{r-1}]\). On the other hand, by setting equations (6.6) and (6.7) equal to zero, we obtain

\[
Q = \sqrt{\frac{2D\left(A + \frac{S}{m} + R(L)\right)}{n\left(m\left(1 - \frac{D}{P}\right) - 1 + \frac{2D}{P}\right)c_v + C_v}}
\]  
\hspace{1cm} (6.10)

\[
S = \frac{abQm}{D}
\]  
\hspace{1cm} (6.11)

For a fixed \(m\) and \(L \in [L_i, L_{r-1}]\), by solving equations (6.10) and (6.11) we can obtain the values of \((Q^*, S^*)\). Moreover, it can be verified that the Second Order Sufficient Conditions (SOSCs) are satisfied as follows. For a fixed \(m, L \in [L_i, L_{r-1}]\) the Hessian Matrix \(ITC(Q, L, m)\) is positive definite \(Q^*, S^*\). The proof is shown in appendix (E).

Further, based on the convexity behavior of the objective function with respect to the decision variable, we establish the following iterative algorithm is designed to find the optimal values of order quantity \(Q\), lead time \(L\), setup cost \(S\) and the total number of deliveries \(m\) in order to minimize the integrated total cost \(ITC(Q, L, m, \theta)\).
6.6 ALGORITHM FOR NON-DEFECTIVE ITEMS WITH SETUP COST REDUCTION

Step 1. set \( m = 1 \).

Step 2. For each \( L, i = 1, 2, 3...n \), perform (2.1)-(2.4)

2.1. Start with \( S_{i1} = S_0 \).

2.2. Substitute \( S_{i1} \) into Eq. (6.10) and evaluate \( Q_{i1} \).

2.3. Utilizing \( Q_{i1} \) determine \( S_{i2} \) from equation (6.11).

2.4. repeat Steps (2.1)-(2.3) until no change occurs in the values of \( Q_{i}, S_{i} \). Denote the solution \((Q_{i}', S_{i}')\).

Step 3. Compare \( S_{i}' \) with \( S_0 \)

3.1 If \( S_{i}' < S_0 \) then the current solution is optimal for the given \( L \).

We denote the optimal solution by a \((Q_0, S_0)\), i.e., if \((Q_0, S_0) = (Q_{i}', S_{i}')\) then go to step (4).

3.2 If \( S_{i}' \geq S_0 \), then for this given \( L \), \( S_{i}' = S_0 \) and utilize Eq. (6.11) (replace \( S \) by \( S_0 \)) to determine the new \( Q_0 \) by procedure similar to the one in step 2.

Step 4. Utilize equation. (6.3) to calculate the corresponding \( ITC(Q_0, S_0, L, m) \).

Step 5. Find \( \min 1, 2, 3...n ITC(Q_0, S_0, L, m) \).

Step 6. If \( ITC(Q^*, S^*, L^*, m^*) = \min 1, 2, 3...n ITC(Q_0, S_0, L, m) \), then \((Q^*_m, S^*_m, L^*_m) \) is the optimal solution for a fixed \( m \).

Step 7. Set \( m = m + 1 \) and repeat steps (2)–(6) to get \( ITC(Q^*_m, S^*_m, L^*_m, m) \).

Step 8. If \( ITC(Q^*_m, S^*_m, L^*_m, m) \geq ITC(Q^*_m, S^*_m, L^*_m, m + 1) \), then go to step 7, otherwise go to step 9.

Step 9. Set \( ITC(Q^*, S^*, L^*, m) = ITC(Q^*_m, S^*_m, L^*_m, m + 1) \).
then \((Q^*, S^*, L^*, m^*)\) is the optimal solution.

### 6.7 MODEL FORMULATION FOR DEFECTIVE ITEMS

We note that the equation (6.1) the possible relationship between quality and lot size is ignored and hence no quality-related cost is considered. However, as mentioned previously, this may not be realistic. We shall then modify the equation (6.1) by considering the possible relationship between quality and lot size.

Specifically, the following assumption of imperfect production process as in [132] will be employed to formulate this relationship. That is, while producing a lot, the process can go out-of-control probability \(\theta\) (in general, \(\theta\) is very small and close to zero). Once out-of-control, the process produces defective units and continues to do so until the entire lot is produced.

