CHAPTER SEVEN

$M^{[X]}/G/1$ Queue with Two Stage Heterogeneous Service, Random Breakdown, Delayed Repairs and Extended Server Vacations with Bernoulli Schedule
7.1 Introduction

Server vacation models are useful for the systems in which the server wants to utilize the idle time for different purposes. In fact, queueing systems with server breakdowns are very common in communication systems and manufacturing systems. The study on two phases queueing system with vacation have become an interesting area in queueing theory. Many researchers have put their efforts in this area by considering various aspects like two phase queueing system with Bernoulli feedback, random break downs, Bernoulli vacation etc.

Also queueing systems with breakdowns have been studied by several authors including Federgruen and Green (1986), Tang (1997), Li et al. (1997),

Recently Maraghi et al. (2009) have studied some queueing systems with vacations and breakdowns. Thangaraj and Vanitha (2010a) have obtained transient solution for M/G/1 queue with two-stage heterogeneous service with compulsory server vacation and random breakdowns. Khalaf et al. (2010) studied an $M^{[X]}/G/1$ queue with Bernoulli schedule general vacation times, random breakdowns, general delay times for repairs to start and general repair times. They have obtained steady state results in terms of the probability generating functions for the number of customers in the queue. Choudhury and Madan (2005) and Madan (2000a) have studied two stage service with server vacations.

In this chapter, we consider $M^{[X]}/G/1$ queue with two stage service, random breakdown, delayed repairs and extended server vacations. Customers arrive at the system in batches of variable size in a compound Poisson process and they are provided one by one service on a first come - first served basis. The server provides two stages of service which is essential for all customers with service times having general (arbitrary) distribution. As soon as the second stage of a customer’s service is completed, the server will take a vacation with probability $p$ or may continue to stay in the system with probability $1 - p$. On completion of first phase of vacation, the server has the further option of taking an extended vacation. We assume that with probability $r$ the server takes an extended vacation and with probability $1 - r$ rejoins the system immediately after completion of phase one vacation. The system may break down at random and breakdowns are assumed to occur according to a Poisson. Further, we assume that once the system breaks down, its repairs do not start immediately and there is a delay time, the customer whose service is
interrupted comes back to the head of the queue. Repair times, delay times and vacation times follow general (arbitrary) distribution.

Here we derive time dependent probability generating functions in terms of Laplace transforms. We also derive the average queue size, system size and average waiting time in the queue, the system. Some particular cases and numerical results are also discussed.

The rest of this chapter is organized as follows. The mathematical description of our model is given in section 7.2. Definitions and equations governing the system are given in section 7.3. The time dependent solution have been obtained in section 7.4 and corresponding steady state results have been derived explicitly in section 7.5. Average queue size, system size and average waiting time in the queue, system are computed in section 7.6. Some particular cases and numerical results are discussed in section 7.7 and 7.8 respectively.

7.2 Mathematical description of the model

We assume the following to describe the queueing model of our study.

a) Customers arrive at the system in batches of variable size in a compound Poisson process and they are provided one by one service on a first come - first served basis. Let \( \lambda c_i dt \) (\( i = 1, 2, \ldots \)) be the first order probability that a batch of \( i \) customers arrives at the system during a short interval of time \( (t, t + dt] \), where \( 0 \leq c_i \leq 1 \), \( \sum_{i=1}^{\infty} c_i = 1 \) and \( \lambda > 0 \) is the arrival rate of batches.

b) The server provides two stages of service which is essential for all customers. The service time follows a general (arbitrary) distribution with distribution function \( B_i(s) \) and density function \( b_i(s) \). Let \( \mu_i(x)dx \) be the conditional probability density of service completion during the interval
\((x, x + dx]\), given that the elapsed service time is \(x\), so that

\[
\mu_i(x) = \frac{b_i(x)}{1 - B_i(x)}, \quad i = 1, 2,
\]

and therefore,

\[
b_i(s) = \mu_i(s)e^{-\int_0^s \mu_i(x)dx}, \quad i = 1, 2.
\]

c) As soon as the second stage of a customer’s service is completed, the server
will take a vacation with probability \(p\) or may continue to stay in the
system with probability \(1 - p\). On completion of first phase of vacation,
the server has the further option of taking an extended vacation. We
assume that with probability \(r\), the server takes an extended vacation and
with probability \(1 - r\) rejoins the system immediately after completion
of phase one vacation.

d) The server’s vacation time follows a general (arbitrary) distribution with
distribution function \(V(t)\) and density function \(v(t)\). Let \(\beta_i(x)dx\) be the
conditional probability of a completion of a vacation during the interval
\((x, x + dx]\), given that the elapsed vacation time is \(x\), so that

\[
\beta_i(x) = \frac{v(x)}{1 - V(x)}, \quad i = 1, 2,
\]

and therefore,

\[
v(t) = \beta_i(t)e^{-\int_0^t \beta_i(x)dx}, \quad i = 1, 2.
\]

e) The system may break down at random and breakdowns are assumed to
occur according to a Poisson stream with mean breakdown rate \(\alpha > 0\).
Further we assume that once the system breaks down, its repairs do not
start immediately and there is a delay time, the customer whose service
is interrupted comes back to the head of the queue.
f) The delay times follow a general (arbitrary) distribution with distribution function $F(x)$ and density function $f(x)$. Let $\theta(x)dx$ be the conditional probability of a completion of a delay during the interval $(x, x + dx]$, given that the elapsed delay time is $x$, so that

$$\theta(x) = \frac{f(x)}{1 - F(x)}$$

and therefore,

$$f(t) = \theta(t)e^{-\int_0^t \theta(x)dx}.$$ 

g) The duration of repairs follows a general (arbitrary) distribution with distribution function $G(x)$ and density function $g(x)$. Let $\gamma(x)dx$ be the conditional probability of a completion of repairs during the interval $(x, x + dx]$, given that the elapsed repair time is $x$, so that

$$\gamma(x) = \frac{g(x)}{1 - G(x)}$$

and therefore,

$$g(t) = \gamma(t)e^{-\int_0^t \gamma(x)dx}.$$ 

i) Various stochastic processes involved in the system are assumed to be independent of each other.

7.3 Definitions and equations governing of the system

We define

$$P_n^{(1)}(x, t) = \text{Probability that at time } t, \text{ the server is active providing first}$$
stage of service and there are \( n \ (n \geq 0) \) customers in the queue excluding the one being served and the elapsed service time is \( x \). Accordingly, \( P_n^{(1)}(t) = \int_0^\infty P_n^{(1)}(x,t)dx \) denotes the probability that at time \( t \) there are \( n \) customers in the queue excluding the one customer in the first stage of service irrespective of the value of \( x \).

\[ P_n^{(2)}(x,t) = \text{Probability that at time } t, \text{ the server is active providing second stage of service and there are } n \ (n \geq 0) \text{ customers in the queue excluding the one being served and the elapsed service time is } x. \]  Accordingly, 

\[ P_n^{(2)}(t) = \int_0^\infty P_n^{(2)}(x,t)dx \]  denotes the probability that at time \( t \) there are \( n \) customers in the queue excluding the one customer in the second stage of service irrespective of the value of \( x \).

\[ V_n^{(1)}(x,t) = \text{Probability that at time } t, \text{ the server is under phase one vacation with elapsed vacation time is } x \text{ and there are } n \ (n \geq 0) \text{ customers in the queue. Accordingly } \]

\[ V_n^{(1)}(t) = \int_0^\infty V_n^{(1)}(x,t)dx \]  denotes the probability that at time \( t \) there are \( n \) customers in the queue and the server is under phase one vacation irrespective of the value of \( x \).

\[ V_n^{(2)}(x,t) = \text{Probability that at time } t, \text{ the server is under extended vacation with elapsed vacation time is } x \text{ and there are } n \ (n \geq 0) \text{ customers in the queue. Accordingly } \]

\[ V_n^{(2)}(t) = \int_0^\infty V_n^{(2)}(x,t)dx \]  denotes the probability that at time \( t \) there are \( n \) customers in the queue and the server is under extended vacation irrespective of the value of \( x \).

\[ D_n(x,t) = \text{Probability that at time } t, \text{ there are } n \ (n \geq 0) \text{ customers in the queue and the server is inactive due to system breakdown and waiting for repairs to start with elapsed delay time is } x. \]  Accordingly 

\[ D_n(t) = \int_0^\infty D_n(x,t)dx \]  denotes the probability that at time \( t \), there are \( n \) customers in the queue and the server is waiting for repairs to start irrespective of the value of \( x \).

\[ R_n(x,t) = \text{probability that at time } t, \text{ there are } n \ (n \geq 0) \text{ customers in the queue, and the server is under repair with elapsed repair time is } x. \]
Accordingly \( R_n(t) = \int_0^\infty R_n(x,t) \, dx \) denotes the probability that at time \( t \), there are \( n \) customers in the queue and the server is under repair irrespective of the value of \( x \).

