CHAPTER 3

VENDOR-BUYER INTEGRATED INVENTORY MODEL WITH QUALITY IMPROVEMENT AND NEGATIVE EXPONENTIAL LEAD TIME CRASHING COST

3.1 INTRODUCTION

Inventory is the important role in business organization. Inventories play an extremely important role in a nation’s economy. Inventory or stock refers to the goods and materials that a business holds for the ultimate purpose of future use. Inventory management involves a retailer seeking to acquire and maintain a proper merchandise assortment while ordering, shipping, handling and related costs are kept in check. It also involves systems and processes that identify inventory requirements, set targets, provide replenishment techniques, report actual and projected inventory status and handle all functions related to the tracking and management of material.

The scope of inventory management concerns the fine lines between replenishment lead time, carrying costs of inventory, asset management, inventory forecasting, inventory valuation, inventory visibility, future inventory price forecasting, physical inventory, available physical space for inventory, quality management, replenishment, returns and defective goods, and demand forecasting. Balancing these competing requirements leads to optimal inventory levels, which is an ongoing process as the business needs shift and react to the wider environment.
In the past, most of the inventory model researches considered only the independent viewpoint. However, in supply chain environment, the coordination of all the partners is the key to efficient management of a supply chain to achieve global optimally. Research on coordinating supply chains is currently very popular. During the last few years, the concept of integrated the vendor and the buyer inventory management has attracted considerable attention, accompanying the growth of Supply Chain Management.

Recognizing the strategic vendor-buyer partnerships as a fundamental driver for the success of the supply chain, increasing attention has been located on the integrated vendor-buyer inventory models. Most of the inventory models considered so far assumed a single facility (e.g., a buyer or a vendor) managing its inventory policy in order to minimize its own cost or maximize its own profit. This kind of one side optimal strategy seems not suitable for today’s global markets anymore. In order to face the fierce competition, companies search for new technologies and strategies to allow them to reduce costs and better compete in global markets.

In the modern global competitive market, the buyer and vendor should be treated as strategic partners in the supply chain with a long term cooperative relationship. Facing a competitive commercial environment, many vendors and buyers would like to establish long term cooperative relationships to obtain stable sources of supply and demand to gain the optimum profit from each other. Thus, the way to determine the optimal order quantity for the integrated vendor and the buyer system has become an important issue.

Lead time is usually defined for a stock-pile-up inventory or stock shipment inventory. Lead time can be reduced by various operational
improvements, such as better scheduling, redesigning the process, etc., However, it can also be reduced by organizational means such as using performance system that give managers incentive to reduce lead time. In most deterministic and stochastic inventory models encountered in the literature, the optimal policy is determined with the assumption that lead time is independent parameter. However, lead time is composed of many controllable components such as run time, setup time, waiting time, moving time and lot size inspection time.

Lead time may influence customer service and impact inventory costs. These effects of lead time are well known but are too general to be used in practical ways. In fact, under practical situation, lead time should be reduced. Consequently, it is important to know how and to what extent each of the many components of the manufacturing lead time influence the level of inventory in order to select the most cost effective inventory model.

Keeping in mind the widespread application of the JIT manufacturing philosophy and lead time management being one of its most effective methods of implementation, the lead time has been assumed to be an added control parameter. Also, as it may not be always possible to resolve the lead time into all its components and estimate their individual crashing costs, the crashing cost has been considered as a negative exponential function of the lead time. It helps in the speedy delivery of items which in turn reflects the efficiency of the supplier in managing the unexpected fluctuations in demand and short product life cycle. Reduction in lead time also leads to various other advantages such as reduced safety stock level, reduced loss due to stock out and provides a competitive advantage thereby yielding a smoother and more efficient supply chain. As a drastic reduction in lead time
could be achieved by using a negative exponential function of lead time, we have expressed lead time crashing cost as a negative exponential function.

To the best of our knowledge, the authors has developed a single vendor and a single buyer integrated inventory model with quality improvement and negative exponential lead time crashing cost function for non-defective and defective items. The objective of this chapter is to find out an optimal inventory strategy that can minimize the value of the integrated total cost in a supply chain system.

The rest of this chapter is organized as follows: Problem description is given in section 3.2. In Section 3.3, we provide the fundamental assumption and notations. Section 3.4 describes model development for non-defective items. Solution procedure for non-defective items is given in the section 3.5. In Section 3.6, an efficient algorithm is developed to obtain the optimal solution for non-defective items. A numerical example is provided in Section 3.7 to illustrate for non-defective items.

Section 3.8, describes model development for defective items under investment in quality improvement. Solution procedure for defective items under the investment for quality improvement is given in the section 3.9. In Section 3.10, an efficient algorithm is developed to obtain the optimal solution for defective items under investment in quality improvement. A numerical example is provided in Section 3.11 to illustrate the results. Finally, we draw conclusion in Section 3.12.

3.2 PROBLEM DESCRIPTION

In this chapter, we have considered the single vendor and single buyer integrated inventory model with lead time reduction for non-
defective and defective items under investment for quality improvement. Here, the buyers lead time can be shortened by paying an additional crashing cost which is a negative exponential lead time function.

The objective of this chapter is to minimize the integrated total cost incurred by the single vendor and the single buyer. An iterative method is developed to find the optimal solution. The optimal solution of the integrated inventory model is illustrated with the help of numerical examples along with graphical representations are provided to illustrate the model.

