CHAPTER 5

INTEGRATED INVENTORY MODEL

INvolving INVESTMENT FOR QUALITY

IMPROVEMENT IN A SUPPLY CHAIN SYSTEM

5.1 INTRODUCTION

Definitions of a Supply Chain virtually universally encompass the following three functions: (1) supply of materials to a manufacturer; (2) the manufacturing process; and (3) the distribution of finished goods through a network of distributors and retailers to a final customer.

In real SC problems, lead time has an important effect on manager’s decisions about when to order and how many products should be ordered in each ordering interval. In some problems, lead time is controllable; it means that the lead time could be reduced by paying a penalty cost. For example suppose that lead time contains elements such as time for receiving raw material, setup time, processing time and transportation time then each of these components can be shortened if its related penalty cost is paid.

Supply Chain Management is the coordination of production, inventory, location and transportation among the participants in a supply chain to achieve the best mix of responsiveness and efficiency for the market being served. SCM is becoming an important part of a manufacturers work with suppliers and has recently received significant attention from researchers. Effective SCM has become a potentially valuable way of securing competitive advantage and improving organizational performance since competition is no longer between organizations, but among supply chains. As competition in the 1990’s
intensified and markets became global, so did the challenges associate with getting a product and service to the right place at the right time to deliver the item at the lowest cost. Organizations began to realize that it is not enough to improve inefficiencies within an organization, but their whole supply chain has to be made competitive. The understanding and practicing of SCM has become an essential prerequisite for staying competitive in the global race and for enhancing beneficially.

In recent times, the role of SCM endures some dramatic changes. Satisfying the retailer’s demand is not only the prime interest of the manufacturer, but also maintaining the quality of the supplied product. The product quality can be defined as the fulfillment of customer expectations. If customer expectations are not fulfilled then the product is termed as low-quality product. Customer expectations may vary from product to product.

For example, for a mechanical or in electronic product, these are performance, reliability, safety, and appearance; for pharmaceutical products, physical and chemical characteristics, medicinal effect, toxicity, taste, and shelf life may be important; for a food product, expectations include taste, nutritional properties, texture, and shelf life and so on. Product quality is very important for the company. Bad quality products affect the customer’s confidence, image and sales of the company. Product quality is equally important for consumers who are ready to pay high price, but in return, expect high quality. If they are not satisfied with the quality of the product of a company, they purchase from the competitors. Nowadays, very good quality international products are available in the local markets. So, if the domestic companies don’t improve product quality, they will struggle to survive in the market. Thus, product quality has significant impact on the life
and performance of a supply chain. Compared to the traditional business model, the supply chain system can integrate the upstream and downstream companies between enterprises and use the resources more efficiently to create more income.

Quality improvement can be defined as a systematic approach to make changes that lead to better patient outcomes, stronger system performance, and enhanced professional development. In the current SCM environment, companies are using JIT production to gain and maintain a competitive advantage. JIT requires a spirit of cooperation between the buyer and the vendor and it has been shown that forming a partnership between the buyer and the vendor are helpful in achieving tangible benefits for both parties. In this complex environment, successful companies have devoted considerable attention to reduce inventory cost and improve the quality simultaneously.

Quality improvement is an important role in the competitive business environment of the twenty-first century, and firms are using improved quality as a strategic weapon to enhance their competitiveness against rival firms. As the procedure from raw materials to products is not within a single firm but throughout a supply chain, quality of a manufacturer’s products depends on not only its own process quality but also the quality of its suppliers. Quality practices must advance from traditional firm central and product based mindsets to an interorganizational supply chain orientation involving customers, suppliers, and other partners.

Therefore, to the best of our knowledge, the author has incorporated investment for quality improvement in an integrated inventory model involving in the supply chain system. We have developed the simple algorithm to find the optimal order quantity,
number of deliveries and process quality that can minimize the integrated total cost of the single vendor and the single buyer.

The remainder of this chapter is organized as follows: Problem description is given in section 5.2. In section 5.3, the fundamental assumption and notations are provided in section 5.4, describes the model development. In section 5.5, the solution procedure is given. An efficient algorithm is developed in section 5.6 to obtain the optimal solution. A numerical example is provided in section 5.7 to illustrate the results. Section 5.8 concludes the chapter.