\( Q(t) \) is the probability that at time \( t \), there are no customers in the system and the server is idle but available in the system.

The system is then governed by the following set of differential-difference equations:

\[
\frac{\partial}{\partial x} P^{(1)}_0(x,t) + \frac{\partial}{\partial t} P^{(1)}_0(x,t) + [\lambda + \mu_1(x) + \alpha] P^{(1)}_0(x,t) = 0 \quad (7.1)
\]

\[
\frac{\partial}{\partial x} P^{(1)}_n(x,t) + \frac{\partial}{\partial t} P^{(1)}_n(x,t) + [\lambda + \mu_1(x) + \alpha] P^{(1)}_n(x,t) = \lambda \sum_{k=1}^{n} c_k P^{(1)}_{n-k}(x,t), \quad n \geq 1 \quad (7.2)
\]

\[
\frac{\partial}{\partial x} P^{(2)}_0(x,t) + \frac{\partial}{\partial t} P^{(2)}_0(x,t) + [\lambda + \mu_2(x) + \alpha] P^{(2)}_0(x,t) = 0 \quad (7.3)
\]

\[
\frac{\partial}{\partial x} P^{(2)}_n(x,t) + \frac{\partial}{\partial t} P^{(2)}_n(x,t) + [\lambda + \mu_2(x) + \alpha] P^{(2)}_n(x,t) = \lambda \sum_{k=1}^{n} c_k P^{(2)}_{n-k}(x,t), \quad n \geq 1 \quad (7.4)
\]

\[
\frac{\partial}{\partial x} V^{(1)}_0(x,t) + \frac{\partial}{\partial t} V^{(1)}_0(x,t) + [\lambda + \beta_1(x)] V^{(1)}_0(x,t) = 0 \quad (7.5)
\]

\[
\frac{\partial}{\partial x} V^{(1)}_n(x,t) + \frac{\partial}{\partial t} V^{(1)}_n(x,t) + [\lambda + \beta_1(x)] V^{(1)}_n(x,t) = \lambda \sum_{k=1}^{n} c_k V^{(1)}_{n-k}(x,t), \quad n \geq 1 \quad (7.6)
\]

\[
\frac{\partial}{\partial x} V^{(2)}_0(x,t) + \frac{\partial}{\partial t} V^{(2)}_0(x,t) + [\lambda + \beta_2(x)] V^{(2)}_0(x,t) = 0 \quad (7.7)
\]

\[
\frac{\partial}{\partial x} V^{(2)}_n(x,t) + \frac{\partial}{\partial t} V^{(2)}_n(x,t) + [\lambda + \beta_2(x)] V^{(2)}_n(x,t) = \lambda \sum_{k=1}^{n} c_k V^{(2)}_{n-k}(x,t), \quad n \geq 1 \quad (7.8)
\]

\[
\frac{\partial}{\partial x} D_0(x,t) + \frac{\partial}{\partial x} D_0(x,t) + [\lambda + \theta(x)] D_0(x,t) = 0 \quad (7.9)
\]
\[
\frac{\partial}{\partial x} D_n(x, t) + \frac{\partial}{\partial x} D_n(x, t) + [\lambda + \theta(x)] D_n(x, t) = \lambda \sum_{k=1}^{n} c_k D_{n-k}(x, t), \\
n \geq 1 \quad (7.10)
\]

\[
\frac{\partial}{\partial x} R_0(x, t) + \frac{\partial}{\partial x} R_0(x, t) + [\lambda + \gamma(x)] R_0(x, t) = 0 \quad (7.11)
\]

\[
\frac{\partial}{\partial x} R_n(x, t) + \frac{\partial}{\partial x} R_n(x, t) + [\lambda + \gamma(x)] R_n(x, t) = \lambda \sum_{k=1}^{n} c_k R_{n-k}(x, t), \\
n \geq 1 \quad (7.12)
\]

\[
\frac{d}{dt} Q(t) = -\lambda Q(t) + (1 - p) \int_{0}^{\infty} P_0^{(2)}(x, t) \mu_2(x) dx \\
+ \int_{0}^{\infty} R_0(x, t) \gamma(x) dx + (1 - r) \int_{0}^{\infty} V_0^{(1)}(x, t) \beta_1(x) dx \\
+ \int_{0}^{\infty} V_0^{(2)}(x, t) \beta_2(x) dx \quad (7.13)
\]

The above set of equations are to be solved subject to the following boundary conditions:

\[
P_n^{(1)}(0, t) = \lambda c_{n+1} Q(t) + (1 - p) \int_{0}^{\infty} P_{n+1}^{(2)}(x, t) \mu_2(x) dx \\
+ (1 - r) \int_{0}^{\infty} V_{n+1}^{(1)}(x, t) \beta_1(x) dx + \int_{0}^{\infty} V_{n+1}^{(2)}(x, t) \beta_2(x) dx \\
+ \int_{0}^{\infty} R_{n+1}(x, t) \gamma(x) dx, \quad n \geq 0 \quad (7.14)
\]

\[
P_n^{(2)}(0, t) = \int_{0}^{\infty} P_n^{(1)}(x, t) \mu_1(x) dx, \quad n \geq 0 \quad (7.15)
\]

\[
V_n^{(1)}(0, t) = p \int_{0}^{\infty} P_n^{(2)}(x, t) \mu_2(x) dx, \quad n \geq 0 \quad (7.16)
\]

\[
V_n^{(2)}(0, t) = r \int_{0}^{\infty} V_n^{(1)}(x, t) \beta_1(x) dx, \quad n \geq 0 \quad (7.17)
\]

\[
D_0(0, t) = 0 \quad (7.18)
\]

\[
D_n(0, t) = \alpha \int_{0}^{\infty} P_{n-1}^{(1)}(x, t) dx + \alpha \int_{0}^{\infty} P_{n-1}^{(2)}(x, t) dx, \quad n \geq 1 \quad (7.19)
\]

\[
R_n(0, t) = \int_{0}^{\infty} D_n(x, t) \theta(x) dx, \quad n \geq 0 \quad (7.20)
\]
We assume that initially there are no customers in the system and the server is idle. So the initial conditions are

\[ P_n^{(i)}(0) = V_n^{(i)}(0) = 0, \quad \text{for } i = 1, 2, \quad Q(0) = 1 \text{ and } \]
\[ D_n(0) = 0, R_n(0) = 0 \quad \text{for } n = 0, 1, 2, ... \quad (7.21) \]

7.4 Probability generating functions of the queue length: The time-dependent solution

In this section, we obtain the transient solution for the above set of differential-difference equations.

**Theorem:** The system of differential difference equations to describe an \( M^{[X]}/G/1 \) Queue with Two Stage Heterogeneous Service, Random Breakdown, Delayed Repairs and Extended Server Vacations with Bernoulli Schedule are given by equations (7.1) to (7.20) with initial conditions (7.21) and the generating functions of transient solution are given by equations (7.84) to (7.89).

**Proof:** We define the probability generating functions, for \( i = 1, 2. \)

\[ P^{(i)}(x, z, t) = \sum_{n=0}^{\infty} z^n P_n^{(i)}(x, t); \quad P^{(i)}(z, t) = \sum_{n=0}^{\infty} z^n P_n^{(i)}(t); \]
\[ V^{(i)}(x, z, t) = \sum_{n=0}^{\infty} z^n V_n^{(i)}(x, t); \quad V^{(i)}(z, t) = \sum_{n=0}^{\infty} z^n V_n^{(i)}(t); \]
\[ D(x, z, t) = \sum_{n=0}^{\infty} z^n D_n(x, t); \quad D(z, t) = \sum_{n=0}^{\infty} z^n D_n(t); \quad C(z) = \sum_{n=1}^{\infty} c_n z^n; \]
\[ R(x, z, t) = \sum_{n=0}^{\infty} z^n R_n(x, t); \quad R(z, t) = \sum_{n=0}^{\infty} z^n R_n(t); \quad (7.22) \]
which are convergent inside the circle given by $|z| \leq 1$ and define the Laplace transform of a function $f(t)$ as

$$\tilde{f}(s) = \int_{0}^{\infty} e^{-st} f(t) dt, \quad \Re(s) > 0.$$  

Taking the Laplace transform of equations (7.1) to (7.20) and using (7.21), we obtain

$$\frac{\partial}{\partial x} \tilde{P}_0^{(1)}(x,s) + (s + \lambda + \mu_1(x) + \alpha) \tilde{P}_0^{(1)}(x,s) = 0$$  

(7.23)

$$\frac{\partial}{\partial x} \tilde{P}_n^{(1)}(x,s) + (s + \lambda + \mu_1(x) + \alpha) \tilde{P}_n^{(1)}(x,s) = \lambda \sum_{k=1}^{n} c_k \tilde{P}_{n-k}^{(1)}(x,s), \quad n \geq 1$$  