The expected number of defective items in a run of size \(mQ\) with a given probability \(\theta\) that the process can go out of control is \(\frac{m^2Q^2\theta}{2}\) (for more details, see [115], [117], [118], [119], [122], [132], [150], [152] and [191]). Thus, the defective cost per unit time is given by \(\frac{gmQD\theta}{2}\). Hence the total cost for vendor and buyer incorporating the defective cost per year can be represented by

\[
ITC(Q, L, m) = \frac{DA}{Q} + \frac{D}{Q} R(L) + \left(\frac{D}{mQ}\right) S + \left(\frac{Q}{2}\left(m\left(1 - \frac{D}{P}\right) - 1 + \frac{2D}{P}\right) rC_v + rC_b\right)
+ \left(\frac{Q}{2} + k\sigma\sqrt{L}\right) rC_v + \frac{gmQD\theta}{2}
\]

(6.12)

Building upon the equation (6.12) we wish to study the effects of investment for quality improvement. We note that quality improvement requires the cooperation of the vendor. Also, in JIT systems, it can be
observed that buyers often devote much time and effort to establishing a long-term relationship with their suppliers and even provide the financial support to help their suppliers improve the production process.

The total cost in (6.12) does not include any investment on the part of the vendor to improve the process quality. However, as mentioned earlier, process quality is an important tool in the hands of the decision maker since its control is needed to lower associated costs incurred and the production of smaller batch sizes of better quality products. Therefore, it is quite appropriate for the vendor to make an investment to try and reduce the number of defective items produced. Assuming a logarithmic investment function of the process quality is the form is given below

\[ I(\theta) = q \ln \left( \frac{\theta_0}{\theta} \right) \text{ for } 0 < \theta \leq \theta_0 \]  

(6.13)

where \( \theta_0 \) is the original probability that the production process can go out-of-control and \( q = 1/\xi \) with \( \xi \) denoting the percentage decrease \( \theta \) per dollar increase in investment \( I(\theta) \).

The investment \( I(S) \) and \( I(\theta) \) required to reduce the setup cost, process quality from the original level \( S_0, \theta_0 \) to target level \( S, \theta \) where \( I(S) \) and \( I(\theta) \) are functions which are convex and strictly decreasing. Here, investments \( I(S) \) and \( I(\theta) \) are one-time investment costs whose benefits will extend indefinitely into the future.

Thus, the integrated total cost of such investment is \( I(S) \) and \( I(\theta) \), where \( \alpha \) is the annual fractional cost of capital investment (e.g., interest rate). This logarithmic investment function is consistent with the Japanese experience as reported in Hall [53]; and has been used by Porteus [132] and others (Kim et al. [87], Coates et al. [27] and
Annadurai and Uthayakumar [2] and Annadurai and Uthayakumar [3]).
Then our objective is to minimize the inventory integrated total cost per unit time denoted by $ITC(Q, L, m, \theta, S)$, namely, the sum of the capital investment cost for reducing the setup cost $S$ and process quality $\theta$ is expressed in equations (6.2) and (6.13) and the inventory integrated total cost is expressed in equation (6.12) by optimizing $Q$, $S$, $\theta$ and $L$
subject to the constrained on $0 < S \leq S_0$, $0 < \theta \leq \theta_0$. 

$ITC(Q, L, m, S, \theta) = \frac{DA}{Q} + \frac{D}{Q} R(L) + \left(\frac{D}{mQ}\right) S + rC_s \left(\frac{Q}{2} \left( m \left(1 - \frac{D}{P}\right) - 1 + \frac{2D}{P} \right) \right) + \left( \frac{Q}{2} + k\sigma \sqrt{L} \right) rC_v$

$+ \alpha b \ln \left( \frac{S}{S_o} \right) + \frac{gmQD\theta}{2} + \alpha q \ln \left( \frac{\theta}{\theta} \right)$

$= \frac{D}{Q} \left( A + \frac{S}{m} + R(L) \right) + \frac{Q}{2} \left( m \left(1 - \frac{D}{P}\right) - 1 + \frac{2D}{P} \right) rC_v + rC_b + gmD\theta$

$+ rC_p k\sigma \sqrt{L} + \alpha b \ln \left( \frac{S}{S_o} \right) + \alpha q \ln \left( \frac{\theta}{\theta} \right)$

(6.14)

Subject to $0 < A \leq A_0$ and $0 < \theta \leq \theta_0$ where $\alpha$ is the annual fractional cost of capital investment (e.g., interest rate).