3.3 ASSUMPTIONS AND NOTATIONS

To establish the mathematical models, following assumptions and notations are used throughout this chapter.

3.3.1 Assumptions

To develop the model, we adopt the following assumptions.

1. The supply chain involves only one item, the single vendor and the single buyer.

2. The buyer orders a lot of size $Q$ and the vendor manufactures $mQ$ with a finite production rate $P$ ($P > D$) at one setup but ship in quantity $Q$ to the buyer over $m$ times. The vendor incurs a set up cost $S$ for each production run and the buyer incurs an ordering cost $A$ for each order of quantity $Q$.

3. The demand $X$ during lead time $L$ follows a normal distribution with mean $\mu L$ and standard deviation $\sigma \sqrt{L}$.

4. The reorder point (ROP) equals the sum of the expected demand during lead time and the safety stock. The reorder point $R = \text{expected demand during lead time} + \text{safety stock (SS)}$, and $SS = k \times$
(standard deviation of lead time), that is, $R = DL + k\sigma\sqrt{L}$ where $k$ is safety factor.

5. The inventory is continuously reviewed. The buyers place an order when the on hand inventory reaches the reorder point $R$.

6. If a shortened lead time is requested then the extra costs incurred by the vendor will be fully transferred to the buyer. Therefore, lead time crash cost is the buyer’s cost component. (Jha and Shanker [80]).

7. If the buyer is not eager to add an extra cost to control the lead time, he can obtain his items exactly on the existing lead time ($L = L_e$) and hence, crashing cost is zero. Otherwise, the lead time $L$ of the buyer should be within the interval $[L_s, L_e]$, that is $L_s \leq L < L_e$.

8. The buyer lead time is controllable and reducible by adding additional crashing cost which is determined by the length of negative exponential function. Hence, the lead time crashing cost is $R(L) = \begin{cases} 0 & \text{if } L = L_e \\ \alpha e^{-\gamma L} & \text{if } L_s \leq L < L_e \end{cases}$ where $\alpha$ and $\gamma$ are constants.

9. The relationship between lot size and quality is formulated as follows: while the vendor is producing a lot, the process can go out of control with a given probability $\theta$ each time when another unit is produced. The process is assumed to be in control in the beginning of the production process. Once out of control, the process produces defective items and continues to do so until the entire lot is produced. (This assumption is in line with Porteus [132]).

3.3.2 Notations

To develop the proposed model, we adopt the following notations.
Existing lead time for the items to arrive in the buyer’s inventory

Shortest lead time for the items to arrive in the buyer’s inventory

Buyer’s total cost

Vendor’s total cost

Safety factor

Capital investment required to reduce process quality \( \theta, 0 < \theta \leq \theta_0 \)

Percentage decrease in out of control probability per dollar increase in the investment to reduce the out of control probability.

Vendor’s fractional opportunity cost of capital per unit time.

3.4 MODEL DEVELOPMENT FOR NON-DEFECTIVE ITEMS

Buyer’s Total Cost \((TC_b)\)

The buyer places an order after every \(Q\) units, therefore for average cycle time of \(\frac{Q}{D}\), the expected ordering cost per unit time = \(\frac{A}{Q} = \frac{AD}{Q}\).

The expected net inventory level just before the arrival of procurement is the safety stock, \(s = R - DL\). The expected net inventory level immediately after arrival of procurement is \(Q + s\). Hence, the average inventory over the cycle can be approximated by \(\left(\frac{Q}{2}\right) + s\), i.e. \(\left(\frac{Q}{2}\right) + k\sigma\sqrt{L}\). So buyer’s holding cost per unit time is \(\left(\frac{Q}{2} + k\sigma\sqrt{L}\right)rc_b\)

Lead time crashing cost per unit time \(\frac{D}{Q}R(L) = \frac{Dae^{-\mu}}{Q}, L \in [L_e, L_s]\). The total cost of the buyer \((TC_b)\) is given by

\(TC_b(Q,L) = \text{Ordering cost} + \text{holding cost} + \text{lead time crashing cost}\)
$$\frac{DA}{Q} + \left( \frac{Q}{2} + k\sigma \sqrt{L} \right) c_v + \frac{D\omega - \mu}{Q}$$

**Vendor’s Total Cost (TC_v)**

Once the buyer orders a lot size of $Q$ units, the vendor produces the items in a lot size of $mQ$ units in each production cycle of length $\frac{mQ}{D}$ with a constant production rate $P$ unit per unit time and the buyer will receive the supply in $m$ lots each of size $Q$ units. Figure (3.1) depicts the pattern of stock for the vendor and indicates the expected inventory per unit time. Figure (3.2) leads to the first lot size of $Q$ units is ready for delivery after time $\frac{Q}{P}$ just after the start of the production. During the production period, $\frac{mQ}{P}$ the vendor’s inventory is building up at a constant rate, and simultaneously supplies a lot of size $Q$ units to the buyer on expected every $\frac{Q}{D}$ units of time.