5.2 PROBLEM DESCRIPTION

We have concerned a single vendor and a single buyer integrated inventory model involving investment for quality improvement in a supply chain system. Here we have used exponential lead time crashing cost for defective items in this chapter.

In this chapter, the capital investment for the quality improvement is assumed to be a logarithmic function. A solution procedure is developed to find the optimal solution.

The main aim of this chapter is to minimize the integrated total cost in defective items. Graphical representation is presented to illustrate in this chapter. Finally, a numerical example is presented to illustrate the solution procedure.

5.3 ASSUMPTIONS AND NOTATIONS

First of all, the following assumptions and notations are employed throughout this chapter so as to develop the model.

5.3.1 Assumptions

We adopt the following assumptions are used.
1. There is a single vendor and a single buyer and they deal with a single product considered in this chapter.

2. The buyer orders a lot of size $Q$ and the vendor manufactures $mQ$ with a finite production rate $P$ ($P > D$) at one setup, but in shipment quantity $Q$ to the buyer over $m$ times. The vendor incurs a setup cost $S$ for each production run and the buyer incurs an ordering cost $A$ for each order quantity $Q$.

3. The demand $X$ during lead time $L$ follows a normal distribution with mean $\mu L$ and standard deviation $\sigma \sqrt{L}$.

4. The inventory is continuously reviewed. The buyer places an order when the on hand inventory reaches the reorder point $R$.

5. The buyer places an order when the inventory position reaches the reorder point $R$. The reorder point $R = \text{expected demand during lead time + safety stock}$, that is, $R = DL + k\sigma \sqrt{L}$ where $k$ is a safety factor and $\sigma$ is the standard deviation.

6. The extra cost incurred by the vendor will be transferred to the buyer if shortened lead time is requested.

7. If the buyer is not eager to add an extra cost to control the lead time, he can obtain his items exactly on the existing lead time ($L = L_e$) and hence, crashing cost is zero. Otherwise, the lead time $L$ of the buyer should be within the interval $[L_e, L_r]$, that is $L_e \leq L < L_r$.

8. The crashing costs were observed to grow with lead time by a proportion which can be approximated by an exponential function. Therefore, the lead time crashing cost per order $R(L)$ is assumed to be an exponential function of $L$ and is defined as
The vendor uses automation procedure (automatically detects the defective item by machine, no human inspector is needed to inspect the defectiveness of items) to detect the imperfect production. As a result, if the system moves to out-of control state from in-control state, it will continue production of defective items until the whole lot is produced.

The relationship between lot size and quality is formulated as follows: while the vendor is producing a lot, the process can go out of control with a given probability $\theta$ each time when another unit is produced. The process is assumed to be in control in the beginning of the production process. Once out of control, the process produces defective items and continues to do so until the entire lot is produced. (Yang and Pan [191]).

Additional investment is a good strategy to reduce imperfect production during the out-of-control state. For instance, to reduce an out-of-control probability from 0.00002 to 0.000018, an investment of $200 may be needed, and again another $200 can be used to reduce it to 0.000016, and so on. Therefore, the best way to reduce the imperfect production is by using some initial investment. We assume a capital investment $q(\theta)$ (refer to Porteus [132]) to improve the process quality and reduce out-of-control probability as $q(\theta) = q \ln(\theta_0 / \theta)$ for $0 < \theta \leq \theta_0$, where $\theta_0$ is the current probability that the production process can go out of control, and $q = 1/\xi$ with $\xi$ meaning the percentage decrease in $\theta$ per dollar increase in $q(\theta)$. From the investment function if $q(\theta) = 0$,
then there is no investment for quality improvement. If there is at least some investment, then the value of $\theta$ will be reduced for every stage, which indicates the improvement of product quality. The benefit of using the logarithmic function is that it is convex within the range defined for the investment function. (Sarkar et al. [150]).

5.3.2 Notations

To establish the proposed model, the following notations are used.