6.8 SOLUTION PROCEDURE FOR DEFECTIVE ITEMS WITH SETUP COST REDUCTION UNDER INVESTMENT FOR QUALITY IMPROVEMENT

To solve the above non-linear programming problem, temporarily ignores the constraint $0 < S \leq S_0$, $0 < \theta \leq \theta_0$ and relax the integer requirement on $m$ (the number of deliveries from the vendor to the buyer during one production cycle). For a fixed $Q, S, \theta$ and $L \in [L_i, L_{i-1}]$, $ITC(Q, L, m, S, \theta)$ can be proved to be a convex function of $m$. Consequently, the search for the optimal deliveries $m^*$ is reduced to find a local minimum.
Property 1. For a fixed $Q, S, \theta$ and $L \in [L_i, L_{i-1}]$, $ITC(Q, L, S, \theta, m)$ is convex in $m$. Taking the first and second order partial derivatives of $ITC(Q, L, S, \theta, m)$ with respect to $m$, we have

\[
\frac{\partial ITC(Q, L, S, \theta, m)}{\partial m} = -\frac{DS}{Qm^2} + \frac{Q}{2} \left[ rC_v \left( 1 - \frac{D}{P} \right) + gD \theta \right] \quad (6.15)
\]

\[
\frac{\partial^2 ITC(Q, L, S, \theta, m)}{\partial^2 m} = \frac{2DS}{Qm^3} > 0 \quad (6.16)
\]

Therefore, $ITC(Q, L, S, \theta, m)$ is convex in $m$, for a fixed $Q, \theta, S$ and $L \in [L_i, L_{i-1}]$.

Next, the first order partial derivatives of $ITC(Q, L, S, \theta, m)$ with respect to $Q, \theta, S$ and $L \in [L_i, L_{i-1}]$ are respectively taken for a fixed $m$. This process yields

\[
\frac{\partial ITC(Q, \theta, m, S, L)}{\partial Q} = -\frac{D}{Q^2} \left( A + \frac{S}{m} + R(L) \right) + \frac{1}{2} \left( \left( m \left( 1 - \frac{D}{P} \right) - 1 + \frac{2D}{P} \right) rC_v + rC_b + gmd \theta \right) \quad (6.17)
\]

\[
\frac{\partial ITC(Q, \theta, m, S, L)}{\partial \theta} = \frac{QgmD}{2} - \frac{a \theta}{Q} \quad (6.18)
\]

\[
\frac{\partial ITC(Q, \theta, m, S, L)}{\partial S} = \frac{D}{Qm} - \frac{a \theta}{S} \quad (6.19)
\]

\[
\frac{\partial ITC(Q, \theta, m, S, L)}{\partial L} = \frac{Dc_i}{Q} + \frac{rC_p k \sigma L^{\frac{1}{2}}}{2} \quad (6.20)
\]

Furthermore, for a fixed $(Q, S, \theta, m)$, $ITC(Q, L, S, \theta, m)$ is noted to be a concave function in $L \in [L_i, L_{i-1}]$, because

\[
\frac{\partial^2 ITC(Q, \theta, L, S, m)}{\partial L^2} = -\frac{rC_p k \sigma L^{\frac{3}{2}}}{4} < 0 \quad (6.21)
\]
Hence, for a fixed \((Q,S,\theta,m)\) the minimum total cost per unit time occurs at the end points of the interval \(L \in [L_{i}, L_{i-1}]\). On the other hand, by setting equations (6.17) - (6.19) equal to zero, we obtain

\[
Q = \frac{2D\left(A + \frac{S}{m} + R(L)\right)}{rC_{v}\left(m\left(1-\frac{D}{P}\right)-1+\frac{2D}{P}\right)+rC_{b}} + gmD\theta
\]  
(6.22)

\[
S = \frac{\alpha bQm}{D}
\]  
(6.23)

\[
\theta = \frac{2\alpha q}{gmDQ}
\]  
(6.24)

For a fixed \(m\) and \(L \in [L_{i}, L_{i-1}]\), by solving equations (6.22)-(6.24), we can obtain the values of \((Q^*, \theta^*, S^*)\). Moreover, it can be verified that the Second Order Sufficient Conditions (SOSCs) are satisfied as follows. For a fixed \(m\), \(L \in [L_{i}, L_{i-1}]\) the Hessian Matrix \(ITC(Q,L,\theta)\) is positive definite \(Q^*, S^*, \theta^*\) and \(L^*\). The proof is shown in appendix (F).