Subsequently, during the non-production period the vendor continues his deliveries to the buyer on expected every $\frac{Q}{D}$ units of time until the inventory level falls to zero. So the vendor’s expected on hand inventory is evaluated as the difference of the vendor’s accumulated inventory and the buyer’s accumulated inventory. Hence, the vendor’s expected inventory per unit time is given by

$$= \left[ mQ \left( \frac{Q}{P} + (m-1) \frac{Q}{D} \right) - \frac{m^2 Q^2}{2P} \right] - \left[ \frac{Q^2}{2} \left( 1 + 2 + \ldots + (m-1) \right) \right] \frac{D}{mQ}$$

$$= \frac{Q}{2} \left[ m \left( 1 - \frac{D}{P} \right) - 1 + \frac{2D}{P} \right]$$
So the vendor’s holding cost per unit time is \( r_c \frac{Q}{2} \left[ m \left( 1 - \frac{D}{P} \right) - 1 + \frac{2D}{P} \right] \).

**Figure (3.1) Inventory pattern for the vendor**

On the other hand for the vendor, since \( S \) is the vendor setup cost per setup and the production quantity for a vendor in a lot is \( mQ \) units, the vendor total cost per unit time is given by \( \left( \frac{D}{mQ} \right) S \).
The total cost of the vendor’s ($TC_v$) is given by

$$TC_v(Q,m) = \text{Setup cost} + \text{holding cost}$$

$$= \left( \frac{D}{mQ} \right) S + r_c \left( \frac{Q}{2} \left( m \left( 1 - \frac{D}{P} \right) - 1 + \frac{2D}{P} \right) \right)$$

**Integrated Total Cost (ITC)**

If the buyer’s order quantity is $Q$ and the vendor’s lot size is $mQ$, then the integrated total cost is given by

$$ITC(Q,L,m) = TC_b + TC_v$$

$$= \frac{DA}{Q} + \left( \frac{Q}{2} + k\sigma \sqrt{L} \right) r_b + \frac{D}{Q} R(L) + \left( \frac{D}{mQ} \right) S + r_c \left( \frac{Q}{2} \left( m \left( 1 - \frac{D}{P} \right) - 1 + \frac{2D}{P} \right) \right)$$

$$= \frac{D}{Q} \left( A + \frac{S}{m} + R(L) \right) + \frac{Q}{2} r \left( \left( m \left( 1 - \frac{D}{P} \right) - 1 + \frac{2D}{P} \right) c_v + c_b \right)$$

$$+ r_c k\sigma \sqrt{L}$$

(3.1)

**3.5 SOLUTION PROCEDURE FOR NON-DEFECTIVE ITEMS**

To solve the above non-linear programming problem, for a fixed $Q$ and $L \in [L_s, L_e]$, $ITC(Q,L,m)$ can be proved to be a convex function of $m$. Consequently, the search for the optimal deliveries $m^*$ is reduced to find a local minimum.

**Property 1.** For a fixed $Q$ and $L \in [L_s, L_e]$, $ITC(Q,L,m)$ is convex in $m$.

Taking the first and second order partial derivatives of $ITC(Q,L,m)$ with respect to $m$, we have

$$\frac{\partial ITC(Q,L,m)}{\partial m} = - \frac{DS}{Qm^2} + \frac{Qr}{2} \left[ c_v \left( 1 - \frac{D}{P} \right) \right]$$

(3.2)

and

$$\frac{\partial^2 ITC(Q,L,m)}{\partial^2 m} = \frac{2DS}{Qm^3} > 0$$

(3.3)

Therefore, $ITC(Q,L,m)$ is convex in $m$, for a fixed $Q$ and $L \in [L_s, L_e]$. 
Next, the first order partial derivatives of $ITC(Q,L,m)$ with respect to $Q$ and $L \in [L_c, L_s]$ are taken for a fixed $m$, respectively. This process yields

$$\frac{\partial ITC(Q,L,m)}{\partial Q} = -\frac{D}{Q^2} \left( A + \frac{S}{m} + R(L) \right) + \frac{L}{2} \left( m \left( 1 - \frac{D}{P} \right) - 1 \right) \left( c_v + c_b \right)$$ (3.4)

$$\frac{\partial ITC(Q,L,m)}{\partial L} = -\frac{\gamma \sigma L}{Q} + \frac{r \sigma_k \sigma L^{1/2}}{2}$$ (3.5)

By setting equation (3.4) equal to zero, we obtain

$$Q^* = \sqrt{\frac{2D \left( A + \frac{S}{m} + \alpha \sigma L \right)}{\left( m \left( 1 - \frac{D}{P} \right) - 1 \right) \left( c_v + c_b \right)}} \quad L \in [L_c, L_s]$$ (3.6)

Theoretically, for a fixed $m$ and $L \in [L_c, L_s]$ from (3.6) we can obtain the values of $Q^*$. Moreover, it can be verified that the second order sufficient conditions are satisfied as follows. For a fixed $m$, the Hessian Matrix $ITC(Q,L,m)$ is positive definite $Q^*, L^*$. The proof is shown in appendix (A).

Further, based on the convexity behavior of the objective function with respect to the decision variable, we establish the following algorithm is designed to find the optimal values of order quantity $Q$, lead time $L$ and total number of deliveries $m$ which minimizes the integrated total cost $ITC(Q, L, m)$.