$L_e$ Existing lead time for the items to arrive in the buyer’s inventory

$L_s$ Shortest lead time for the items to arrive in the buyer’s inventory

$TC_{vb}$ Total cost of the vendor and the buyer

$r$ Annual inventory holding cost per dollar invested in stocks

$i$ Vendor’s fractional opportunity cost of capital per unit time.(e.g. interest rate).

$q$ Percentage decrease in out of control probability per dollar increase in the investment to reduce the out of control probability.

5.4 MODEL DEVELOPMENT FOR DEFECTIVE ITEMS UNDER INVESTMENT FOR QUALITY IMPROVEMENT

Total Cost for Vendor and buyer ($TC_{vb}$)

The buyer places an order after every $Q$ units, therefore for average cycle time of $\frac{Q}{D}$, the expected ordering cost ordering cost per unit time

$$= \frac{A}{Q} = \frac{AD}{Q}.$$  The expected net inventory level just before the arrival of procurement is the safety stock, $s = R - DL$. The expected net inventory level immediately after arrival of procurement is $Q + s$. Hence, the
average inventory over the cycle can be approximated by \( \left( \frac{Q}{2} \right) + s \), i.e. \( \left( \frac{Q}{2} \right) + k\sigma \sqrt{L} \). So buyer’s holding cost per unit time is \( \left( \frac{Q}{2} + k\sigma \sqrt{L} \right) r_{cb} \)

Lead time crashing cost per unit time \( \frac{D}{Q} R(L) = \frac{D_{c\ell}}{L} \), \( L \in [L_c, L_r] \)

Vendor setup cost per year = \( \left( \frac{D}{mQ} \right) S \).

The vendor–buyer integrated system is designed for a vendor’s production situation in which the vendor begins to produce items at a constant production rate \( P \) where the buyer orders a lot size \( Q \) and a finite number of units are added to the inventory until the production run has been completed. The vendor produces the item in a lot of size \( mQ \) in each production cycle of length \( \frac{mQ}{D} \) and the buyer will receive the supply in \( m \) lots, each of size \( Q \).

The first lot of size \( Q \) is ready for delivery after time \( \frac{Q}{P} \) just after the start of the production and then the vendor continues making the delivery on an average of every \( \frac{Q}{D} \) units of time until the inventory level falls to zero (see figures 5.1 to 5.3).

So the vendor’s expected on hand inventory is evaluated as the difference of the vendor’s accumulated inventory and the buyer’s accumulated inventory. Hence, the vendor’s expected inventory per unit time is given by

\[
\begin{align*}
&= \left[ mQ \left( \frac{Q}{P} + (m-1) \frac{Q}{D} \right) - \frac{m^2 Q^2}{2P} \right] - \left[ \frac{Q^2}{D} (1 + 2 + \ldots + (m-1)) \right] \frac{D}{mQ} \\
&= \frac{Q}{2} \left[ m \left( 1 - \frac{D}{P} \right) - 1 + 2 \frac{D}{P} \right].
\end{align*}
\]
So the vendor’s holding cost per unit time is 
\[ rc_v \frac{Q}{2} \left[ m \left( 1 - \frac{D}{P} \right) - 1 + \frac{2D}{P} \right] \]

**Figure (5.1) Inventory pattern for the buyer**

**Figure (5.2) Inventory pattern for the vendor**
Figure (5.3) Inventory pattern for the vendor and the buyer

Accordingly, the vendor and buyer total cost \( TC_{vb} \) per unit time is given by

\[
TC_{vb}(Q, L, m) = \frac{D}{Q} \left( A + \frac{S}{m} + R(L) \right) + \frac{Q}{2} r \left( \left( m \left( 1 - \frac{D}{P} \right) - 1 + \frac{2D}{P} \right) c_v + c_b \right) + r c_b k \sigma \sqrt{L} .
\]  

(5.1)

After a long production process, the system can get out of control state with a probability \( \theta \) is very small and close to zero and once it is out of control, it products imperfect items \( \theta \) generally, \( \theta \) continuously until the entire lot is produced. Due to this relationship between lot size \( mQ \) and out of control product quality, \( \theta \) the expected numbers of defective items during a production run cycle is approximated by \( \frac{m^2 Q^2 \theta}{2} \) (see Porteus [132] for detail derivation).

