Chapter VII

Generalized Lambert Summability of Orthogonal Series
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7.1 Introduction
Let \( \{ \varphi_n(\theta) \} \), \( n = 0, 1, \ldots \) be an orthonormal system defined in an interval \((a, b)\).

Let
\[
\sum_{n=0}^{\infty} c_n \varphi_n(\theta) \tag{7-1}
\]
be an orthogonal series, where \( \{ c_n \} \) be a sequence of real numbers.

Let \( f(\theta) \in L^2(a, b) \). The orthogonal expansion of \( f(\theta) \) is
\[
f \sim \sum_{n=0}^{\infty} c_n \varphi_n(\theta) \tag{7-2}
\]
where, \( c_n \) is determined by
\[
c_n = \int_a^b f(\theta) \varphi_n(\theta) d\theta;
\]

Let
\[
S_n(\theta) = \sum_{n=1}^{\infty} c_n \varphi_n(\theta)
\]
be a sequence of partial sums of series (7-1).

The \((L, 1)\) sum of the series (7-1) is given by
\[
\lim_{x \to 1} (1 - x) \sum_{n=1}^{\infty} \frac{n c_n \varphi_n}{1 - x^n} x^n
\]
(See Bellman, R. 1943)

The \((L, \alpha)\) sum of the series (7-1) is
\[
\lim_{x \to 1} (1 - x) \sum_{n=1}^{\infty} c_n \varphi_n(\theta) \left( \frac{n(1 - x)}{1 - x^n} \right)^\alpha x^n
\]
where, \( \alpha \) is any real number and \( 0 < x < 1 \).
(See Zhogin, I. 1969)

7.2 Generalized Lambert summability of Orthogonal series
Richard Bellman (Bellman, R. 1943) has proved the following theorem:
Theorem 7.1
Lambert summability of an orthogonal expansion (7-2) implies the convergence of partial sums $S_{2^n}(\theta)$ of orthogonal expansion (7-2).

We would like to generalize the Theorem 7.1 for generalized Lambert summability of an orthogonal expansion.

Our theorem is as follows:

**Theorem 7A**
Generalized Lambert summability of an orthogonal expansion (7-2) implies the convergence of partial sum $S_{2^n}(\theta)$ of an orthogonal expansion (7-2).

**7.3 Proof of Theorems**

**Proof of Theorem 7A**
Let $x = 1 - \frac{1}{2^n}$

Define

$$U_n(\theta) = \sum_{k=1}^{\infty} c_k \varphi_k(\theta) \left( \frac{k(1-x)}{1-x^k} \right)^\alpha x^k - S_{2^n}(\theta)$$

Hence,

$$U_n(\theta) = \sum_{k=1}^{2^n} c_k \varphi_k(\theta) \left( \frac{k(1-x)}{1-x^k} \right)^\alpha x^k - \sum_{k=1}^{2^n} c_k \varphi_k(\theta) + \sum_{k=2^n+1}^{\infty} c_k \varphi_k(\theta) \left( \frac{k(1-x)}{1-x^k} \right)^\alpha x^k$$

$$= \sum_{k=1}^{2^n} c_k \varphi_k(\theta) \left[ \left( \frac{k(1-x)}{1-x^k} \right)^\alpha x^k - 1 \right] + \sum_{k=2^n+1}^{\infty} c_k \varphi_k(\theta) \left( \frac{k(1-x)}{1-x^k} \right)^\alpha x^k$$

$$:= T_n(\theta) + V_n(\theta)$$

where,

$$T_n(\theta) = \sum_{k=1}^{2^n} c_k \varphi_k(\theta) \left[ \left( \frac{k(1-x)}{1-x^k} \right)^\alpha x^k - 1 \right]$$

$$V_n(\theta) = \sum_{k=2^n+1}^{\infty} c_k \varphi_k(\theta) \left( \frac{k(1-x)}{1-x^k} \right)^\alpha x^k$$

We may arrive at our conclusion if, we prove that

$$\lim_{n \to \infty} U_n(\theta) = 0$$
Now, we consider the series
\[ \sum_{n=1}^{\infty} [U_n(\theta)]^2. \]
It is sufficient to prove
\[ \sum_{n=1}^{\infty} \int_{a}^{b} [U_n(\theta)]^2 d\theta < \infty \]
for convergence almost everywhere in \( \theta \).
Now,
\[ \sum_{n=1}^{\infty} \int_{a}^{b} [U_n(\theta)]^2 d\theta = \sum_{n=1}^{\infty} \int_{a}^{b} [T_n(\theta) + V_n(\theta)]^2 d\theta \]
\[ \leq 2 \sum_{n=1}^{\infty} \int_{a}^{b} [T_n(\theta)]^2 d\theta + 2 \sum_{n=1}^{\infty} \int_{a}^{b} [V_n(\theta)]^2 d\theta \]
\[ := 2I_1 + 2I_2 \quad (7-3) \]
Now, we shall show the convergence of \( I_1 \).
Here,
\[ I_1 = \sum_{n=1}^{\infty} \int_{a}^{b} [T_n(\theta)]^2 d\theta \]
\[ = \sum_{n=1}^{\infty} \int_{a}^{b} \left[ \sum_{k=1}^{2^n} c_k \varphi_k(\theta) \left( \frac{k(1-x)}{1-x^k} \right)^x - 1 \right]^2 d\theta \]
Hence, by orthonormality, we have
\[ I_1 \leq \sum_{n=1}^{\infty} \left[ \sum_{k=1}^{2^n} c_k^2 \left( \frac{k(1-x)}{1-x^k} \right)^x - 1 \right]^2 \]
Now, for \( 0 \leq x \leq 1 \)
\[ \frac{1 - x^k}{1 - x} \leq k \]
So,
\[ 1 - x^k \leq k(1 - x), \]
So,
\[ 1 - x^k \geq 1 - \left( \frac{k(1-x)}{1-x^k} \right)^\alpha \quad \text{x}^k \geq 0, \]