### 3.6 ALGORITHM FOR NON-DEFECTIVE ITEMS

**Step 1.** Let $m = 1$.

**Step 2.** Perform step (2.1)-(2.2) for all integer values of $L$ in the interval $[L_c, L_s]$.

2.1. Compute $Q$ from equation (3.6).
2.2. Compute the corresponding $ITC(Q, L, m)$, by putting $Q$ in equation (3.1).

**Step 3.** Let $ITC(Q^*, L^*, m) = \min \{ITC(Q, L, m)\}$, then $(Q^*, L^*)$ is an optimal solution for a fixed $m$.

**Step 4.** Set $m = m + 1$ and repeat steps (2)-(3) to get $ITC(Q^*, L^*, m)$.

**Step 5.** If $ITC(Q^*, L^*, m) \leq ITC(Q^*, L^*, m - 1)$; go to step 4, otherwise go to step 6.

**Step 6.** Set $ITC(Q^*, L^*, m^*) = ITC(Q^*, L^*, m - 1)$, then $(Q^*, L^*, m^*)$ is the optimal solution.

### 3.7 Numerical Example for Non-Defective Items

In this section, a numerical example is given to illustrate the above solution procedure. The solution to this example is obtained by using the computer MatLab 2008 software. We use the same numerical example as in Pan and Yang [123] is taken. Consider a single vendor and a single buyer integrated inventory system in which following data: $D = 1000 \text{ units/year}$, $k = 2.33$, $c_v = 20 \text{ units}$, $r = 0.2$, $P = 3200 \text{ units/year}$, $A = $25/\text{order}$, $c_b = 25/\text{units}$, $S = $400/\text{setup}$, $\sigma = 7/\text{units/week}$, lead time crashing cost

$$R(L) = \begin{cases} 0 & \text{if } L = 6 \\ \alpha e^{-\gamma L} & \text{if } 1 \leq L < 6 \end{cases}$$

where $\alpha = 156$, $\gamma = 1$.

Applying the solution procedure of the proposed algorithm, the computational results are demonstrated in table (3.1). The optimal solutions from table (3.1), can be read off as lead time $L^* = 4 \text{ weeks}$, order quantity $Q^* = 133 \text{ units}$, number of deliveries $m^* = 4$ and the corresponding integrated total cost $ITC^* = 2089$. 


A graphical representation is presented to show the convexity of $ITC(Q^*, L^*, m)$ in figure (3.3) and the graphical representation of the integrated total cost for different number of deliveries $m$ is shown in figures (3.4).

**Table (3.1) Optimal solution for different values of lead time in non-defective items**

<table>
<thead>
<tr>
<th></th>
<th>$L=1$</th>
<th>$L=2$</th>
<th>$L=3$</th>
<th>$L=4$</th>
<th>$L=5$</th>
<th>$L=6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q$</td>
<td>393</td>
<td>2537</td>
<td>378</td>
<td>2477</td>
<td>370</td>
<td>2476</td>
</tr>
<tr>
<td>$ITC$</td>
<td>2537</td>
<td>378</td>
<td>2477</td>
<td>370</td>
<td>2476</td>
<td>369</td>
</tr>
<tr>
<td>$Q$</td>
<td>2537</td>
<td>378</td>
<td>2477</td>
<td>370</td>
<td>2476</td>
<td>369</td>
</tr>
<tr>
<td>$ITC$</td>
<td>378</td>
<td>2477</td>
<td>370</td>
<td>2476</td>
<td>369</td>
<td>2505</td>
</tr>
</tbody>
</table>

$R(L)=\alpha e^{-\gamma L}$, where $\alpha=156$, $\gamma=1$ and $1 \leq L < 6$

$R(L)=0$

Figure (3.3) Graph representing the convexity of $ITC$
Figure (3.4) Graphical representation of the optimal solution in ITC

3.8 MODEL DEVELOPMENT FOR DEFECTIVE ITEMS UNDER INVESTMENT FOR QUALITY IMPROVEMENT

The integrated inventory model is designed for vendor production situations in which, once an order is placed, production begins and a constant number of units are added to the inventory each day until the production run has been completed.

After a long production process, the system can get out-of-control state with a probability $\theta$ (general $\theta$ is very small and close to zero) and once it is in out-of-control, it produce imperfect items continuously until the entire lot is produced. Due to this relationship between lot size and out-of-control product quality, the expected number of defective items in a run of size $mQ$ can be evaluated as $m^2Q^2\theta$. Suppose $g$ is the cost of replacing a defective unit and the production quantity for the buyer in a lot of $mQ$. Then it’s expected defective cost per unit time is given by $\frac{gmQD\theta}{2}$. And also considers a possible relationship between lot size and
quality by incorporating a quality-related cost. In an imperfect production process there is a certain probability \( \theta \) that a system may go to out-of-control state \( \theta \) is provided and considered to be very small and close to zero.

Considering an investment for the vendor’s capital investment \( I(\theta) \), in improving process quality (reducing out-of-control probability from \( \theta_0 \) to \( \theta \)) is given by a logarithmic function, \( I(\theta) = q \ln \left( \frac{\theta_0}{\theta} \right) \), for \( 0 < \theta \leq \theta_0 \) (This investment function has also been used in Porteus [131], Keller and Noori [84], Hong and Hayya [62]).

It is to be noted that lower value of the probability \( \theta \) gives higher value of quality level, where \( \theta_0 \) is the initial probability that the production process may go to out of control state and \( q = \frac{1}{\xi} \) with \( \xi \) denoting the percentage decrease in \( \theta \) per dollar increase in investment \( I(\theta) \). See the example (Porteus [131] Yang and Pan [191], Yang et al. [193], Hong and Hayya [62], Keller and Noori [84], Ouyang and Chang [109], Ouyang et al. [115] and Ouyang et al. [120]).