Suppose \( g \) is the cost of replacing a defective unit, and the production quantity for the supplier in a lot of \( mQ \). Therefore, the expected annual defective cost is \( \frac{gmQD\theta}{2} \)
\[ TC_{vb}(Q, L, m) = \frac{D}{Q} \left( A + \frac{S}{m} + R(L) \right) + \frac{Q}{2} \left( m \left( 1 - \frac{D}{P} \right) - 1 + \frac{2D}{P} \right) c_v + c_b \]
\[ + r_c b k \sigma \sqrt{L} + \frac{gmQD\theta}{2}. \tag{5.2} \]

### 5.4.1 Investment for Quality Improvement

During long production processes, machinery systems may produce low quality products, which may result in revenue loss and an impugned industry reputation. Therefore, to maintain the brand image of the industry, firms may choose to make some initial investments that improve the quality of products. Although this strategy for quality improvement may lead to increases in total system costs, setup cost reduction counter balances the added expense such that the total cost of the system is maintained.

Based on equation (5.2), we wish to analyse the effect of investment on quality improvement. Consequently, the objective of the integrated model is to minimize the sum of the ordering/setup cost, holding cost, defective cost, process quality and lead time crashing cost by simultaneously determining the optimal values of $Q$, $m$, $\theta$ and $L$, subject to the constraint, $0 < \theta \leq \theta_0$. Thus, the integrated total cost per year is given by

\[ ITC(Q, L, m, \theta) = TC_{vb}(Q, L, m) + i q \ln \frac{\theta_0}{\theta} \]
\[ \frac{D}{Q} \left( A + \frac{S}{m} + R(L) \right) + \frac{Q}{2} \left( m \left( 1 - \frac{D}{P} \right) - 1 + \frac{2D}{P} \right) c_v + c_b \]
\[ + r_c b k \sigma \sqrt{L} + \frac{gmQD\theta}{2} + i q \ln \frac{\theta_0}{\theta}. \tag{5.3} \]

Where $i$ is the Vendor’s annual fractional cost of capital investment (e.g., interest rate).
5.5 SOLUTION PROCEDURE FOR DEFECTIVE ITEMS

To solve the above non-linear programming problem, \( ITC(Q,L,m,\theta) \) can be proved to be a convex function of \( m \) for a fixed \( Q,\theta \) and \( L\in[L_s,L_e] \). Consequently, the search for the optimal deliveries \( m^* \) is reduced to find a local minimum.

**Property 1.** For a fixed \( Q,\theta \) and \( L\in[L_s,L_e] \), \( ITC(Q,L,m,\theta) \) is convex in \( m \).

Taking the first and second order partial derivatives of \( ITC(Q,L,m,\theta) \) with respect to \( m \), we have

\[
\frac{\partial ITC(Q,L,m,\theta)}{\partial m} = -\frac{DS}{Qm^2} + \frac{Q}{2} \left[ r c_i \left( 1 - \frac{D}{P} \right) + gD\theta \right] \tag{5.4}
\]

and

\[
\frac{\partial^2 ITC(Q,L,m,\theta)}{\partial^2 m} = \frac{2DS}{Qm^3} > 0 \tag{5.5}
\]

Therefore, \( ITC(Q,L,m,\theta) \) is convex in \( m \), for a fixed \( Q,\theta \) and \( L \).

Next, in order to find the integrated total cost for this non-linear programming problem, ignore the constraint \( 0 < \theta \leq \theta_0 \) for the moment and minimize the integrated total cost function over \( Q,\theta \) and \( L \) with classical optimization techniques by taking the first order partial derivatives of \( ITC(Q,L,m,\theta) \) with respect to \( Q,\theta \) and \( L\in[L_s,L_e] \) as follows

\[
\frac{\partial ITC(Q,L,m,\theta)}{\partial Q} = -\frac{D}{Q^2} \left( A + \frac{S}{m} + R(L) \right)
+ \frac{r}{2} \left[ m \left( 1 - \frac{D}{P} \right) - 1 + \frac{2D}{P} \right] c_i + c_b \tag{5.6}
\]

\[
\frac{\partial ITC(Q,L,m,\theta)}{\partial \theta} = \frac{gmDQ}{2} - \frac{iq}{\theta} \tag{5.7}
\]
By setting equations (5.6) and (5.7) equal to zero, we obtain

\[ Q = \frac{2D \left( A + \frac{S}{m} + \frac{C}{T} \right)}{\sqrt{m \left( 1 - \frac{D}{P} \right) - 1 + \frac{2D}{P} \left( c_v + c_b \right) + gmD \theta}} \]  