Further, based on the convexity behavior of the objective function with respect to the decision variable, we establish the following algorithm is designed to find the optimal values of order quantity \(Q\), setup cost \(S\), process quality \(\theta\), lead time \(L\) and the total number of deliveries \(m\) which minimizes the integrated total cost \(ITC(Q,\theta,L,S,m)\).

6.9 ALGORITHM FOR DEFECTIVE ITEMS WITH SETUP COST REDUCTION UNDER INVESTMENT FOR QUALITY IMPROVEMENT

**Step 1.** Set \(m=1\), since \(m\) is an integer

**Step 2.** For each \(L_{i}, i=1,2,3,...,n\), perform (2.1)-(2.4)

2.1. Start with \(\theta_{i} = \theta_{0}, S_{i} = S_{0}\).
2.2. Substitute $\theta_i, S_i$ into Eq. (6.21) and evaluate $Q_i$.

2.3. Utilizing $Q_i$ determines $S_{i2}$ and $\theta_{i2}$ from equations (6.22) and (6.23) respectively.

2.4. Repeat steps (2.1)-(2.3) until no change occurs in the values of $Q_i, \theta_i$ and $S_i$. Denote by $(Q'_i, \theta'_i, S'_i)$.

Step 3. Compare $S'_i$ with $S_0$ and $\theta'_i$ with $\theta_0$, respectively.

3.1 If $S'_i < S_0$ and $\theta'_i < \theta_0$, then the current solution is optimal for the given $L_i$. We denote the optimal solution by $(Q^0_i, \theta^0_i, S^0_i)$. If $(Q^0_i, \theta^0_i, S^0_i) = (Q'_i, \theta'_i, S'_i)$, then go to step (5).

3.2 If $S'_i \geq S_0$ and $\theta'_i < \theta_0$, then for this given $L_i$, let $S^0_i = S_0$ and utilize equation (6.21) (replace $S$ by $S_0$) and equation (6.23) to determine the new $(Q'_i, \theta'_i)$ by a procedure similar to the one in step 2 the result is denoted by $(Q^*_i, \theta^*_i)$. If $\theta^*_i < \theta_0$, then optimal solution is obtained, i.e., if $(Q^0_i, S^0_i, \theta^0_i) = (Q^*_i, S_0, \theta^*_i)$, then go to step 5; otherwise, go to step 4.

3.3 If $S'_i < S_0$ and $\theta'_i \geq \theta_0$, then for this given $L_i$, let $\theta^0_i = \theta_0$ and utilize equation (6.21) (replace $\theta$ by $\theta_0$) and equation (6.22) to determine the new $(Q'_i, S'_i)$ by a procedure similar to the one in step 2 the result is denoted by $(Q^*_i, S^*_i)$. If $S^*_i < S_0$, then optimal solution is obtained, i.e., if $(Q^0_i, S^0_i, \theta^0_i) = (Q^*_i, S^*_i, \theta^0_i)$, then go to step 5; otherwise, go to step 4.

3.4 If $S'_i \geq S_0$ and $\theta'_i \geq \theta_0$, then go to step 4.

Step 4. For given $L_i$, Let $\theta^0_i = \theta_0$ and $S^0_i = S_0$ and utilize equation (6.21) (replace $\theta$ by $\theta_0$ and $S$ by $S_0$) to determine the corresponding optimal solution $Q^0_i$ by a procedure similar to the one step 2.
Step 5. Utilize equation (6.14) to calculate the corresponding $\text{ITC}(Q^0_i, S^0_i, \theta^0_i, L_i, m)$.