So, the integrated total cost incorporating the defective cost per unit time and process quality per unit time involved the total cost for buyer and vendor can be represented by

\[
ITC(Q, L,m,\theta) = TC_b + TC_v + \frac{gmQD\theta}{2} + iq \ln \left( \frac{\theta_0}{\theta} \right)
\]

\[
= \frac{D}{Q} \left( A + \frac{S}{m} + R(L) \right) + \frac{Q}{2} \left( m \left( 1 - \frac{D}{P} \right) - 1 + \frac{2D}{P} \right) \left( r_c + r_v \right) + r_c k \sigma \sqrt{L}
\]

\[
+ \frac{gmQD\theta}{2} + iq \ln \left( \frac{\theta_0}{\theta} \right)
\]

Subject to \( 0 < \theta \leq \theta_0 \), where \( i \) is the fractional opportunity cost of capital per unit time.
3.9 SOLUTION PROCEDURE FOR DEFECTIVE ITEMS UNDER INVESTMENT FOR QUALITY IMPROVEMENT

To solve the above non-linear programming problem, for fixed \( Q, \theta \) and \( L \in [L_i, L_s], \) \( ITC(Q, L, m, \theta) \) can be proved to be a convex function of \( m \). Consequently, the search for the optimal deliveries \( m^* \) is reduced to find a local minimum.

**Property 2.** For a fixed \( Q, \theta \) and \( L \in [L_i, L_s], \) \( ITC(Q, L, m, \theta) \) is convex in \( m \).

Taking the first and second order partial derivatives of \( ITC(Q, L, m, \theta) \) with respect to \( m \), we have

\[
\frac{\partial ITC(Q, L, m, \theta)}{\partial m} = -\frac{DS}{Qm^2} + \frac{Qr}{2} \left[ c_i \left(1 - \frac{D}{P}\right) + gD\theta \right]
\]

and

\[
\frac{\partial^2 ITC(Q, L, m, \theta)}{\partial^2 m} = \frac{2DS}{Qm^3} > 0
\]

Therefore, \( ITC(Q, L, m, \theta) \) is convex in \( m \), for a fixed \( Q, \theta \) and \( L \).

Next, the first order partial derivatives of \( ITC(Q, L, m, \theta) \) with respect to \( Q, \theta \) and \( L \in [L_i, L_s] \) are taken for fixed \( m \) respectively. This process yields

\[
\frac{\partial ITC(Q, L, m, \theta)}{\partial Q} = -\frac{D}{Q^2} \left( A + \frac{S}{m} + R(L) \right)
\]

\[
+ \frac{r}{2} \left( m \left(1 - \frac{D}{P}\right) - 1 + \frac{2D}{P} \right) c_i + c_b + gmd\theta
\]

(3.10)

\[
\frac{\partial ITC(Q, L, m, \theta)}{\partial \theta} = \frac{gmDQ}{2} - \frac{iq}{\theta}
\]

(3.11)

\[
\frac{\partial ITC(Q, L, m, \theta)}{\partial L} = -\frac{\gamma \alpha e^{-Md}}{Q} + \frac{rc_b k\sigma L^{\frac{1}{2}}}{2}
\]

(3.12)

By setting equations (3.10) and (3.11) equal to zero, we obtain...
\[ Q^* = \left[ \frac{2D\left( A + \frac{S}{m} + R(L) \right)}{r\left( m \left( 1 - \frac{D}{P} \right) - 1 + \frac{2D}{P} \right) c_v + c_b \right)} + gmD \theta \right] L \in [L_r, L_s] \] (3.13)

\[ \theta^* = \frac{2iq}{gmDQ} \] (3.14)

Theoretically, for a fixed \( m \) and \( L \in [L_r, L_s] \), from (3.13) & (3.14) we can obtain the values of \( Q^*, \theta^* \). Moreover, it can be verified that the second order sufficient conditions are satisfied as follows. For a fixed \( m \) the Hessian Matrix \( ITC(Q,L,\theta) \) is positive definite \( Q^*, \theta^* \) and \( L^* \). The proof is shown in appendix (B).

Further, based on the convexity behavior of the objective function with respect to the decision variable, we establish the following iterative algorithm is designed to find the optimal values of order quantity \( Q \), process quality \( \theta \), lead time \( L \) and total number of deliveries \( m \) which minimizes the integrated total cost \( ITC(Q,L,m) \).

### 3.10 ALGORITHM FOR DEFECTIVE ITEMS UNDER INVESTMENT FOR QUALITY IMPROVEMENT

**Step 1.** Let \( m = 1 \) and set \( \theta = \theta_0 \).

**Step 2.** Perform step (2.1) and (2.5) for all integer values of \( L \) in this interval \( [L_r, L_s] \).

1. **2.1.** Use \( m, \theta \) to compute \( Q \) from equation (3.13).
2. **2.2.** Use \( Q \) and \( m \) to compute \( \theta \) from equation (3.14).
3. **2.3.** Repeat steps (2.1)-(2.2) until no change occurs in the values of \( Q \) and \( \theta \). Denote these solutions by \( Q^\ast \) and \( \theta^\ast \) respectively.
4. **2.4.** Compare \( \theta^\ast \) with \( \theta_0 \).
(i) If $\theta^* < \theta_0$, then the solution is optimal for given $[L_1, L_2]$. Denote the optimal solution by $(Q^*, \theta^*, L^*)$.