(7.24)

$$\frac{\partial}{\partial x} \tilde{P}_0^{(2)}(x,s) + (s + \lambda + \mu_2(x) + \alpha) \tilde{P}_0^{(2)}(x,s) = 0$$  

(7.25)

$$\frac{\partial}{\partial x} \tilde{P}_n^{(2)}(x,s) + (s + \lambda + \mu_2(x) + \alpha) \tilde{P}_n^{(2)}(x,s) = \lambda \sum_{k=1}^{n} c_k \tilde{P}_{n-k}^{(2)}(x,s), \quad n \geq 1$$  

(7.26)

$$\frac{\partial}{\partial x} \tilde{V}_0^{(1)}(x,s) + (s + \lambda + \beta_1(x)) \tilde{V}_0^{(1)}(x,s) = 0$$  

(7.27)

$$\frac{\partial}{\partial x} \tilde{V}_n^{(1)}(x,s) + (s + \lambda + \beta_1(x)) \tilde{V}_n^{(1)}(x,s) = \lambda \sum_{k=1}^{n} c_k \tilde{V}_{n-k}^{(1)}(x,s), \quad n \geq 1$$  

(7.28)

$$\frac{\partial}{\partial x} \tilde{V}_0^{(2)}(x,s) + (s + \lambda + \beta_2(x)) \tilde{V}_0^{(2)}(x,s) = 0$$  

(7.29)

$$\frac{\partial}{\partial x} \tilde{V}_n^{(2)}(x,s) + (s + \lambda + \beta_2(x)) \tilde{V}_n^{(2)}(x,s) = \lambda \sum_{k=1}^{n} c_k \tilde{V}_{n-k}^{(2)}(x,s), \quad n \geq 1$$  

(7.30)

$$\frac{\partial}{\partial x} \tilde{D}_0(x,s) + (\lambda + \theta(x)) \tilde{D}_0(x,s) = 0$$  

(7.31)

$$\frac{\partial}{\partial x} \tilde{D}_n(x,s) + (\lambda + \theta(x)) \tilde{D}_n(x,s) = \lambda \sum_{k=1}^{n} c_k \tilde{D}_{n-k}(x,s), \quad n \geq 1$$  

(7.32)

$$\frac{\partial}{\partial x} \tilde{R}_0(x,s) + (\lambda + \gamma(x)) \tilde{R}_0(x,s) = 0$$  

(7.33)
\[
\frac{\partial}{\partial x} \bar{R}_n(x,s) + (\lambda + \gamma(x)) \bar{R}_n(x,s) = \lambda \sum_{k=1}^{n} c_k \bar{R}_{n-k}(x,s), \quad n \geq 1 \tag{7.34}
\]

\[
(s + \lambda) \bar{Q}(s) = 1 + (1 - p) \int_{0}^{\infty} \bar{P}_0^{(2)}(x,s) \mu_2(x) dx \\
+ (1 - r) \int_{0}^{\infty} \bar{V}_0^{(1)}(x,s) \beta_1(x) dx \\
+ \int_{0}^{\infty} \bar{R}_0(x,s) \gamma(x) dx + \int_{0}^{\infty} \bar{V}_0^{(2)}(x,s) \beta_2(x) dx \tag{7.35}
\]

\[
\bar{P}_n^{(1)}(0,s) = \lambda c_{n+1} \bar{Q}(s) + (1 - p) \int_{0}^{\infty} \bar{P}_{n+1}^{(2)}(x,s) \mu_2(x) dx \\
+ (1 - r) \int_{0}^{\infty} \bar{V}_{n+1}^{(1)}(x,s) \beta_1(x) dx + \int_{0}^{\infty} \bar{V}_{n+1}^{(2)}(x,s) \beta_2(x) dx \\
+ \int_{0}^{\infty} \bar{R}_{n+1}(x,s) \gamma(x) dx, \quad n \geq 0 \tag{7.36}
\]

\[
\bar{P}_n^{(2)}(0,s) = \int_{0}^{\infty} \bar{P}_n^{(1)}(x,s) \mu_1(x) dx, \quad n \geq 0 \tag{7.37}
\]

\[
\bar{V}_n^{(1)}(0,s) = p \int_{0}^{\infty} \bar{P}_n^{(2)}(x,s) \mu_2(x) dx, \quad n \geq 0 \tag{7.38}
\]

\[
\bar{V}_n^{(2)}(0,s) = r \int_{0}^{\infty} \bar{V}_n^{(1)}(x,s) \beta_1(x) dx, \quad n \geq 0 \tag{7.39}
\]

\[
\bar{D}_0(0,s) = 0 \tag{7.40}
\]

\[
\bar{D}_n(0,s) = \alpha \int_{0}^{\infty} \bar{P}_{n-1}(x,s) dx + \alpha \int_{0}^{\infty} \bar{P}_{n-1}^{(2)}(x,s) dx, \quad n \geq 1 \tag{7.41}
\]

\[
\bar{R}_n(0,s) = \int_{0}^{\infty} \bar{D}_n(x,s) \theta(x) dx, \quad n \geq 0 \tag{7.42}
\]

Now multiplying equations (7.24), (7.26), (7.28), (7.30), (7.32) and (7.34) by \( z^n \) and summing over \( n \) from 1 to \( \infty \), adding to equations (7.23), (7.25), (7.27), (7.29), (7.31), (7.33) and using the generating functions defined in (7.22), we get

\[
\frac{\partial}{\partial x} \bar{P}^{(1)}(x,z,s) + \left[ s + \lambda - \lambda C(z) + \mu_1(x) + \alpha \right] \bar{P}^{(1)}(x,z,s) = 0 \tag{7.43}
\]
\[
\frac{\partial}{\partial x} \tilde{P}^{(2)}(x, z, s) + [s + \lambda - \lambda C(z) + \mu_2(x) + \alpha] \tilde{P}^{(2)}(x, z, s) = 0 \tag{7.44}
\]
\[
\frac{\partial}{\partial x} \tilde{V}^{(1)}(x, z, s) + [s + \lambda - \lambda C(z) + \beta_1(x)] \tilde{V}^{(1)}(x, z, s) = 0 \tag{7.45}
\]
\[
\frac{\partial}{\partial x} \tilde{V}^{(2)}(x, z, s) + [s + \lambda - \lambda C(z) + \beta_2(x)] \tilde{V}^{(2)}(x, z, s) = 0 \tag{7.46}
\]
\[
\frac{\partial}{\partial x} \tilde{D}(x, z, s) + [s + \lambda - \lambda C(z) + \theta(x)] \tilde{D}(x, z, s) = 0 \tag{7.47}
\]
\[
\frac{\partial}{\partial x} \tilde{R}(x, z, s) + [s + \lambda - \lambda C(z) + \gamma(x)] \tilde{R}(x, z, s) = 0 \tag{7.48}
\]

For the boundary conditions, we multiply both sides of equation (7.36) by \(z^n\) summing over \(n\) from 0 to \(\infty\) and use the equations (7.22), we get

\[
z \tilde{P}^{(1)}(0, z, s) = \lambda C(z) \bar{Q}(s) + (1 - p) \int_{0}^{\infty} \tilde{P}^{(2)}(x, z, s) \mu_2(x) dx
\]
\[
- (1 - p) \int_{0}^{\infty} \tilde{P}^{(2)}_0(x, s) \mu_1(x) dx
\]
\[
+ (1 - r) \int_{0}^{\infty} \tilde{V}^{(1)}(x, z, s) \beta_1(x) dx
\]
\[
- (1 - r) \int_{0}^{\infty} \tilde{V}^{(1)}_0(x, s) \beta_1(x) dx
\]
\[
+ \int_{0}^{\infty} \tilde{V}^{(2)}(x, z, s) \beta_2(x) dx - \int_{0}^{\infty} \tilde{V}^{(2)}_0(x, s) \beta_2(x) dx
\]
\[
+ \int_{0}^{\infty} \tilde{R}(x, z, s) \gamma(x) dx - \int_{0}^{\infty} \tilde{R}_0(x, s) \gamma(x) dx
\]

Using equation (7.35), the above equation becomes

\[
z \tilde{P}^{(1)}(0, z, s) = (1 - s \bar{Q}(s)) + \lambda (C(z) - 1) \tilde{Q}(s)
\]
\[
+ (1 - p) \int_{0}^{\infty} \tilde{P}^{(2)}(x, z, s) \mu_2(x) dx
\]
\[
+ (1 - r) \int_{0}^{\infty} \tilde{V}^{(1)}(x, z, s) \beta_1(x) dx
\]
\[
+ \int_{0}^{\infty} \tilde{V}^{(2)}(x, z, s) \beta_2(x) dx + \int_{0}^{\infty} \tilde{R}(x, z, s) \gamma(x) dx \tag{7.49}
\]
Performing similar operation on equations (7.37) to (7.42), we get

\[ \tilde{P}^{(2)}(0, z, s) = \int_{0}^{\infty} \tilde{P}^{(1)}(x, z, s) \mu_1(x) dx \] (7.50)

\[ \tilde{V}^{(1)}(0, z, s) = p \int_{0}^{\infty} \tilde{P}^{(2)}(x, z, s) \mu_2(x) dx \] (7.51)

\[ \tilde{V}^{(2)}(0, z, s) = r \int_{0}^{\infty} \tilde{V}^{(1)}(x, z, s) \beta_1(x) dx \] (7.52)

\[ \tilde{D}(0, z, s) = \alpha z \int_{0}^{\infty} \tilde{P}^{(1)}(x, z, s) dx + \alpha z \int_{0}^{\infty} \tilde{P}^{(2)}(x, z, s) dx \] (7.53)

\[ \tilde{R}(0, z, s) = \int_{0}^{\infty} \tilde{D}(x, z, s) \theta(x) dx \] (7.54)

Integrating equation (7.43) between 0 and \( x \), we get

\[ \tilde{P}^{(1)}(x, z, s) = \tilde{P}^{(1)}(0, z, s) e^{-[s+\lambda-\lambda C(z)+\alpha]x-\int_{0}^{x} \mu_1(t) dt} \] (7.55)

where \( P^{(1)}(0, z, s) \) is given by equation (7.49).