(5.9)

\[ \theta = \frac{2iq}{gmDQ} \]  

(5.10)

Theoretically, for a fixed \( m \) and \( L \in [L_c, L_s] \), from (5.9) & (5.10) we can obtain the values of \( Q^*, \theta^* \). Moreover, it was found that the second order sufficient conditions are satisfied as follows. For a fixed \( m \), the Hessian Matrix \( ITC(Q, L, \theta, m) \) is positive definite \( Q^*, \theta^* \) and \( L^* \). The proof is shown in appendix (D).

Further, based on the convexity behavior of the objective function with respect to the decision variable, we establish the following algorithm is designed to find the optimal values of order quantity \( Q \), process quality \( \theta \), lead time \( L \) and total number of deliveries \( m \) which minimizes the integrated total cost \( ITC(Q, L, m) \).

5.6 ALGORITHM FOR DEFECTIVE ITEMS

Step 1. Let \( m = 1 \) and set \( \theta = \theta_0 \).

Step 2. Perform steps (2.1) and (2.5) for all integer values of \( L \) in the interval \([L_c, L_s] \).

2.1. Use \( m, \theta \) to compute \( Q \) from equation (5.9).

2.2. Use \( Q \) and \( m \) to compute \( \theta \) from equation (5.10).

2.3. Repeat steps (2.1) - (2.2) until no change occurs in the values
of $Q$ and $\theta$. Denote by $Q^*$ and $\theta^*$ respectively.

2.4. Compare $\theta^*$ with $\theta_0$

(i) If $\theta^* < \theta_0$, then the solution is optimal for a given $L \in [L_\nu, L_\lambda]$. Denote the optimal solution by $(Q^*, \theta^*, L^*)$.

(ii) If $\theta^* \geq \theta_0$, then take $\theta = \theta_0$ and utilize equation Eq. (5.9) (replace to $\theta^*$ by $\theta_0$) to determine new $Q^*$ similar to the one in step 2. The result is denoted by $(Q^*, \theta^*)$.

2.5. Compute the corresponding $ITC(Q, L, m, \theta)$, by putting $Q, \theta$ in equation (5.2), then go to step 4.

Step 4. Let $ITC(Q^*, \theta^*, L^*, m)$ = minimum of $ITC(Q, L, m, \theta)$, then $(Q^*, L^*, \theta^*)$ is an optimal solution for a fixed $m$.

Step 5. Set $m = m + 1$ and repeat steps (2)-(3) to get $ITC(Q^* L^*, \theta^*, m_0)$.

Step 6. If $ITC(Q^*, L^*, \theta^*, m) \leq ITC(Q^*, L^*, \theta^*, m-1)$; go to step 5, otherwise go to step 7.

Step 7. Set $ITC(Q^*, L^*, \theta^*, m^*) = ITC(Q^*, L^*, \theta^*, m-1)$, then $(Q^*, L^*, \theta^*, m^*)$ is the optimal solution.

5.7 NUMERICAL EXAMPLE

In this section, a numerical example is given to illustrate the above solution procedure. We consider the numerical example with the following data: $D = 600$ units/year, $P = 2000$ units/year, $c_b = 100$/units, $i = 0.1$, $A = $200/order, $g = $15/units, $S = $1500/setup, $r = 0.2$, $q = 400$, $l(\theta) = i \ln \left( \frac{\theta}{\theta_0} \right)$, $i = 0.1$, $c_v = 70$/unit, $k = 1.31$, lead time crashing cost

$$ R(L) = \begin{cases} 0 & \text{if } L = 6 \\ C & \text{if } 1 \leq L < 6 \end{cases} $$

where $C = 5$. 