(ii) If $\theta^* \geq \theta_0$, then take $\theta^* = \theta_0$ and utilize equation (3.13) (replace $\theta^*$ by $\theta_0$) to determine new $Q^*$ similar to the one in step 2. The result is denoted by $(Q^*, \theta^*)$.

2.5. Compute the corresponding $ITC(Q, L, m, \theta)$, by putting $Q, \theta$ in equation (3.7).

Step 3. Let $ITC(Q^*, \theta^*, m, L^*) = \min ITC(Q, L, m, \theta)$, then $(Q^*, L^*, \theta^*)$ is an optimal solution for a fixed $m$.

Step 4. Set $m = m + 1$ and repeat steps (2)-(3) to get $ITC(Q^*, L^*, \theta^*, m)$.

Step 5. If $ITC(Q^*, L^*, \theta^*, m) \leq ITC(Q^*, L^*, \theta^*, m - 1)$; go to step 4, otherwise go to step 6.

Step 6. $ITC(Q^*, \theta^*, m^*, L^*) = ITC(Q^*, \theta^*, m - 1, L^*)$, then $(Q^*, L^*, m^*, \theta^*)$ is the optimal solution.

3.11 NUMERICAL EXAMPLE FOR DEFECTIVE ITEMS UNDER INVESTMENT FOR QUALITY IMPROVEMENT

In this section, a numerical example is given to illustrate the above solution procedure. The solution to this example is obtained by using the computer MatLab 2008 software. We use the same numerical example as in Pan and Yang [123] to verify the results obtained by this chapter. Consider a single vendor and a single buyer integrated inventory system in which following data: $D = 1000$ units/year, $r = 0.2$, $P = 3200$ units/year, $c_v = 20$ units, $k = 2.33$, $A = $25/order, $c_b = 25$ units, $\theta_0 = 0.0002$, $\sigma = 7$ units/week, $i = 0.1$, $I(\theta) = iq\ln\left(\frac{\theta_0}{\theta}\right)$, $q = 400$, $r = 0.2$, $S = $400/setup, $g = $15/units, lead time crashing cost
\[ R(L) = \begin{cases} 
0 & \text{if } L = 6 \\
\alpha e^{-\gamma L} & \text{if } 1 \leq L < 6
\end{cases} \]

where \( \alpha = 156, \gamma = 1 \).

Applying the solution procedure of the proposed algorithm, the computational results are demonstrated in table (3.2). The optimal solutions from table (3.2), can be read off as lead time \( L' = 4 \) weeks, order quantity \( Q^* = 130 \) units, number of deliveries \( m^* = 4 \), process quality \( \theta^* = 0.000010256 \) and the corresponding integrated total cost \( ITC^* = 2248 \).

The result of solution procedure is summarized in table (3.2). A graphical representation is presented to show the convexity of \( ITC(Q^*, L', m) \) in figure (3.5) and the graphical representation of the integrated total cost for different number of deliveries \( m \) is shown in figures (3.6).

**Table (3.2) Optimal solutions for different values of lead time in defective items**

<table>
<thead>
<tr>
<th>( m )</th>
<th>( L = 1 )</th>
<th>( L = 2 )</th>
<th>( L = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Q )</td>
<td>( \Theta )</td>
<td>( ITC )</td>
<td>( Q )</td>
</tr>
<tr>
<td>1</td>
<td>387</td>
<td>0.000013781</td>
<td>2684</td>
</tr>
<tr>
<td>2</td>
<td>246</td>
<td>0.000010840</td>
<td>2493</td>
</tr>
<tr>
<td>3</td>
<td>188</td>
<td>0.0000094563</td>
<td>2496</td>
</tr>
<tr>
<td>4</td>
<td>156</td>
<td>0.0000085470</td>
<td>2618</td>
</tr>
<tr>
<td>5</td>
<td>135</td>
<td>0.0000079012</td>
<td>2618</td>
</tr>
</tbody>
</table>
Continuation of table (3.2)

<table>
<thead>
<tr>
<th>Q</th>
<th>Θ</th>
<th>ITC</th>
<th>Q</th>
<th>Θ</th>
<th>ITC</th>
<th>Q</th>
<th>Θ</th>
<th>ITC</th>
</tr>
</thead>
<tbody>
<tr>
<td>364</td>
<td>0.000014652</td>
<td>2621</td>
<td>363</td>
<td>0.000014962</td>
<td>2635</td>
<td>362</td>
<td>0.000014733</td>
<td>2649</td>
</tr>
<tr>
<td>221</td>
<td>0.000012066</td>
<td>2341</td>
<td>220</td>
<td>0.000012121</td>
<td>2352</td>
<td>219</td>
<td>0.000012177</td>
<td>2365</td>
</tr>
<tr>
<td>162</td>
<td>0.000010974</td>
<td>2266</td>
<td>161</td>
<td>0.000011042</td>
<td>2274</td>
<td>161</td>
<td>0.000011042</td>
<td>2285</td>
</tr>
<tr>
<td>130</td>
<td>0.000010256</td>
<td>2248</td>
<td>129</td>
<td>0.000010336</td>
<td>2253</td>
<td>129</td>
<td>0.000010336</td>
<td>2263</td>
</tr>
<tr>
<td>110</td>
<td>0.0000096970</td>
<td>2253</td>
<td>109</td>
<td>0.0000097859</td>
<td>2256</td>
<td>108</td>
<td>0.0000098765</td>
<td>2264</td>
</tr>
</tbody>
</table>