Step 6. Find $\min_{1, 2, 3, \ldots, n} \text{ITC}(Q^0_i, S^0_i, \theta^0_i, L_i, m)$

Step 7. If $\text{ITC}(Q^*, S^*, \theta^*, L^*, m) = \min_{1, 2, 3, \ldots, n} \text{ITC}(Q^0_i, S^0_i, \theta^0_i, L_i, m)$, then $(Q^*_{(m)}, S^*_{(m)}, \theta^*_{(m)}, L^*_{(m)})$ is the optimal solution for a fixed $m$.

Step 8. Set $m = m + 1$ and repeat Steps (2)-(7) to get $\text{ITC}(Q^*_{(m)}, S^*_{(m)}, \theta^*_{(m)}, L^*_{(m)}, m)$.

Step 9. If $\text{ITC}(Q^*_{(m)}, S^*_{(m)}, \theta^*_{(m)}, L^*_{(m)}, m) \geq \text{ITC}(Q^*_{(m)}, S^*_{(m)}, \theta^*_{(m)}, L^*_{(m)}, m + 1)$, then go to step 8, otherwise go to step 10.

Step 10. Set $\text{ITC}(Q^*, S^*, \theta^*, L^*, m) = \text{ITC}(Q^*_{(m+1)}, S^*_{(m+1)}, \theta^*_{(m+1)}, L^*_{(m+1)}, m + 1)$, then $(Q^*, S^*, \theta^*, L^*, m^*)$ is the optimal solution.

6.10 NUMERICAL EXAMPLES FOR BOTH NON-DEFECTIVE AND DEFECTIVE ITEMS

Non-defective items

We provide a numerical example to illustrate the solution procedure. Let us use the same numerical example as in Pan and Yang [123] to prove the results obtained by the following data:

- $D = 1000$ units/year, $P = 3200$ units/year, $\alpha = 0.1$, $k = 2.33$, $C_v = $20/unit,
- $A = $25/order, $S_0 = $400/setup, $I(S) = b \ln \left( \frac{S_0}{S} \right)$, $b = 3500$, $\sigma = 7$ units/week.

And the lead time has three components with data as shown in table (6.1).
Applying the solution procedure of the proposed algorithm, the computational results are demonstrated in table (6.2). The optimal solutions from table (6.2) can be read off as number of deliveries $m^* = 2$, optimal lead time $L^* = 6$ weeks, order quantity $Q^* = 125$ unit, setup cost $S^* = $88 and the corresponding integrated total cost $ITC = $1855.

A graphical representation is presented to show the convexity of $ITC(Q,m,S,L)$ in figure (6.3) and the graphical representation of the integrated total cost for different number of deliveries $m$ is shown in figure (6.4).

**Table (6.1) Lead time data for the illustrative example**

<table>
<thead>
<tr>
<th>Lead time component ($i$)</th>
<th>Normal duration $b_i$ (days)</th>
<th>Minimum duration $a_i$ (days)</th>
<th>Unit crashing cost $c_i$ ($$/days)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20</td>
<td>6</td>
<td>0.1</td>
</tr>
<tr>
<td>2</td>
<td>20</td>
<td>6</td>
<td>1.2</td>
</tr>
<tr>
<td>3</td>
<td>16</td>
<td>9</td>
<td>5.0</td>
</tr>
</tbody>
</table>

**Table (6.2) Summary of optimal solution in non-defective items**

<table>
<thead>
<tr>
<th>$i$</th>
<th>$m=1$</th>
<th>$m=2$</th>
<th>$m=3$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$L$</td>
<td>$R(L)$</td>
<td>$Q^*$</td>
</tr>
<tr>
<td>0</td>
<td>8</td>
<td>0.0</td>
<td>162</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>1.4</td>
<td>163</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>18.2</td>
<td>186</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>53.2</td>
<td>224</td>
</tr>
</tbody>
</table>
Defective items

We provide a numerical example to illustrate the solution procedure. Let us use the same numerical example as in Pan and Yang [123] to prove the results obtained by the following data: $D = 1000 \text{ units/year}$, $P = 3200 \text{ units/year}$, $\alpha = 0.1$, $k = 2.33$, $r = 0.2$, 

Figure (6.3) Graph representing the convexity of ITC

Figure (6.4) Graphical representation of the optimal solution in ITC
\[ C_v = \$20 \text{/ unit}, \quad C_s = \$25 \text{/ units}, \quad A = \$25 \text{/ order}, \quad S_o = \$400 \text{/ setup}, \]
\[ \sigma = 7 \text{ unit/week}, \quad I(\theta) = \alpha q \ln \left( \frac{\theta_0}{\theta} \right), \quad q = 400, \quad I(S) = \alpha b \ln \left( \frac{S_0}{S} \right), \quad b = 3500, \]
\[ g = \$15 \text{/ per defective units}. \]

And the lead time has three components with data as shown in table (6.1).