Applying the solution procedure of the proposed algorithm, the computational results are demonstrated in table (5.1). The optimal solutions from table (5.1) can be read off as lead time \( L^* = 2 \) weeks, order quantity \( Q^* = 139 \) units, number of deliveries \( m^* = 3 \), process quality \( \theta^* = 0.000021316 \) and the corresponding integrated total cost \( ITC^* = 6507 \).

A graphical representation is presented to show the convexity of \( ITC(Q^*,L^*,m) \) in figure (5.4) and the graphical representation of the integrated total cost for different number of deliveries \( m \) is shown in figure (5.5).

**Table (5.1) Optimal solutions for different values of lead time in defective items**

<table>
<thead>
<tr>
<th>m</th>
<th>( L=1 )</th>
<th>( L=2 )</th>
<th>( L=3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q</td>
<td>( \Omega )</td>
<td>ITC</td>
<td>Q</td>
</tr>
<tr>
<td>1</td>
<td>301</td>
<td>0.000029531</td>
<td>7627</td>
</tr>
<tr>
<td>2</td>
<td>196</td>
<td>0.000022676</td>
<td>7005</td>
</tr>
<tr>
<td>3</td>
<td>152</td>
<td>0.000019493</td>
<td>6994</td>
</tr>
<tr>
<td>4</td>
<td>127</td>
<td>0.000017498</td>
<td>7142</td>
</tr>
<tr>
<td>5</td>
<td>110</td>
<td>0.000016162</td>
<td>7348</td>
</tr>
</tbody>
</table>

\( R(L)=e^{CL} \), where \( C=5 \) and \( 1 \leq L < 6 \)
Continuation of table (5.1)

<table>
<thead>
<tr>
<th>L = 4</th>
<th>L = 5</th>
<th>L = 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q</td>
<td>Θ</td>
<td>ITC</td>
</tr>
<tr>
<td>289</td>
<td>0.000030757</td>
<td>7515</td>
</tr>
<tr>
<td>182</td>
<td>0.000024420</td>
<td>6728</td>
</tr>
<tr>
<td>138</td>
<td>0.000021471</td>
<td>6577</td>
</tr>
<tr>
<td>113</td>
<td>0.000019666</td>
<td>6600</td>
</tr>
<tr>
<td>97</td>
<td>0.000018328</td>
<td>6692</td>
</tr>
</tbody>
</table>

Figure (5.4) Graph representing the convexity of ITC
In this chapter, we have formulated the single vendor and the single buyer integrated inventory model for defective items in a supply chain system. In this chapter, we have created the investment for quality improvement under a supply chain system in defective items. Here the lead time crashing cost is assumed to be an exponential function of the lead time.

One logarithmic investment function for quality improvement incorporated in this chapter. The main contribution of this chapter is to minimize the integrated total cost. A solution procedure is developed to find the optimal solution. The proposed integrated inventory model is useful particularly for JIT inventory systems where the vendor and the buyer form a strategic alliance for profit sharing.

The return on investment for quality improvement is substantial and many papers have shown that improving quality could reduce waste, in other words, cut the cost. In addition, the probability of defects also
makes a great impact on the inventory policy regarding production cycle and lot size, so it is important to always take quality issues into account for any business in a competitive supply chain environment nowadays.

Also, it shows when there is an investing option of improving the process quality, it is advisable to invest. Using the model obtained in this chapter, managers can quickly respond to customer’s demand by efficiently determining the appropriate ordering policy.

The numerical example is given to illustrate the integrated inventory model. We have developed the mathematical model and solution procedure is framed to determine the optimal solutions.

APPENDIX D

We want to prove that the Hessian Matrix of $ITC(Q, L, \theta, m)$ at the point $(Q^*, L^*, \theta^*)$ for a fixed $m$ is positive definite. We first obtain the Hessian Matrix $H$ as follows