$R(L) = \alpha e^{-\gamma L}$, $\alpha = 156$, $\gamma = 1$ and $1 \leq L < 6$

$R(L) = 0$

Figure (3.5) Graph representing the convexity of ITC
3.12 CONCLUSION

The purpose of this chapter is to present a single vendor and a single buyer integrated inventory model with lead time reduction for non-defective items and defective items under investment for quality improvement under a supply chain system. Here, the lead time crashing cost is considered as a negative exponential function. In our model, the process quality is assumed to be a logarithmic function. Aim of this chapter is to minimizing the integrated total inventory cost of the single vendor and single buyer.

In addition, numerical examples are given to illustrate the results for both non-defective items and defective items. A solution procedure to find the order quantity, number of delivers, process quality, lead time and integrated total cost is given. The optimal solution of the model is illustrated with the help of numerical examples.
APPENDIX A

We first obtain the Hessian Matrix \( H \) as follows

\[
H = \begin{bmatrix}
\frac{\partial^2 ITC(Q, L, m)}{\partial Q^2} & \frac{\partial^2 ITC(Q, L, m)}{\partial Q \partial L} \\
\frac{\partial^2 ITC(Q, L, m)}{\partial L \partial Q} & \frac{\partial^2 ITC(Q, L, m)}{\partial L^2}
\end{bmatrix}
\]

where

\[
\frac{\partial^2 ITC(Q, L, m)}{\partial Q^2} = \frac{2D}{Q^3} \left( A + \frac{S}{m} + \alpha e^{-q} \right)
\]

\[
\frac{\partial^2 ITC(Q, L, m)}{\partial Q \partial L} = \frac{\partial^2 ITC(Q, L, m)}{\partial L \partial Q} = \frac{\gamma \alpha e^{-q} D}{Q^2}
\]

\[
\frac{\partial^2 ITC(Q, L, m)}{\partial L^2} = D \gamma^2 \alpha e^{-q} - \frac{r_c k \sigma L^2}{4}
\]

We proceed by evaluating the principal minor determinant of the Hessian Matrix \( H \) at point \((Q', L')\). The first principal minor determinant of \( H \) then becomes

\[
|H_{11}| = \frac{2D}{Q^3} \left( A + \frac{S}{m} + \alpha e^{-q} \right) > 0.
\]

\[
|H_{22}| = \frac{2D}{Q^3} \left( A + \frac{S}{m} + \alpha e^{-q} \right) \left( \gamma^2 \alpha e^{-q} - \frac{1}{4} \frac{r_c k \sigma L^2}{4} - \left( \frac{\gamma \alpha e^{-q} D}{Q^2} \right)^2 \right)
\]

\[
= \frac{2D}{Q^3} \left( A + \frac{S}{m} + \alpha e^{-q} \right) \left( \frac{4D \gamma^2 \alpha e^{-q} - Q r_c k \sigma L^2}{4Q} \right) - \left( \frac{\gamma \alpha e^{-q} D}{Q^2} \right)^2
\]

\[
= \frac{8D^2 \left( A + \frac{S}{m} + \alpha e^{-q} \right) \gamma^2 \alpha e^{-q} - 2D Q r_c k \sigma L^2 \left( A + \frac{S}{m} + \alpha e^{-q} \right)}{4Q^4} - \left( \frac{\gamma \alpha e^{-q} D}{Q^2} \right)^2
\]

\[
= \frac{2D \left( A + \frac{S}{m} + \alpha e^{-q} \right) \left( 4D \gamma^2 \alpha e^{-q} - Q r_c k \sigma L^2 \right)}{4Q^4} - \left( \frac{\gamma \alpha e^{-q} D}{Q^2} \right)^2
\]
\[ D \left( A + \frac{S}{m} + \alpha e^{-\mathcal{L}} \left( 4D\gamma^2 e^{-\mathcal{L}} - Qrc_b k\sigma L^2 \right) \right) = \frac{2Q^4}{2Q^4} - \frac{\gamma e^{-\mathcal{L}} D}{Q^2} \]

\[ D \left( A + \frac{S}{m} + \alpha e^{-\mathcal{L}} \left( 4D\gamma^2 e^{-\mathcal{L}} - Qrc_b k\sigma L^2 \right) \right) - 2\left( \gamma e^{-\mathcal{L}} D \right)^2 > 0 \]

Since \( 4D\gamma^2 e^{-\mathcal{L}} > Qrc_b k\sigma L^2 \)

\[ D \left( A + \frac{S}{m} + \alpha e^{-\mathcal{L}} \left( 4D\gamma^2 e^{-\mathcal{L}} - Qrc_b k\sigma L^2 \right) \right) > 2\left( \gamma e^{-\mathcal{L}} D \right)^2. \]

Therefore \(|H_{22}| > 0\). Hence for a fixed \( m \), the Hessian Matrix is positive definite and \( ITC(Q,L,m) \) is convex with respect to \( (Q,L) \).

