Historical Background

The fundamental idea of a Finsler space may be traced back to the famous lecture of Riemann: "Über die Hypothesen, welche der Geometrie zugrunde liegen." In this memoir of 1854 Riemann discusses various possibilities by means of which an n-dimensional manifold may be endowed with a metric, and pays particular attention to a metric defined by the positive square root of a positive definite quadratic differential form. Thus the foundations of Riemannian geometry are laid; nevertheless, it is also suggested that the positive fourth root of a fourth order differential form might serve as a metric function. These functions have three properties in common: they are positive, homogeneous of the first degree in the differentials, and are also convex in the latter. It would seem natural, therefore, to introduce a further generalization to the effect that the distance $ds$ between two neighbouring points represented by the coordinates $x$ and $x + dx$ be defined by one functions $F(x, dx)$:

$$ds = F(x, dx),$$

where this function satisfies these three properties.
It is remarkable that the first systematic study of manifolds endowed with such a metric was delayed by more than 60 years. It was an investigation of this kind which formed the subject matter of the thesis of Finsler in 1918, after whom such spaces were eventually named. It would appear that this new impulse was derived almost directly from the calculus of variations, with particular reference to the new geometrical background which was introduced by Caratheodory in connection with problems in parametric form. The kernel of these methods is the so called indicatrix, while the property of convexity is of fundamental importance with regard to the necessary conditions for a minimum in the calculus of variations. In fact, the remarkable affinity between some aspects of differential geometry and the calculus of variations had been noticed some years prior to the publication of Finsler's thesis, in particular by Bliss, Landsberg and Blaschke. Both Bliss and Landsberg introduced (distinct) definitions of angle in terms of invariants of a parametric problem in the calculus of variations, while an analytic study of such invariants had been made by E. Noether and A. Underhill. Yet the geometrical theories of Bliss and Landsberg were developed against an Euclidean background and cannot, therefore, be regarded as fulfilling the true objectives of the generalisation of Riemann's proposal.
Clearly, Finsler's thesis must be regarded as the first step in this direction.

A few years later, however, the general development took a curious turn away from the basic aspects and methods of the theory as developed by Finsler. The latter did not make use of the tensor calculus, being guided in principle by the notions of the calculus of variations; and in 1925 the methods of the tensor calculus were applied to the theory independently but almost simultaneously by Synge, Taylor and Berwald. It was found that the second derivatives of the half of the square of $F(x, dx) \left( \frac{1}{2} F^2 (x^l, dx^l) \right)$ with respect to the differentials, $dx$, served admirably as components of a metric tensor in analogy with Riemannian geometry, and from the differential equations of the geodesics connection coefficients could be derived by means of which a generalization of Levi-Civita's parallel displacement could be defined. While the corresponding covariant derivatives as introduced by Synge and Taylor coincide, the theory of Berwald shows a marked difference, in the sense that in his geometry the lemma of Ricci (which in Riemannian geometry implies the vanishing of the covariant derivative of the metric tensor) is no longer valid. Nevertheless, Berwald continued to develop his theory with particular reference to the theory of curvature as well as to two-dimensional spaces. The significance of his work was
enhanced by the advent of the general geometry of paths (a
generalization of the so-called Non-Riemannian geometry) due
to Douglas and Knebelman, for the initial approach of
Berwald was such as to establish a close affinity between
these branches of metric and non-metric differential geometry.

Again, the theory took a new and unexpected turn in 1934
when E. Cartan published his tract on Finsler spaces. He
showed that it was indeed possible to define connection
coefficients and hence a covariant derivative such that the
preservation of Ricci's lemma was ensured. On this basis
Cartan developed a theory of curvature, and practically all
subsequent investigations concerning the geometry of Finsler
spaces were dominated by this approach. Several
Mathematicians expressed the opinion that the theory had thus
attained its final form. To a certain extent this was correct, but
not altogether so, as we shall now indicate.

The above-mentioned theories make use of a certain device
which basically involves the consideration of a space whose
elements are not the points of the underlying manifold, but the
line-elements of the later, which form a (2n-1)-dimensional
variety. This facilitates the introduction of what Cartan calls
the "Euclidean connection", which, by means of certain
postulates, may be derived uniquely from the fundamental
metric function $F(x, dx)$. The method also depends on the introduction of a so-called "element of support", namely, that at each point a previously assigned direction must be given, which then serves as directional argument in all functions depending on direction as well as position. Thus, for instance, the length of a vector and the vector obtained from it by an infinitesimal parallel displacement depend on the arbitrary choice of the element of support. It is this device which led to the development of Finsler geometry in terms of direct generalizations of the methods of Riemannian geometry.

It was felt, however, that the introduction of the element of support was undesirable from a geometrical point of view, while the natural link with the calculus of variations was seriously weakened. This view was expressed independently by several authors, in particular by Wagner, Busemann and the present writer. It was emphasized that the natural local metric of a Finsler space is a Minkowskian one, and that the arbitrary imposition of a Euclidean metric would to some extent obscure some of the most interesting characteristics of the Finsler space. Thus at the beginning of the present decade further theories were put forward. The rejection of the use of the element of support, however desirable from a geometrical point of view, led to new difficulties: for instance, the natural orthogonality between two vectors is not in general symmetric,
while the analytical difficulties are certainly enhanced, particularly since Ricci's lemma cannot be generalized as before. Fortunately, from the point of view of differential invariants, there exist marked similarities between all these theories, which is a perfectly natural phenomenon and could have been expected. It is in the application and in the interpretation of these invariants that the two points of view appear to be irreconcilable.

Essentially, a Finsler manifold is a manifold $M$ where each tangent space is equipped with a Minkowski norm, that is, a norm that is not necessarily induced by an inner product. (Here, a Minkowski norm has no relation to indefinite inner products.) This norm also induces a canonical inner product. However, in sharp contrast to the Riemannian case, these Finsler-inner products are not parameterized by points of $M$, but by directions in $TM$. Thus one can think of a Finsler manifold as a space where the inner product does not only depend on where you are, but also in which direction you are looking. Despite this quite large step away from Riemannian geometry, Finsler geometry contains analogues for many of the natural objects in Riemannian geometry.

For example, length, geodesics, curvature, connections, covariant derivative, and structure equations all generalize. However, normal coordinates do not [Run59]. Let us also
point out that in Finsler geometry the unit spheres do not need to be ellipsoids. Finsler geometry is named after Paul Finsler who studied it in his doctoral thesis in 1917. Presently Finsler geometry has found an abundance of applications in both physics and practical applications [101, 102].