Again integrating equation (7.55) by parts with respect to \( x \), yields

\[ \tilde{P}^{(1)}(z, s) = \tilde{P}^{(1)}(0, z, s) \left[ 1 - \tilde{B}_1(s + \lambda - \lambda C(z) + \alpha) \right] \] (7.56)

where

\[ \tilde{B}_1(s + \lambda - \lambda C(z) + \alpha) = \int_{0}^{\infty} e^{-[s+\lambda-\lambda C(z)+\alpha]x} dB_1(x) \]

is the Laplace-Stieltjes transform of the first stage of service time \( B_1(x) \). Now multiplying both sides of equation (7.55) by \( \mu_1(x) \) and integrating over \( x \), we obtain

\[ \int_{0}^{\infty} \tilde{P}^{(1)}(x, z, s) \mu_1(x) dx = \tilde{P}^{(1)}(0, z, s) \tilde{B}_1[s + \lambda - \lambda C(z) + \alpha] \] (7.57)
Similarly, on integrating equations (7.44) to (7.48) from 0 to \(x\), we get

\[
P^{(2)}(x, z, s) = P^{(2)}(0, z, s) e^{-[s+\lambda-\lambda C(z)+\alpha]x - \int_0^x \mu_2(t) dt}
\] (7.58)

\[
V^{(1)}(x, z, s) = V^{(1)}(0, z, s) e^{-[s+\lambda-\lambda C(z)]x - \int_0^x \beta_1(t) dt}
\] (7.59)

\[
V^{(2)}(x, z, s) = V^{(2)}(0, z, s) e^{-[s+\lambda-\lambda C(z)]x - \int_0^x \beta_2(t) dt}
\] (7.60)

\[
D(x, z, s) = D(0, z, s) e^{-[s+\lambda-\lambda C(z)]x - \int_0^x \theta(t) dt}
\] (7.61)

\[
R(x, z, s) = R(0, z, s) e^{-[s+\lambda-\lambda C(z)]x - \int_0^x \gamma(t) dt}
\] (7.62)

where \(P^{(2)}(0, z, s)\), \(V^{(1)}(0, z, s)\), \(V^{(2)}(0, z, s)\), \(D(0, z, s)\) and \(R(0, z, s)\) are given by equations (7.50) to (7.54).

Again integrating equations (7.58) to (7.62) by parts with respect to \(x\), yields

\[
P^{(2)}(z, s) = P^{(2)}(0, z, s) \left[ \frac{1 - \tilde{B}_2(s + \lambda - \lambda C(z) + \alpha)}{s + \lambda - \lambda C(z) + \alpha} \right]
\] (7.63)

\[
V^{(1)}(z, s) = V^{(1)}(0, z, s) \left[ \frac{1 - \tilde{V}_1(s + \lambda - \lambda C(z))}{s + \lambda - \lambda C(z)} \right]
\] (7.64)

\[
V^{(2)}(z, s) = V^{(2)}(0, z, s) \left[ \frac{1 - \tilde{V}_2(s + \lambda - \lambda C(z))}{s + \lambda - \lambda C(z)} \right]
\] (7.65)

\[
D(z, s) = D(0, z, s) \left[ \frac{1 - \tilde{F}(s + \lambda - \lambda C(z))}{s + \lambda - \lambda C(z)} \right]
\] (7.66)

\[
R(z, s) = R(0, z, s) \left[ \frac{1 - \tilde{G}(s + \lambda - \lambda C(z))}{s + \lambda - \lambda C(z)} \right]
\] (7.67)

where

\[
\tilde{B}_2(s + \lambda - \lambda C(z) + \alpha) = \int_0^\infty e^{-[s+\lambda-\lambda C(z)+\alpha]x} dB_2(x)
\]

\[
\tilde{V}_1(s + \lambda - \lambda C(z)) = \int_0^\infty e^{-[s+\lambda-\lambda C(z)]x} dV_1(x)
\]
\[ \tilde{V}_2(s + \lambda - \lambda C(z)) = \int_0^\infty e^{-(s+\lambda-\lambda C(z))x} dV_2(x) \]

\[ \tilde{F}(s + \lambda - \lambda C(z)) = \int_0^\infty e^{-(s+\lambda-\lambda C(z))x} dF(x) \]

\[ \tilde{G}(s + \lambda - \lambda C(z)) = \int_0^\infty e^{-(s+\lambda-\lambda C(z))x} dG(x) \]

are the Laplace-Stieltjes transform of the second stage of service time \( B_2(x) \), phase one vacation time \( V_1(x) \), extended vacation time \( V_2(x) \), delay time \( F(x) \) and repair time \( G(x) \).

Now multiplying both sides of equations (7.58) to (7.62) by \( \mu_2(x), \beta_1(x), \beta_2(x), \theta(x) \) and \( \gamma(x) \) and integrating over \( x \), we obtain

\[ \int_0^\infty \tilde{P}^{(2)}(x, z, s) \mu_2(x) dx = \tilde{P}^{(2)}(0, z, s) \tilde{B}_2[s + \lambda - \lambda C(z) + \alpha] \] (7.68)

\[ \int_0^\infty \tilde{V}^{(1)}(x, z, s) \beta_1(x) dx = \tilde{V}^{(1)}(0, z, s) \tilde{V}_1[s + \lambda - \lambda C(z)] \] (7.69)

\[ \int_0^\infty \tilde{V}^{(2)}(x, z, s) \beta_2(x) dx = \tilde{V}^{(2)}(0, z, s) \tilde{V}_2[s + \lambda - \lambda C(z)] \] (7.70)

\[ \int_0^\infty \tilde{D}(x, z, s) \theta(x) dx = \tilde{D}(0, z, s) \tilde{F}[s + \lambda - \lambda C(z)] \] (7.71)

\[ \int_0^\infty \tilde{R}(x, z, s) \gamma(x) dx = \tilde{R}(0, z, s) \tilde{G}[s + \lambda - \lambda C(z)] \] (7.72)

Now, using equation (7.57) in (7.50), we get

\[ \tilde{P}^{(2)}(0, z, s) = \tilde{B}_1(a) \tilde{P}^{(1)}(0, z, s) \] (7.73)
By using equations (7.68) and (7.73) in (7.51), we get

\[ \bar{V}^{(1)}(0, z, s) = p\bar{B}(a)\bar{P}^{(1)}(0, z, s) \] (7.74)

Now using equations (7.69) and (7.74) in (7.52), we get

\[ \bar{V}^{(2)}(0, z, s) = rp\bar{B}(a)\bar{V}^{(1)}(0, z, s) \] (7.75)

Similarly, using equations (7.55), (7.58) and (7.73) in (7.53), we get

\[ \bar{D}(0, z, s) = \alpha z \left[ \frac{1 - \bar{B}(a)}{a} \right] \bar{P}^{(1)}(0, z, s) \] (7.76)

Now using equations (7.71) and (7.76) in (7.54), we get

\[ \bar{R}(0, z, s) = \alpha z \bar{F}(b) \left[ \frac{1 - \bar{B}(a)}{a} \right] \bar{P}^{(1)}(0, z, s) \] (7.77)

Using equations (7.68), (7.69), (7.70), (7.72) in (7.49), we get

\[ z\bar{P}^{(1)}(0, z, s) = [1 - s\bar{Q}(s)] + (1 - p)\bar{B}_2(a)\bar{P}^{(2)}(0, z, s) \]
\[ + \lambda(C(z) - 1)\bar{Q}(s) + (1 - r)\bar{V}^{(1)}(0, z, s) \]
\[ + \bar{V}_2(b)\bar{V}^{(2)}(0, z, s) + \bar{G}(b)\bar{R}(0, z, s) \]

where

\[ a = s + \lambda - \lambda C(z) + \alpha, \quad b = s + \lambda - \lambda C(z) \text{ and } \bar{B}(a) = \bar{B}_1(a)\bar{B}_2(a). \]