**APPENDIX B**

We first obtain the Hessian Matrix \( H \) as follows

\[
H = \begin{bmatrix}
\frac{\partial^2 ITC(Q,L,m,\theta)}{\partial Q^2} & \frac{\partial^2 ITC(Q,L,m,\theta)}{\partial Q \partial L} & \frac{\partial^2 ITC(Q,L,m,\theta)}{\partial Q \partial \theta} \\
\frac{\partial^2 ITC(Q,L,m,\theta)}{\partial L \partial Q} & \frac{\partial^2 ITC(Q,L,m,\theta)}{\partial L^2} & \frac{\partial^2 ITC(Q,L,m,\theta)}{\partial L \partial \theta} \\
\frac{\partial^2 ITC(Q,L,m,\theta)}{\partial \theta \partial Q} & \frac{\partial^2 ITC(Q,L,m,\theta)}{\partial \theta \partial L} & \frac{\partial^2 ITC(Q,L,m,\theta)}{\partial \theta^2}
\end{bmatrix}
\]

Where

\[
\frac{\partial^2 ITC(Q,L,m,\theta)}{\partial Q^2} = \frac{2D}{Q^4} \left( A + \frac{S}{m} + \alpha e^{-\mathcal{L}} \right)
\]

\[
\frac{\partial^2 ITC(Q,L,m,\theta)}{\partial \theta^2} = \frac{iq}{\theta^2}
\]

\[
\frac{\partial^2 ITC(Q,L,m,\theta)}{\partial L^2} = \frac{\gamma^2 e^{-\mathcal{L}} D}{Q} - \frac{rc_b k\sigma L^2}{4}
\]

\[
\frac{\partial^2 ITC(Q,L,m,\theta)}{\partial Q \partial L} = \frac{\partial^2 ITC(Q,L,m,\theta)}{\partial L \partial Q} = \frac{\gamma e^{-\mathcal{L}} D}{Q^2}
\]
\[
\frac{\partial^2 ITC(Q, L, m, \theta)}{\partial Q \partial \theta} = \frac{\partial^2 ITC(Q, L, m, \theta)}{\partial \theta \partial Q} = \frac{gmD}{2}
\]
\[
\frac{\partial^2 ITC(Q, L, m, \theta)}{\partial L \partial \theta} = \frac{\partial^2 ITC(Q, L, m, \theta)}{\partial \theta \partial L} = 0
\]

We proceed by evaluating the principal minor determinant of the Hessian Matrix \(H\) at point \((Q^*, L^*, \theta^*)\). The first principal minor determinant of \(H\) then becomes
\[
|H_{11}| > 0.
\]
\[
|H_{22}| > 0.
\]

Hence for a fixed \(m\), the Hessian Matrix is positive definite and
\(ITC(Q, L, m, \theta)\) is convex with respect to \((Q, L)\).
\[
|H_{33}| = \frac{\partial^2 ITC(Q, L, m, \theta)}{\partial Q^2} \left( \frac{\partial^2 ITC(Q, L, m, \theta)}{\partial L^2} \frac{\partial^2 ITC(Q, L, m, \theta)}{\partial \theta^2} - \left( \frac{\partial^2 ITC(Q, L, m, \theta)}{\partial \theta L} \right)^2 \right)
\]
\[
- \frac{\partial^2 ITC(Q, L, m, \theta)}{\partial L \partial Q} \left( \frac{\partial^2 ITC(Q, L, m, \theta)}{\partial L Q} \frac{\partial^2 ITC(Q, L, m, \theta)}{\partial \theta^2} - \frac{\partial^2 ITC(Q, L, m, \theta)}{\partial \theta L} \frac{\partial^2 ITC(Q, L, m, \theta)}{\partial Q \theta} \right)
\]
\[
+ \frac{\partial^2 ITC(Q, L, m, \theta)}{\partial \theta Q} \left( \frac{\partial^2 ITC(Q, L, m, \theta)}{\partial Q \theta} \frac{\partial^2 ITC(Q, L, m, \theta)}{\partial L \theta} - \frac{\partial^2 ITC(Q, L, m, \theta)}{\partial L^2} \frac{\partial^2 ITC(Q, L, m, \theta)}{\partial Q \theta} \right)
\]

If
\[
\frac{2D}{Q^3} \left( A + \frac{S}{m} + \alpha e^{-\gamma L} \right) \left( \frac{iq}{\theta^2} \right) \left( \gamma^2 \alpha e - \frac{\gamma L D}{Q} - \frac{1}{4} rc_b k \sigma L^2 \right)^2 + \left( \frac{gmD}{2} \right)^2 \left( \gamma^2 \alpha e - \frac{\gamma L D}{Q} - \frac{1}{4} rc_b k \sigma L^2 \right)^2
\]

Then
\[
\frac{2D}{Q^3} \left( A + \frac{S}{m} + \alpha e^{-\gamma L} \right) \left( \frac{iq}{\theta^2} \right) \left( \gamma^2 \alpha e - \frac{\gamma L D}{Q} - \frac{1}{4} rc_b k \sigma L^2 \right)^2 - \left( \frac{gmD}{2} \right)^2 \left( \gamma^2 \alpha e - \frac{\gamma L D}{Q} - \frac{1}{4} rc_b k \sigma L^2 \right)^2 > 0
\]

Therefore \(|H_{33}| > 0\). Hence for a fixed \(m\), the Hessian Matrix is positive definite and \(ITC(Q, L, m, \theta)\) is convex with respect to \((Q, L, \theta)\).