Applying the solution procedure of the proposed algorithm, the computational results are demonstrated in table (6.3). The optimal solutions from table (6.3) can be read off as number of deliveries \( m^* = 2 \), optimal lead time \( L^* = 6 \) weeks, order quantity \( Q^* = 118 \text{ unit} \), setup cost \( S^* = \$83 \), process quality \( \theta^* = 0.000022409 \) and the corresponding integrated total cost \( ITC = \$1984 \).

A graphical representation is presented to show the convexity of \( ITC(Q, \theta, m, S, L) \) in figure (6.5) and the graphical representation of the integrated total cost for different number of deliveries \( m \) is shown in figure (6.6).

### Table (6.3) Summary of optimal solution in defective items

<table>
<thead>
<tr>
<th>( i )</th>
<th>( L )</th>
<th>( R(L) )</th>
<th>( Q^* )</th>
<th>( S^* )</th>
<th>( \Theta^* )</th>
<th>( ITC )</th>
<th>( Q^* )</th>
<th>( S^* )</th>
<th>( \Theta^* )</th>
<th>( ITC )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>8</td>
<td>0.0</td>
<td>153</td>
<td>54</td>
<td>0.000034858</td>
<td>2036</td>
<td>117</td>
<td>86</td>
<td>0.000022792</td>
<td>2003</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>1.4</td>
<td>154</td>
<td>54</td>
<td>0.000034632</td>
<td>2014</td>
<td>118</td>
<td>83</td>
<td>0.000022409</td>
<td>1984</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>18.2</td>
<td>177</td>
<td>62</td>
<td>0.000030132</td>
<td>2079</td>
<td>138</td>
<td>97</td>
<td>0.000019324</td>
<td>2078</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>53.2</td>
<td>216</td>
<td>76</td>
<td>0.000024691</td>
<td>2235</td>
<td>171</td>
<td>120</td>
<td>0.000015595</td>
<td>2282</td>
</tr>
</tbody>
</table>
Continuation of table (6.3)

<table>
<thead>
<tr>
<th>m=3</th>
<th>Q*</th>
<th>S*</th>
<th>θ*</th>
<th>ITC</th>
</tr>
</thead>
<tbody>
<tr>
<td>97</td>
<td>102</td>
<td>0.000018328</td>
<td>2023</td>
<td></td>
</tr>
<tr>
<td>99</td>
<td>104</td>
<td>0.000017957</td>
<td>2006</td>
<td></td>
</tr>
<tr>
<td>116</td>
<td>122</td>
<td>0.000015326</td>
<td>2126</td>
<td></td>
</tr>
<tr>
<td>145</td>
<td>152</td>
<td>0.000012261</td>
<td>2376</td>
<td></td>
</tr>
</tbody>
</table>

Figure (6.5) Graph representing the convexity of ITC
6.11 CONCLUSION

Setup cost is the cost incurred to configure a machine for a production run. The main purpose of this chapter is to present a single vendor and a single buyer integrated inventory model with setup cost reduction and quality improvement for non-defective items and defective items.

The mathematical model is derived to investigate the effects of the best decisions when capital investment strategies are adopted. Two logarithmic investment functions for process quality and setup cost reduction respectively were incorporated in this chapter.

We have developed an algorithm to minimize the integrated total cost of the single vendor and the single buyer by simultaneously optimizing the order quantity, lead time, the number of deliveries, process quality and setup cost.

An iterative algorithm was devised to determine the optimal solution. Furthermore, numerical examples are given to illustrate the
results for both non-defective items and defective items. Graphical representation is presented to illustrate the numerical examples.

Hence, this chapter may applicable for manufacturing companies in order to reduce setup cost, process quality in order to minimize the integrated total cost in defective items and non-defective items in a supply chain system.