Similarly using equations (7.73) to (7.75) and (7.77) in the above equation, we get

\[ \bar{P}^{(1)}(0, z, s) = \frac{[1 - s\bar{Q}(s)] + \lambda[C(z) - 1]\bar{Q}(s)}{Dr} \] (7.78)
Thus

\[ Dr = z - \bar{B}(a)[1 - p + p\bar{V}_1(b)(1 - r + r\bar{V}_2(b))] \]

\[-\frac{\alpha z}{a} \bar{F}(b)\bar{G}(b)[1 - \bar{B}(a)] \]

Substituting the value of \( \bar{P}^{(1)}(0, z, s) \) from equation (7.78) into equations (7.73) to (7.77), we get

\[
\bar{P}^{(2)}(0, z, s) = \bar{B}_1(a) \left[ \frac{(1 - s\bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s)}{Dr} \right] \tag{7.79}
\]

\[
\bar{V}^{(1)}(0, z, s) = p\bar{B}(a) \left[ \frac{(1 - s\bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s)}{Dr} \right] \tag{7.80}
\]

\[
\bar{V}^{(2)}(0, z, s) = rp\bar{B}(a)\bar{V}_1(b) \left[ \frac{(1 - s\bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s)}{Dr} \right] \tag{7.81}
\]

\[
\bar{D}(0, z, s) = \alpha z \left\{ \frac{1 - \bar{B}(a)}{a} \right\} \left[ \frac{(1 - s\bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s)}{Dr} \right] \tag{7.82}
\]

\[
\bar{R}(0, z, s) = \alpha z\bar{F}(b) \left\{ \frac{1 - \bar{B}(a)}{a} \right\} \left[ \frac{(1 - s\bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s)}{Dr} \right] \tag{7.83}
\]

Using equations (7.78) to (7.83) in (7.56), (7.63) to (7.67), we get

\[
\bar{P}^{(1)}(z, s) = \frac{1 - \bar{B}_1(a)}{a} \left[ \frac{(1 - s\bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s)}{Dr} \right] \tag{7.84}
\]

\[
\bar{P}^{(2)}(z, s) = \bar{B}_1(a) \left[ \frac{1 - \bar{B}_2(a)}{a} \right] \left[ \frac{(1 - s\bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s)}{Dr} \right] \tag{7.85}
\]

\[
\bar{V}^{(1)}(z, s) = p\bar{B}(a) \left\{ \frac{1 - \bar{V}_1(b)}{b} \right\} \left[ \frac{(1 - s\bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s)}{Dr} \right] \tag{7.86}
\]

\[
\bar{V}^{(2)}(z, s) = rp\bar{B}(a)\bar{V}_1(b) \left\{ \frac{1 - \bar{V}_2(b)}{b} \right\} \left[ \frac{(1 - s\bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s)}{Dr} \right] \tag{7.87}
\]

\[
\bar{D}(z, s) = \alpha z \left\{ \frac{1 - \bar{F}(b)}{b} \right\} \left\{ \frac{1 - \bar{B}(a)}{a} \right\} \left[ \frac{(1 - s\bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s)}{Dr} \right] \tag{7.88}
\]

\[
\bar{R}(z, s) = \alpha z\bar{F}(b) \left\{ \frac{1 - \bar{G}(b)}{b} \right\} \left\{ \frac{1 - \bar{B}(a)}{a} \right\} \left[ \frac{(1 - s\bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s)}{Dr} \right] \tag{7.89}
\]

Thus \( \bar{P}^{(1)}(z, s), \bar{P}^{(2)}(z, s), \bar{V}^{(1)}(z, s), \bar{V}^{(2)}(z, s) \bar{D}(z, s) \) and \( \bar{R}(z, s) \) are
completely determined from equations (7.84) to (7.89) which completes the proof of the theorem.

7.5 The steady state results

In this section, we shall derive the steady state probability distribution for our queueing model. These probabilities are obtained by suppressing the argument \( t \) wherever it appears in the time-dependent analysis. This can be obtained by applying the Tauberian property,

\[
\lim_{s \to 0} s \bar{f}(s) = \lim_{t \to \infty} f(t)
\]

In order to determine \( \bar{P}^{(1)}(z, s) \), \( \bar{P}^{(2)}(z, s) \), \( \bar{V}^{(1)}(z, s) \), \( \bar{V}^{(2)}(z, s) \) \( \bar{D}(z, s) \) and \( \bar{R}(z, s) \) completely, we have yet to determine the unknown \( Q \) which appears in the numerators of the right hand sides of equations (7.84) to (7.89). For that purpose, we shall use the normalizing condition

\[
P^{(1)}(1) + P^{(2)}(1) + V^{(1)}(1) + V^{(2)}(1) + D(1) + R(1) + Q = 1
\]

The steady state probabilities for an \( M^{(X)}/G/1 \) queue with two stage heterogeneous service, random breakdown, delayed repair and extended server vacation with Bernoulli schedule are given by

\[
P^{(1)}(1) = \frac{\lambda E(I) [1 - \bar{B}_1(\alpha)] Q}{\alpha dr_1}
\]

\[
P^{(2)}(1) = \frac{\lambda E(I) \bar{B}_1(\alpha) [1 - \bar{B}_2(\alpha)] Q}{\alpha dr_1}
\]

\[
V^{(1)}(1) = \frac{\lambda p E(I) \bar{B}_1(\alpha) \bar{B}_2(\alpha) E(V) Q}{dr_1}
\]

\[
V^{(2)}(1) = \frac{\lambda pr E(I) \bar{B}_1(\alpha) \bar{B}_2(\alpha) E(eV) Q}{dr_1}
\]
\[ D(1) = \frac{\lambda E(I)(1 - \bar{B}_1(\alpha)\bar{B}_2(\alpha))E(D)Q}{dr_1} \]
\[ R(1) = \frac{\lambda E(I)(1 - \bar{B}_1(\alpha)\bar{B}_2(\alpha))E(R)Q}{dr_1} \]

where

\[ dr_1 = -\lambda p\bar{B}_1(\alpha)\bar{B}_2(\alpha)E(I)[E(V) + rE(eV)] \]
\[ -\frac{\lambda E(I)}{\alpha}(1 - \bar{B}_1(\alpha)\bar{B}_2(\alpha))(\alpha + \lambda E(I)) \]
\[ -\lambda E(I)(1 - \bar{B}_1(\alpha)\bar{B}_2(\alpha))(E(D) + E(R)) \]

\( P^{(1)}(1), \ P^{(2)}(1), \ V^{(1)}(1), \ V^{(2)}(1), \ D(1), \ R(1) \) and \( Q \) are the steady state probabilities that the server is providing first stage of service, second stage of service, server under phase one vacation, extended vacation, delay time, repair time and server under idle respectively without regard to the number of customers in the system.

Thus multiplying both sides of equations (7.84) to (7.89) by \( s \), taking limit as \( s \to 0 \), applying Tauberian property and simplifying, we obtain

\[ P^{(1)}(z) = \frac{[\bar{B}_1(f_1(z)) - 1](f_2(z))Q}{f_1(z)dr} \quad (7.90) \]
\[ P^{(2)}(z) = \frac{f_2(z)\bar{B}_1(f_1(z))\bar{B}_2(f_1(z)) - 1]Q}{f_1(z)dr} \quad (7.91) \]
\[ V^{(1)}(z) = \frac{p\bar{B}_1(f_1(z))\bar{B}_2(f_1(z))\bar{V}_1(f_2(z)) - 1]Q}{dr} \quad (7.92) \]
\[ V^{(2)}(z) = \frac{pf_1(z)\bar{B}_2(f_1(z))\bar{V}_1(f_2(z))\bar{V}_2(f_2(z)) - 1]Q}{dr} \quad (7.93) \]
\[ D(z) = \frac{\alpha z[\bar{F}(f_2(z)) - 1][1 - \bar{B}_1(f_1(z))\bar{B}_2(f_1(z))]Q}{f_1(z)dr} \quad (7.94) \]
\[ R(z) = \frac{\alpha z\bar{F}(f_2(z))\bar{G}(f_2(z)) - 1][1 - \bar{B}_1(f_1(z))\bar{B}_2(f_1(z))]Q}{f_1(z)dr} \quad (7.95) \]
where

\[
    dr = z - \bar{B}_1(f_1(z))\bar{B}_2(f_1(z))[1 - p + p\bar{V}_1(f_2(z))(1 - r + r\bar{V}_2(f_2(z)))]
\]

\[
    - \left(\frac{\alpha z}{f_1(z)}\bar{F}(f_2(z))\bar{G}(f_2(z))[1 - \bar{B}_1(f_1(z))\bar{B}_2(f_1(z))]ight),
\]

\[
f_1(z) = \lambda - \lambda C(z) + \alpha \quad \text{and} \quad f_2(z) = \lambda - \lambda C(z).
\]