$$H = \begin{bmatrix}
\frac{\partial^2 ITC(Q, L, m, \theta)}{\partial Q^2} & \frac{\partial^2 ITC(Q, L, m, \theta)}{\partial Q \partial L} & \frac{\partial^2 ITC(Q, L, m, \theta)}{\partial Q \partial \theta} \\
\frac{\partial^2 ITC(Q, L, m, \theta)}{\partial L \partial Q} & \frac{\partial^2 ITC(Q, L, m, \theta)}{\partial L^2} & \frac{\partial^2 ITC(Q, L, m, \theta)}{\partial L \partial \theta} \\
\frac{\partial^2 ITC(Q, L, m, \theta)}{\partial \theta \partial Q} & \frac{\partial^2 ITC(Q, L, m, \theta)}{\partial \theta \partial L} & \frac{\partial^2 ITC(Q, L, m, \theta)}{\partial \theta^2}
\end{bmatrix}$$

where

$$\frac{\partial^2 ITC(Q, L, m, \theta)}{\partial Q^2} = \frac{2D}{Q^3} \left( A + \frac{S}{m} + e^\frac{C}{L} \right)$$

$$\frac{\partial^2 ITC(Q, L, m, \theta)}{\partial \theta^2} = \frac{iq}{\theta^2}$$

$$\frac{\partial^2 ITC(Q, L, m, \theta)}{\partial Q \partial L} = \frac{2D}{Q^3} \left( A + \frac{S}{m} \right)$$

$$\frac{\partial^2 ITC(Q, L, m, \theta)}{\partial Q \partial \theta} = \frac{2D}{Q^3} \left( A + \frac{S}{m} \right)$$

$$\frac{\partial^2 ITC(Q, L, m, \theta)}{\partial L \partial Q} = \frac{2D}{Q^3} \left( A + \frac{S}{m} \right)$$

$$\frac{\partial^2 ITC(Q, L, m, \theta)}{\partial L \partial \theta} = \frac{2D}{Q^3} \left( A + \frac{S}{m} \right)$$

$$\frac{\partial^2 ITC(Q, L, m, \theta)}{\partial \theta \partial L} = \frac{2D}{Q^3} \left( A + \frac{S}{m} \right)$$

$$\frac{\partial^2 ITC(Q, L, m, \theta)}{\partial \theta^2} = \frac{2D}{Q^3} \left( A + \frac{S}{m} \right)$$
\[
\frac{\partial^2 ITC(Q, L, m, \theta)}{\partial L^2} = \frac{D}{Q} \left( \frac{C}{2CeL} + \frac{C^2 eL}{L^3} + \frac{CE}{L^4} \right) - \frac{r_c k\alpha L - \frac{3}{2}}{4}
\]

\[
\frac{\partial^2 ITC(Q, L, m, \theta)}{\partial Q \partial L} = \frac{\partial^2 TRC(Q, L, m, \theta)}{\partial L \partial Q} = \frac{DCE}{Q^3 L^3}
\]

\[
\frac{\partial^2 ITC(Q, L, m, \theta)}{\partial Q \partial \theta} = \frac{\partial^2 ITC(Q, L, m, \theta)}{\partial \theta \partial Q} = \frac{g m D}{2}
\]

\[
\frac{\partial^2 ITC(Q, L, m, \theta)}{\partial L \partial \theta} = \frac{\partial^2 ITC(Q, L, m, \theta)}{\partial \theta \partial L} = 0
\]

We proceed by evaluating the principal minor determinant of the Hessian Matrix \( H \) at point \((Q', L', \theta')\). The first principal minor determinant of \( H \) then becomes

\[
|H_{11}| = \frac{2D}{Q^3} \left( A + S \frac{m}{m} + \frac{eL}{L} \right) > 0.
\]

\[
|H_{22}| = \frac{2D}{Q^3} \left( A + S \frac{m}{m} + \frac{eL}{L} \right) \left\{ \frac{D}{Q} \left( \frac{C}{2CeL} + \frac{C^2 eL}{L^3} + \frac{CE}{L^4} \right) - \frac{r_c k\alpha L - \frac{3}{2}}{4} \right\} - \left( \frac{DCE}{Q^3 L} \right)^2
\]

\[
= \frac{2D}{Q^3} \left( A + S \frac{m}{m} + \frac{eL}{L} \right) \left\{ \frac{2DCE}{QL^3} + \frac{DC^2 eL}{QL^4} - \frac{r_c k\alpha L - \frac{3}{2}}{4} \right\} - \left( \frac{DCE}{Q^3 L} \right)^2
\]

\[
= \frac{2D}{Q^3} \left( A + S \frac{m}{m} + \frac{eL}{L} \right) \left\{ \frac{2DCE}{QL^3} + \frac{DC^2 eL}{QL^4} \right\} - \frac{2D}{Q^3} \left( A + S \frac{m}{m} + \frac{eL}{L} \right) \left\{ \frac{r_c k\alpha L - \frac{3}{2}}{4} \right\} - \left( \frac{DCE}{Q^3 L} \right)^2 > 0
\]