**APPENDIX E**

We first obtain the Hessian Matrix $H$ as follows

$$H = \begin{bmatrix} \frac{\partial^2 ITC(Q,S,m)}{\partial Q^2} & \frac{\partial^2 ITC(Q,S,m)}{\partial Q \partial S} \\ \frac{\partial^2 ITC(Q,S,m)}{\partial S \partial Q} & \frac{\partial^2 ITC(Q,S,m)}{\partial S^2} \end{bmatrix}$$

where

$$\frac{\partial^2 ITC(Q,S,m)}{\partial Q^2} = \frac{2D}{Q^3} \left( A + \frac{S}{m} + R(L) \right)$$

$$\frac{\partial^2 ITC(Q,S,m)}{\partial Q \partial S} = \frac{\partial^2 ITC(Q,S,m)}{\partial S \partial Q} = -\frac{D}{Q^2 m}$$

$$\frac{\partial^2 ITC(Q,S,m)}{\partial S^2} = \frac{\alpha b}{S^2}$$

We proceed by evaluating the principal minor determinant of the Hessian Matrix $H$ at point $(Q^*, S^*)$. The first principal minor determinant of $H$ then becomes

$$|H_{11}| = \frac{2D}{Q^3} \left( A + \frac{S}{m} + R(L) \right) > 0$$

$$|H_{22}| = \frac{2D}{Q^3} \left( A + \frac{S}{m} + R(L) \right) \left( \frac{\alpha b}{S^2} \right) - \left( -\frac{D}{Q^2 m} \right)^2$$

$$= \frac{2D \alpha b}{Q^3 S} \left( A + \frac{S}{m} + R(L) \right) - \left( \frac{D}{Q^2 m} \right)^2$$

$H_{22} > 0$, provided
\[= \frac{2Dab}{Q^3S} \left( A + \frac{S}{m} + R(L) \right) > \left( \frac{D}{Q^2m} \right)^2\]

Hence for a fixed \(m\), \(L \in [L_i, L_{i-1}]\) the Hessian Matrix is positive definite and \(ITC(Q, L, m)\) is convex with respect to \((Q, S)\).

**APPENDIX F**

We first obtain the Hessian Matrix \(H\) as follows

\[
H = \begin{bmatrix}
\frac{\partial^2 ITC(Q,S,\theta,m)}{\partial Q^2} & \frac{\partial^2 ITC(Q,S,\theta,m)}{\partial Q \partial S} & \frac{\partial^2 ITC(Q,S,\theta,m)}{\partial Q \partial \theta} \\
\frac{\partial^2 ITC(Q,S,\theta,m)}{\partial S \partial Q} & \frac{\partial^2 ITC(Q,S,\theta,m)}{\partial S^2} & \frac{\partial^2 ITC(Q,S,\theta,m)}{\partial S \partial \theta} \\
\frac{\partial^2 ITC(Q,S,\theta,m)}{\partial \theta \partial Q} & \frac{\partial^2 ITC(Q,S,\theta,m)}{\partial \theta \partial S} & \frac{\partial^2 ITC(Q,S,\theta,m)}{\partial \theta^2}
\end{bmatrix}
\]

Where

\[
\frac{\partial^2 ITC(Q,S,\theta,m)}{\partial Q^2} = \frac{2D}{Q^3} \left( A + \frac{S}{m} + R(L) \right)
\]

\[
\frac{\partial^2 ITC(Q,S,\theta,m)}{\partial S^2} = \frac{ab}{S^2}
\]

\[
\frac{\partial^2 ITC(Q,S,\theta,m)}{\partial \theta^2} = \frac{aq}{\theta^2}
\]

\[
\frac{\partial^2 ITC(Q,S,\theta,m)}{\partial Q \partial S} = \frac{\partial^2 ITC(Q,S,\theta,m)}{\partial S \partial Q} = -\frac{D}{Q^2m}
\]

\[
\frac{\partial^2 ITC(Q,S,\theta,m)}{\partial Q \partial \theta} = \frac{\partial^2 ITC(Q,S,\theta,m)}{\partial \theta \partial Q} = \frac{gmD}{2}
\]

\[
\frac{\partial^2 ITC(Q,S,\theta,m)}{\partial S \partial \theta} = \frac{\partial^2 ITC(Q,S,\theta,m)}{\partial \theta \partial S} = 0
\]