Let \(W_q(z)\) denote the probability generating function of the queue size irrespective of the state of the system. Then adding equations (7.90) to (7.95), we obtain

\[
    W_q(z) = P^{(1)}(z) + P^{(2)}(z) + V^{(1)}(z) + V^{(2)}(z) + D(z) + R(z)
\]

\[
    W_q(z) = \frac{[\bar{B}_1(f_1(z)) - 1](f_2(z))Q}{f_1(z)dr} + \frac{(f_2(z))\bar{B}_1(f_1(z))[\bar{B}_2(f_1(z)) - 1]Q}{f_1(z)dr} + \frac{p\bar{B}_1(f_1(z))\bar{B}_2(f_2(z))\bar{V}_1(f_2(z)) - 1]Q}{dr} + \frac{p\alpha\bar{B}_1(f_1(z))\bar{B}_2(f_2(z))\bar{V}_1(f_2(z)) - 1]Q}{dr} + \frac{\alpha z[\bar{F}(f_2(z)) - 1][1 - \bar{B}_1(f_1(z))\bar{B}_2(f_1(z))]Q}{f_1(z)dr} + \frac{\alpha z\bar{F}(f_2(z))[\bar{G}(f_2(z)) - 1][1 - \bar{B}_1(f_1(z))\bar{B}_2(f_1(z))]Q}{f_1(z)dr}
\]

we see that for \(z = 1\), \(W_q(z)\) is indeterminate of the form 0/0. Therefore, we apply L’Hopital’s rule and on simplifying, we obtain

\[
    W_q(1) = \frac{\lambda E(I)Q[1 - \bar{B}_1(\alpha)\bar{B}_2(\alpha))N + p\alpha\bar{B}_1(\alpha)\bar{B}_2(\alpha)M]}{-\lambda E(I)[1 - B_1(\alpha)B_2(\alpha)]N - \lambda p\alpha E(I)B_1(\alpha)B_2(\alpha)M + \alpha B_1(\alpha)B_2(\alpha)}
\]
where

\[ N = 1 + \alpha (E(D) + E(R)) \]
\[ M = E(V) + rE(eV), \]
\[ C(1) = 1, \]
\[ C'(1) = E(I) \]

is mean batch size of the arriving customers, \(-\bar{V}_1'(0) = E(V)\) the mean first phase of vacation time, \(-\bar{V}_2'(0) = E(eV)\) the mean extended vacation time, \(-\bar{F}'(0) = E(D)\) the mean delay time and \(-\bar{G}'(0) = E(R)\) the mean repair time.

Therefore adding \(Q\) to the above equation and equating to 1, simplifying, we get

\[ Q = 1 - \rho \]  

(7.97)

and hence the utilization factor \(\rho\) of the system is given by

\[ \rho = \lambda E(I) \left[ \frac{1}{\alpha B_1(\alpha)B_2(\alpha)} + \frac{(E(D) + (ER))}{B_1(\alpha)B_2(\alpha)} - \frac{1}{\alpha} - E(D) - E(R) + pM \right] \]  

(7.98)

where \(\rho < 1\) is the stability condition under which the steady state exists. Equation (7.97) gives the probability that the server is idle.

Substituting \(Q\) from (7.97) into (7.96), we have completely and explicitly determined \(W_q(z)\), the probability generating function of the queue size.

### 7.6 The average queue size and the average waiting time

Let \(L_q\) denote the mean number of customers in the queue under the steady state. Then

\[ L_q = \frac{d}{dz} W_q(z) \quad \text{at } z = 1 \]
since this formula gives indeterminate of the form $0/0$, then we write $W_q(z)$ given in (7.96) as

$$W_q(z) = \frac{N(z)}{D(z)} Q$$

where

$$N(z) = - [1 - \bar{B}_1(f_1(z))\bar{B}_2(f_1(z))][f_2(z)$$

$$+ \alpha z(1 - \bar{F}(f_2(z))\bar{G}(f_2(z)))] - pf_1(z)\bar{B}_1(f_1(z))\bar{B}_2(f_1(z))$$

$$\times [1 - \bar{V}_1(f_2(z))(1 - r + r\bar{V}_2(f_2(z)))]$$

$$D(z) = f_1(z)[z - \bar{B}_1(f_1(z))\bar{B}_2(f_1(z))(1 - p + p\bar{V}_1(f_2(z))(1 - r + r\bar{V}_2(f_2(z))))$$

$$- \alpha z\bar{F}(f_2(z))\bar{G}(f_2(z))[1 - \bar{B}_1(f_1(z))\bar{B}_2(f_1(z))]$$

$$N'(z) = [\bar{B}_1'(f_1(z))f_1'(z)\bar{B}_2(f_1(z)) + \bar{B}_1(f_1(z))\bar{B}_2'(f_1(z))f_1'(z)]$$

$$\times [f_2(z) + \alpha z(1 - \bar{F}(f_2(z))\bar{G}(f_2(z)))] - [1 - \bar{B}_1(f_1(z))\bar{B}_2(f_1(z))]$$

$$\times [f_2'(z) + \alpha(1 - \bar{F}(f_2(z))\bar{G}(f_2(z)))]$$

$$- \alpha z(\bar{F}'(f_2(z))f_2'(z)\bar{G}(f_2(z)) + \bar{F}(f_2(z))\bar{G}'(f_2(z))f_2'(z))]$$

$$- p[1 - \bar{V}_1(f_2(z))(1 - r + r\bar{V}_2(f_2(z)))]$$

$$\times [f_1'(z)]\bar{B}_1(f_1(z))\bar{B}_2(f_1(z))$$

$$+ f_1(z)\bar{B}_1'(f_1(z))f_1'(z)\bar{B}_2(f_1(z)) + f_1(z)\bar{B}_1(f_1(z))\bar{B}_2'(f_1(z))f_1'(z)]$$

$$+ pf_1(z)\bar{B}_1(f_1(z))\bar{B}_2(f_1(z))[\bar{V}_1'(f_2(z))f_2'(z)(1 - r + r\bar{V}_2(f_2(z))$$

$$+ \bar{V}_1(f_2(z))r\bar{V}_2'(f_2(z))f_2'(z)]$$

$$D'(z) = f_1'(z)[z - \bar{B}_1(f_1(z))\bar{B}_2(f_1(z))(1 - p + p\bar{V}_1(f_2(z))$$

$$(1 - r + r\bar{V}_2(f_2(z)))] + f_1(z)[1 - (\bar{B}_1'(f_1(z))f_1'(z)\bar{B}_2(f_1(z))$$

$$+ \bar{B}_1(f_1(z))\bar{B}_2'(f_1(z))f_1'(z))(1 - p + p\bar{V}_1(f_2(z))$$

$$\times (1 - r + r\bar{V}_2(f_2(z)))] - \bar{B}_1(f_1(z))\bar{B}_2(f_1(z))$$

$$\times (p\bar{V}_1'(f_2(z))f_2'(z)(1 - r + r\bar{V}_2(f_2(z))))$$

$$+ p\bar{V}_1(f_2(z))r\bar{V}_2'(f_2(z))f_2'(z))]$$

$$- \alpha \bar{F}(f_2(z))\bar{G}(f_2(z))[1 - \bar{B}_1(f_1(z))\bar{B}_2(f_1(z))]$$

$$- \alpha z[\bar{F}'(f_2(z))f_2'(z)\bar{G}(f_2(z)) + \bar{F}(f_2(z))\bar{G}'(f_2(z))f_2'(z)]$$
\times [1 - \bar{B}_1(f_1(z))\bar{B}_2(f_1(z))] + \alpha z \bar{F}(f_2(z))\bar{G}(f_2(z)) \\
\times [\bar{B}_1'(f_1(z))f_1'(z)\bar{B}_2(f_1(z)) + \bar{B}_1(f_1(z))\bar{B}_2'(f_1(z))f_1'(z)]