Hence,

\[ I_1 = \sum_{n=1}^{\infty} \int_a^b [T_n(\theta)]^2 d\theta \leq \sum_{n=1}^{\infty} \sum_{k=1}^{2^n} c_k^2 (1 - x^k)^2 \]

\[ \leq \sum_{n=1}^{\infty} \sum_{k=1}^{2^n} k^2 c_k^2 (1 - x)^2 \]

\[ = \sum_{n=1}^{\infty} \sum_{k=1}^{2^n} k^2 c_k^2 \left( 1 - \left( 1 - \frac{1}{2^n} \right) \right)^2 \]

\[ \leq \sum_{n=1}^{\infty} \frac{1}{2^{2n}} \sum_{k=1}^{2^n} k^2 c_k^2 \]

\[ \leq \sum_{k=1}^{2^n} k^2 c_k^2 \sum_{n \geq \log_2 k} 2^{-2n} \]

\[ = O(1) \sum_{k=1}^{2^n} c_k^2 \]

Since, \( f \in L^2(a, b) \), we have

\[ \sum_{k=1}^{2^n} c_k^2 < \infty \]

Hence, \( I_1 < \infty \).

Now, we shall show convergence of \( I_2 \).

\[ \sum_{n=1}^{\infty} \int_a^b [V_n(\theta)]^2 d\theta < \infty \]

\[ I_2 = \sum_{n=1}^{\infty} \int_a^b [V_n(\theta)]^2 d\theta \]

\[ = \sum_{n=1}^{\infty} \int_a^b \left\{ \sum_{k=2^n+1}^{\infty} c_k \varphi_k(\theta) \left( \frac{k(1-x)}{1-x^k} \right)^\alpha x^k \right\}^2 d\theta \]
\[ \leq \sum_{n=1}^{\infty} \sum_{k=2^n+1}^{\infty} k^{2\alpha} c_k^2 \frac{(1-x)^{2\alpha}}{(1-x^k)^{2\alpha} x^{2k}} \]

Since \((1 - 2^{-n})^k\) is decreasing function of \(k\),

\[ \sum_{n=1}^{\infty} \left[ \frac{1}{1 - (1 - 2^{-n})^{2\alpha}} \right] \sum_{k=2^n+1}^{\infty} k^{2\alpha} c_k^2 (1-x)^{2\alpha} x^{2k} \]

\[ = O(1) \sum_{n=1}^{\infty} \sum_{k=2^n+1}^{\infty} k^{2\alpha} c_k^2 (1-x)^{2\alpha} x^{2k} \]

\[ = O(1) \sum_{n=1}^{\infty} \sum_{k=2^n+1}^{\infty} k^{2\alpha} c_k^2 \left( \frac{1}{2^n} \right)^{2\alpha} \left( 1 - \frac{1}{2^n} \right)^{2k} \]

\[ = O(1) \sum_{n=1}^{\infty} \sum_{k=2^n+1}^{\infty} k^{2\alpha} c_k^2 (2^{-n})^{2\alpha} (1 - 2^{-n})^{2k} \]

We can majorize

\[ k^{2\alpha} \sum_{n=1}^{\infty} 2^{-2n\alpha} (1 - 2^{-n})^{2k} \]

by integral,

\[ k^{2\alpha} \int_0^{\infty} 2^{-2\alpha x} (1 - 2^{-x})^{-2k} dx \]

Hence,

\[ I_2 = O(1) \sum_{k=2^n+1}^{\infty} c_k^2 \int_0^{\infty} 2^{-2\alpha x} (1 - 2^x)^{2k} k^{2\alpha} dx \]

\[ = O(1) \sum_{k=2^n+1}^{\infty} c_k^2 \frac{k^{2\alpha}}{(2k + 1)(2k + 2) ... (2k + \alpha)} \]

Since \( \frac{k^{2\alpha}}{(2k + 1)(2k + 2) ... (2k + \alpha)} \) is bounded,

\[ I_2 = O(1) \sum_{k=2^n+1}^{\infty} c_k^2 \]

But \( f \in L^2(a, b) \), so,

\[ \sum_{k=2^n+1}^{\infty} c_k^2 < \infty \]
Hence,

\[ I_2 < \infty \]

Hence, we have proven the convergence of \( I_1 \) and \( I_2 \) separately.

Hence by (7-3) we have

\[
\sum_{n=1}^{\infty} \int_{a}^{b} [U_n(\theta)]^2 d\theta < \infty
\]

Hence,

\[
\sum_{n=1}^{\infty} [U_n(\theta)]^2 < \infty
\]

Therefore,

\[
\lim_{n \to \infty} U_n(\theta) = 0
\]

Hence, the proof.