We proceed by evaluating the principal minor determinant of the Hessian Matrix \(H\) at point \((Q^*, S^*, \theta^*)\). The first principal minor determinant of \(H\) then becomes \(|H_{11}| > 0\).
\[ |H_{22}| > 0. \]

Hence for a fixed \( m, L \in [L, L_{r-1}] \) the Hessian Matrix is positive definite and \( ITC(Q, S, \theta, m) \) is convex with respect to \((Q, S)\).

\[
H = \begin{bmatrix}
\frac{\partial^2 ITC(Q, S, \theta, m)}{\partial Q^2} & \frac{\partial^2 ITC(Q, S, \theta, m)}{\partial Q \partial S} & \frac{\partial^2 ITC(Q, S, \theta, m)}{\partial Q \partial \theta} \\
\frac{\partial^2 ITC(Q, S, \theta, m)}{\partial S^2} & \frac{\partial^2 ITC(Q, S, \theta, m)}{\partial S \partial Q} & \frac{\partial^2 ITC(Q, S, \theta, m)}{\partial S \partial \theta} \\
\frac{\partial^2 ITC(Q, S, \theta, m)}{\partial \theta^2} & \frac{\partial^2 ITC(Q, S, \theta, m)}{\partial \theta \partial Q} & \frac{\partial^2 ITC(Q, S, \theta, m)}{\partial \theta \partial S}
\end{bmatrix}
\]

\[
|H_{33}| = \frac{\partial^2 ITC(Q, S, \theta, m)}{\partial Q^2} \left( \frac{\partial^2 ITC(Q, S, \theta, m)}{\partial S^2} \right) - \left( \frac{\partial^2 ITC(Q, S, \theta, m)}{\partial \theta \partial S} \right)^2
\]

\[
- \frac{\partial^2 ITC(Q, S, \theta, m)}{\partial S \partial Q} \left( \frac{\partial^2 ITC(Q, S, \theta, m)}{\partial Q \partial S} \right) - \frac{\partial^2 ITC(Q, S, \theta, m)}{\partial S \partial \theta} \left( \frac{\partial^2 ITC(Q, S, \theta, m)}{\partial Q \partial \theta} \right)
\]

\[
+ \frac{\partial^2 ITC(Q, S, \theta, m)}{\partial \theta \partial Q} \left( \frac{\partial^2 ITC(Q, S, \theta, m)}{\partial Q \partial \theta} \right) + \frac{\partial^2 ITC(Q, S, \theta, m)}{\partial \theta \partial S} \left( \frac{\partial^2 ITC(Q, S, \theta, m)}{\partial S \partial \theta} \right)
\]

\[
2D \left( A + \frac{S}{m} + R(L) \right) \left( \frac{ab}{S^2} \right) \left( \frac{aQ}{\theta^2} \right) - \left( 0 \right)^2 - \left( \frac{D}{Q^2} \right) \left( \frac{ab}{Q^2} \right) \left( \frac{aQ}{\theta^2} \right) - \left( 0 \right)
\]

\[
+ \left( \frac{g m D}{2} \right) \left( \frac{a b}{S} \right) \left( \frac{g m D}{2} \right)
\]

\[
\frac{2 D \alpha^2 b q}{Q^3 S^2 \theta^2} \left( A + \frac{S}{m} + R(L) \right) - \left( \frac{D^2 a q}{Q^4 m^2 \theta^2} \right) - \left( \frac{g m^2 D^2 a b}{4 S^2} \right)
\]

\[ H_{33} > 0, \text{ provided} \]

\[
\frac{2 D \alpha^2 b q}{Q^3 S^2 \theta^2} \left( A + \frac{S}{m} + R(L) \right) > \left( \frac{D^2 a q}{Q^4 m^2 \theta^2} \right) + \left( \frac{g m^2 D^2 a b}{4 S^2} \right)
\]

Hence for a fixed \( m \) and \( L \in [L, L_{r-1}] \), the Hessian Matrix is positive definite and \( ITC(Q, S, \theta, m) \) is convex with respect to \((Q, S, \theta)\).