Then, we use

\begin{align*}
L_q &= \lim_{z \to 1} \frac{d}{dz} W_q(z) \\
&= \lim_{z \to 1} \left[ \frac{D'(z)N''(z) - N'(z)D''(z)}{2(D'(z))^2} \right] Q \\
&= \left[ \frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2} \right] Q \tag{7.99}
\end{align*}

where primes and double primes in the above equation denote the first and second derivatives at \( z = 1 \) respectively. Carrying out the derivative at \( z = 1 \), we have

\begin{align*}
N'(1) &= \lambda E(I)[(1 - \bar{B}_1(\alpha)\bar{B}_2(\alpha))N + p\alpha \bar{B}_1(\alpha)\bar{B}_2(\alpha)M] \tag{7.100} \\
N''(1) &= (\lambda E(I))^2[\alpha(1 - \bar{B}_1(\alpha)\bar{B}_2(\alpha))[E(D^2) + E(R^2) \\
&\quad + 2E(D)E(R)] + 2[\bar{B}_1'(\alpha)\bar{B}_2(\alpha) + \bar{B}_1(\alpha)\bar{B}_2'(\alpha)]N \\
&\quad + p\alpha \bar{B}_1(\alpha)\bar{B}_2(\alpha)[E(V^2) + 2rE(V)E(eV) + rE(eV^2)] \\
&\quad - 2pM(\bar{B}_1(\alpha)\bar{B}_2(\alpha) + \alpha \bar{B}_1'(\alpha)\bar{B}_2(\alpha) + \alpha \bar{B}_1(\alpha)\bar{B}_2'(\alpha))] \\
&\quad + \lambda E(I(I - 1))[(1 - \bar{B}_1(\alpha)\bar{B}_2(\alpha))N + p\alpha \bar{B}_1(\alpha)\bar{B}_2(\alpha)M] \\
&\quad + 2\alpha E(I)[1 - \bar{B}_1(\alpha)\bar{B}_2(\alpha)](E(D) + E(R)) \tag{7.101} \\
D'(1) &= -\lambda E(I)[1 - \bar{B}_1(\alpha)\bar{B}_2(\alpha)]N \\
&\quad - \lambda p\alpha E(I)\bar{B}_1(\alpha)\bar{B}_2(\alpha)M + \alpha \bar{B}_1(\alpha)\bar{B}_2(\alpha) \tag{7.102} \\
D''(1) &= - (\lambda E(I))^2\bar{B}_1(\alpha)\bar{B}_2(\alpha)[-2pM + \alpha pE(V^2) \\
&\quad + 2rE(V)E(eV) + rE(eV^2)] \\
&\quad - \alpha(E(D^2) + E(R^2) + 2E(D)E(R))] \\
&\quad - \alpha(\lambda E(I))^2[E(D^2) + E(R^2) + 2E(D)E(R)]
\end{align*}
\[ -\lambda E(I(I - 1))[(1 - \bar{B}_1(\alpha)\bar{B}_2(\alpha))N + \alpha p\bar{B}_1(\alpha)\bar{B}_2(\alpha)M] \\
- 2(\lambda E(I))^2[\bar{B}_1'(\alpha)\bar{B}_2(\alpha) + \bar{B}_1(\alpha)\bar{B}_2'(\alpha)](N - \alpha pM) \\
- 2\lambda E(I)[1 + \alpha(E(D) + E(R))(1 - \bar{B}_1(\alpha)\bar{B}_2(\alpha)) \\
+ \alpha(\bar{B}_1'(\alpha)\bar{B}_2(\alpha) + \bar{B}_1(\alpha)\bar{B}_2'(\alpha))] \] (7.103)

where \( E(V^2), E(R^2), E(D^2), E(eV^2) \) are the second moment of phase one vacation time, repair time, delay time and the extended vacation time respectively. \( E(I(I - 1)) \) is the second factorial moment of the batch size of arriving customers.

Then if we substitute the values \( N'(1), N''(1), D'(1), D''(1) \) from equations (7.100) to (7.103) into equation (7.99), we obtain \( L_q \) in the closed form.

Further, we find the average system size \( L \) by using Little’s formula. Thus we have

\[ L = L_q + \rho \] (7.104)

where \( L_q \) has been found by equation (7.99) and \( \rho \) is obtained from equation (7.98).

Let \( W_q \) and \( W \) denote the average waiting time in the queue and in the system respectively. Then by using Little’s formula, we obtain

\[ W_q = \frac{L_q}{\lambda} \]
\[ W = \frac{L}{\lambda} \]

where \( L_q \) and \( L \) have been found in equations (7.99) and (7.104).

### 7.7 Particular cases

**Case 1:** If there is no delay for repairs to start, no extended vacation and no
second stage service i.e, $E(D)=0$, $F(b) = 1$, $r = 0$ and $B_2(\alpha) = 1$. Then our model reduces to a single server $M^{[X]}/G/1$ queue with random breakdown, phase one vacation.

In this case, we find the idle probability $Q$, utilization factor $\rho$ and the average queue size $L_q$ can be simplified to the following expressions.

\[
Q = 1 - \rho \\
\rho = \lambda E(I) \left[ \frac{1}{\alpha B_1(\alpha)} + \frac{E(R)}{B_1(\alpha)} - \frac{1}{\alpha} - E(R) + pE(V) \right] \\
L_q = \left[ \frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2} \right] Q
\]

where

\[
N'(1) = \lambda E(I) [(1 - \tilde{B}_1(\alpha))(1 + \alpha E(R)) + p\alpha \tilde{B}_1(\alpha)E(V)] \\
N''(1) = (\lambda E(I))^2 [\alpha(1 - \tilde{B}_1(\alpha))E(R^2) + 2\tilde{B}_1'(\alpha)E(R) \\
+ \alpha p\tilde{B}_1(\alpha)E(V^2) - 2pE(V)(\tilde{B}_1(\alpha) + \alpha \tilde{B}_1'(\alpha))] \\
+ \lambda E(I(I-I-1))[\alpha(1 - \tilde{B}_1(\alpha))(1 + \alpha E(R)) \\
+ p\alpha \tilde{B}_1(\alpha)E(V)] + 2\lambda E(I)E(R)[1 - \tilde{B}_1(\alpha)]
\]

\[
D'(1) = - \lambda E(I)[1 - \tilde{B}_1(\alpha)][1 + \alpha E(R)] \\
- \lambda p\alpha E(I)\tilde{B}_1(\alpha)E(V) + \alpha \tilde{B}_1(\alpha)
\]

\[
D''(1) = - (\lambda E(I))^2 \tilde{B}_1(\alpha)[\alpha pE(V^2) - 2pE(V) - \alpha E(R^2)] \\
- \alpha(\lambda E(I))^2 E(R^2) - \lambda E(I(I-I-1)) \\
\times [(1 - \tilde{B}_1(\alpha))(1 + \alpha E(R)) + \alpha p\tilde{B}_1(\alpha)(E(V))] \\
- 2(\lambda E(I))^2 \tilde{B}_1'(\alpha)[1 - p\alpha(E(V)) + \alpha E(R)] \\
- 2\lambda E(I)[1 + \alpha E(R)(1 - \tilde{B}_1(\alpha)) + \alpha \tilde{B}_1'(\alpha)]
\]

In the above equations if repair time is exponentially distributed then the result coincide with the result given by Maraghi et al. (2009).
**Case 2:** If there is no extended vacation and no second stage service i.e., \( r = 0 \) and \( \bar{B}_2(\alpha) = 1 \). Then our model reduces to a single server \( M^{[X]}/G/1 \) queue with random breakdown, delayed repairs and phase one vacation.

In this case we find the idle probability \( Q \), utilization factor \( \rho \) and the average queue size \( L_q \) can be simplified to the following expressions.

\[
Q = \frac{1 - \rho}{\rho} = \frac{\lambda E(I) \left[ \frac{1}{\alpha \bar{B}_1(\alpha)} + \frac{(E(D) + (ER))}{\bar{B}_1(\alpha)} - \frac{1}{\alpha} - E(D) - E(R) + pE(V) \right]}{\frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2}} Q
\]

where

\[
N'(1) = \lambda E(I)[(1 - \bar{B}_1(\alpha))(1 + \alpha(E(D) + E(R)))] + p\alpha \bar{B}_1(\alpha)E(V)
\]
\[
N''(1) = (\lambda E(I))^2[\alpha(1 - \bar{B}_1(\alpha))(E(D^2) + E(R^2) + 2E(D)E(R))] + 2\bar{B}_1'(\alpha)[1 + \alpha(E(D) + E(R))]
\]
\[
D'(1) = - \lambda E(I)[1 - \bar{B}_1(\alpha)][1 + \alpha(E(D) + E(R))] - \lambda p\alpha E(I)\bar{B}_1(\alpha)E(V) + \alpha \bar{B}_1(\alpha)
\]
\[
D''(1) = - (\lambda E(I))^2 \bar{B}_1(\alpha)[-2pE(V) + \alpha pE(V^2)] - \alpha(E(D^2) + E(R^2) + 2E(D)E(R)) - \alpha(\lambda E(I))^2[E(D^2) + E(R^2) + 2E(D)E(R)] - \lambda E(I(I - 1))[1 - \bar{B}_1(\alpha))(1 + \alpha(E(D) + E(R)))]
\]
\[+ \alpha p \bar{B}_1(\alpha) E(V)] - 2(\lambda E(I))^2 \bar{B}_1'(\alpha) \]
\[\times [1 - p\alpha E(V) + \alpha (E(D) + E(R))]\]
\[- 2\lambda E(I)[1 + \alpha (E(D) + E(R))(1 - \bar{B}_1(\alpha)) + \alpha (\bar{B}_1'(\alpha))]\]

The above equations coincide with result given by Khalaf et al. (2010).