Since
Therefore \( |H_{22}| > 0 \).

\[
H_{33} = \frac{\partial^2 ITCC(Q, L, m, \theta)}{\partial Q^2} \left( \frac{\partial^2 ITCC(Q, L, m, \theta)}{\partial L^2} \cdot \frac{\partial^2 ITCC(Q, L, m, \theta)}{\partial \theta^2} - \left( \frac{\partial^2 ITCC(Q, L, m, \theta)}{\partial \theta L} \right)^2 \right) \\
- \frac{\partial^2 ITCC(Q, L, m, \theta)}{\partial L \partial Q} \left( \frac{\partial^2 ITCC(Q, L, m, \theta)}{\partial L \partial \theta} - \frac{\partial^2 ITCC(Q, L, m, \theta)}{\partial \theta L} \cdot \frac{\partial^2 ITCC(Q, L, m, \theta)}{\partial Q \partial \theta} \right) \\
+ \frac{\partial^2 ITCC(Q, L, m, \theta)}{\partial \theta \partial Q} \left( \frac{\partial^2 ITCC(Q, L, m, \theta)}{\partial Q \partial L} - \frac{\partial^2 ITCC(Q, L, m, \theta)}{\partial L \partial \theta} \cdot \frac{\partial^2 ITCC(Q, L, m, \theta)}{\partial Q \partial \theta} \right) \\
= \frac{2D}{Q^3} \left( A + \frac{S}{m} + e^T \right) \left[ \frac{C}{2Ce^T L^3} + \frac{C}{e^2 L^4} \right] - \left( \frac{rc_b k \alpha L}{2} \right)^2 \left( \frac{iq}{\theta^2} \right) - (0)^2 \\
- \left( \frac{DQe^T L}{Q^2 L^2} \right) \left( \frac{DQe^T L}{Q^2 L^2} \right) \left( \frac{iq}{\theta^2} \right) - (0)^2 \left( \frac{gmD}{2} \right) \\
+ \left( \frac{gmD}{2} \right) \left( \frac{DQe^T L}{Q^2 L^2} \right) \left( \frac{iq}{\theta^2} \right) - (0)^2 \left( \frac{gmD}{2} \right)
\]
\[
\frac{2D}{Q^3} \left( A + \frac{S}{m} + e^L \right) \left( D \left( \frac{C}{Q} \right) \left( \frac{2CeL}{L^3} + \frac{C}{L^4} \right) - \frac{rc_b k \sigma L}{4} \right) \cdot \frac{-3/2}{i_1^2} \right) - \frac{D C e^T}{Q^2 L^2} \left( \frac{D C e^T}{Q^2 L^2} \right) \left( \frac{-3/2}{i_1^2} \right)
\]

Then

\[
\frac{2D i_q}{Q^3 \theta^2} \left( A + \frac{S}{m} + e^L \right) \left( D \left( \frac{C}{Q} \right) \left( \frac{2CeL}{L^3} + \frac{C}{L^4} \right) - \frac{rc_b k \sigma L}{4} \right) - \frac{D C e^T}{Q^2 L^2} \left( \frac{D C e^T}{Q^2 L^2} \right)
\]

\[
\frac{-\left( \frac{gmD}{2} \right)^2 \left( D \left( \frac{C}{Q} \right) \left( \frac{2CeL}{L^3} + \frac{C}{L^4} \right) - \frac{rc_b k \sigma L}{4} \right) > 0}
\]

Therefore \(|H_{33}| > 0\).

Hence for a fixed \(m\), the Hessian Matrix is positive definite and \(ITC(Q, L, m, \theta)\) is convex with respect to \((Q, L, \theta)\).