**Case 3:** If there is no delay for repairs to start, no extended vacation. Once the system breakdown, if its repairs start immediately and there is no delay time i.e, \(E(D) = 0\), \(\bar{F}(b) = 1\). Once the first phase of vacation finish, the server is ready to start the service and there is no extended vacation time i.e, \(r = 0\).

If \(E(I) = 1\), \(E(I(I - 1)) = 0\) then our model reduces to a single server \(M/G/1\) queue with two stage service with random breakdown, delayed repairs and phase one vacation.

In this case we find the idle probability \(Q\), utilization factor \(\rho\) and the average queue size \(L_q\) can be simplified to the following expressions.

\[Q = 1 - \rho\]
\[\rho = \lambda \left[ \frac{1}{\alpha \bar{B}_1(\alpha) \bar{B}_2(\alpha)} + \frac{E(R)}{\bar{B}_1(\alpha) \bar{B}_2(\alpha)} - \frac{1}{\alpha} - E(R) + pE(V) \right]\]
\[L_q = \left[ \frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2} \right] Q\]

where

\[N'(1) = \lambda [(1 - \bar{B}_1(\alpha) \bar{B}_2(\alpha))(1 + \alpha E(R)) \]
\[+ pa \bar{B}_1(\alpha) \bar{B}_2(\alpha) E(V)]\]
\[N''(1) = \lambda^2 [\alpha(1 - \bar{B}_1(\alpha) \bar{B}_2(\alpha)) E(R^2) \]
\[+ 2(\bar{B}_1'(\alpha) \bar{B}_2(\alpha) + \bar{B}_1(\alpha) \bar{B}_2'(\alpha))(1 + \alpha E(R))\]
\[ D'(1) = -\lambda [1 - \bar{B}_1(\alpha)\bar{B}_2(\alpha)](1 + \alpha E(R)) \]
\[ + \alpha \bar{B}_1(\alpha)\bar{B}_2(\alpha) - \lambda \alpha \bar{B}_1(\alpha)\bar{B}_2(\alpha)E(V) \]
\[ D''(1) = -\lambda^2 \bar{B}_1(\alpha)\bar{B}_2(\alpha)[\alpha p E(V^2) - 2p E(V) - \alpha E(R^2)] - \lambda^2 \alpha E(R^2) \]
\[ - 2\lambda^2 [\bar{B}_1(\alpha)\bar{B}_2(\alpha) + \bar{B}_1(\alpha)\bar{B}_2(\alpha)][1 - \alpha p E(V) + \alpha E(R)] \]
\[ - 2\lambda [1 + \alpha E(R)(1 - \bar{B}_1(\alpha)\bar{B}_2(\alpha))] \]
\[ + \alpha \bar{B}_1'(\alpha)\bar{B}_2(\alpha) + \alpha \bar{B}_1(\alpha)\bar{B}_2'(\alpha)] \]

If repair times are exponentially distributed and \( p = 1 \), then the above results coincide with Thangaraj and Vanitha (2010a).

### 7.8 Numerical results

To numerically illustrate the results obtained in this work, we consider that the service times, vacation times, delay times, extended vacation times and repair times are exponentially distributed. .

In order to see the effect of various parameters on server’s idle time \( Q \), utilization factor \( \rho \) and various other queue characteristics such as \( L, W, L_q, W_q \). We base our numerical example on the result found in case 3.

For this purpose in Table 7.1, we can choose the following values:
\( \mu_1 = 9, \mu_2 = 8, \alpha = 1, \beta = 4, \gamma = 7 \) and \( p = 0.2 \) while \( \lambda \) varies from 0.1 to 10 such that the stability condition is satisfied.

It clearly shows as long as increasing the arrival rate, the server’s idle time decreases while the utilization factor, the average queue size, system size, the average waiting time in the queue and the system of our queueing model are
all increases.

Table 7.1: Computed values of various queue characteristics

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$Q$</th>
<th>$\rho$</th>
<th>$L_q$</th>
<th>$L$</th>
<th>$W_q$</th>
<th>$W$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.9664</td>
<td>0.0336</td>
<td>0.0023</td>
<td>0.0358</td>
<td>0.0230</td>
<td>0.3587</td>
</tr>
<tr>
<td>0.2</td>
<td>0.9329</td>
<td>0.0671</td>
<td>0.0142</td>
<td>0.0813</td>
<td>0.0709</td>
<td>0.4067</td>
</tr>
<tr>
<td>0.3</td>
<td>0.8993</td>
<td>0.1007</td>
<td>0.0329</td>
<td>0.1336</td>
<td>0.1097</td>
<td>0.4455</td>
</tr>
<tr>
<td>0.4</td>
<td>0.8657</td>
<td>0.1343</td>
<td>0.0562</td>
<td>0.1905</td>
<td>0.1406</td>
<td>0.4763</td>
</tr>
<tr>
<td>0.5</td>
<td>0.8321</td>
<td>0.1679</td>
<td>0.0824</td>
<td>0.2503</td>
<td>0.1648</td>
<td>0.5006</td>
</tr>
<tr>
<td>0.6</td>
<td>0.7986</td>
<td>0.2014</td>
<td>0.1100</td>
<td>0.3114</td>
<td>0.1833</td>
<td>0.5191</td>
</tr>
<tr>
<td>0.7</td>
<td>0.7650</td>
<td>0.2350</td>
<td>0.1378</td>
<td>0.3728</td>
<td>0.1969</td>
<td>0.5326</td>
</tr>
<tr>
<td>0.8</td>
<td>0.7314</td>
<td>0.2686</td>
<td>0.1650</td>
<td>0.4336</td>
<td>0.2062</td>
<td>0.5419</td>
</tr>
<tr>
<td>0.9</td>
<td>0.6979</td>
<td>0.3021</td>
<td>0.1908</td>
<td>0.4929</td>
<td>0.2119</td>
<td>0.5477</td>
</tr>
<tr>
<td>1.0</td>
<td>0.6643</td>
<td>0.3357</td>
<td>0.2145</td>
<td>0.5502</td>
<td>0.2145</td>
<td>0.5502</td>
</tr>
</tbody>
</table>

Table 7.2: Computed values of various queue characteristics

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$Q$</th>
<th>$\rho$</th>
<th>$L_q$</th>
<th>$L$</th>
<th>$W_q$</th>
<th>$W$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.8880</td>
<td>0.1120</td>
<td>0.0791</td>
<td>0.1912</td>
<td>0.2638</td>
<td>0.6373</td>
</tr>
<tr>
<td>2</td>
<td>0.9030</td>
<td>0.0970</td>
<td>0.0749</td>
<td>0.1719</td>
<td>0.2498</td>
<td>0.5733</td>
</tr>
<tr>
<td>3</td>
<td>0.9080</td>
<td>0.0920</td>
<td>0.0727</td>
<td>0.1647</td>
<td>0.2423</td>
<td>0.5490</td>
</tr>
<tr>
<td>4</td>
<td>0.9105</td>
<td>0.0875</td>
<td>0.0714</td>
<td>0.1609</td>
<td>0.2379</td>
<td>0.5363</td>
</tr>
<tr>
<td>5</td>
<td>0.9120</td>
<td>0.0880</td>
<td>0.0705</td>
<td>0.1585</td>
<td>0.2350</td>
<td>0.5285</td>
</tr>
<tr>
<td>6</td>
<td>0.9130</td>
<td>0.0870</td>
<td>0.0699</td>
<td>0.1569</td>
<td>0.2330</td>
<td>0.5232</td>
</tr>
<tr>
<td>7</td>
<td>0.9137</td>
<td>0.0863</td>
<td>0.0694</td>
<td>0.1558</td>
<td>0.2316</td>
<td>0.5193</td>
</tr>
<tr>
<td>8</td>
<td>0.9142</td>
<td>0.0858</td>
<td>0.0691</td>
<td>0.1549</td>
<td>0.2305</td>
<td>0.5165</td>
</tr>
<tr>
<td>9</td>
<td>0.9146</td>
<td>0.0854</td>
<td>0.0689</td>
<td>0.1542</td>
<td>0.2297</td>
<td>0.5142</td>
</tr>
<tr>
<td>10</td>
<td>0.9150</td>
<td>0.0850</td>
<td>0.0686</td>
<td>0.1537</td>
<td>0.2289</td>
<td>0.5124</td>
</tr>
</tbody>
</table>

In Table 7.2, we can choose the following arbitrary values: $\mu_1 = 9$, $\mu_2 = 8$, $\alpha = 0.5$, $\lambda = 0.3$, $\gamma = 4$ and $p = 0.1$ while $\beta$ varies from 1 to 10 such that the stability condition is satisfied.

It clearly shows as long as increasing the vacation rate, the server’s idle time increases while the utilization factor, average queue size, system size, average waiting time in the queue and system of our queueing model are all decreases.