CHAPTER-IV

AFFINE, CONFORMAL AND SPECIAL
PROJECTIVE MOTION IN $\mathbb{R}^n - F_n^*$
1. INTRODUCTION:

Ruse, H. S. [7] and Walker, A. G. [12] has studied the recurrent properties of $K_n^*$ space. Wong, A. C. [11] has also obtained various properties of a $K_n^*$ space. Takano, K. [10] has studied the existence of affine motion in a non-Riemannian $K^*$ space and has obtained several results of significance. He has also studied the existence of projective motion in a Riemannian space with bi-recurrent curvature. Yano, K. [13] has defined projective conformal transformation in a Riemannian space. Pande, H. D. and Kumar, A. [5] have also discussed special infinitesimal projective transformation in a Finsler space and have obtained certain theorems therein.

In the chapter under consideration, we have discussed the existence of affine and $R^+$-conformal motions in a $R^+$-recurrent Finsler space. For this purpose, we consider an infinitesimal point transformation \( \bar{x}' = x' + v'(x)dt \), where \( v'(x) \) is a contravariant vector field and \( dt \) is an infinitesimal point constant, such a point transformation considered at each point of \( R^+ \)-recurrent space is called a \( R^+ \)-affine motion iff \( \mathcal{L}_v \Gamma'_{jk} = 0 \), where \( \mathcal{L}_v \) denotes the well known operator of Lie-differentiation. With such as our assumptions we have also derived the complete conditions for the vanishing of \( \mathcal{L}_v R_{jk}' \), where \( R_{jk}' \) is the curvature tensor type entity with respect to the non-symmetric connection \( \Gamma'_{jk} \).

2. EXISTENCE OF AFFINE MOTION IN A R^+-RECURRENT SPACE:

We consider the existence of \( R^+ \)-recurrent space, for this purpose we consider an infinitesimal point transformation

\[
(2.1) \quad \bar{x}' = x' + v'(x)dt,
\]
where $v'(x)$ is an arbitrary contravariant vector field and $dt$ is an
infinitesimal point constant. This point transformation when
considered, at each point in $R^r$-recurrent space, is called a $R^r$-affine
motion when and only when

$$ (2.2) \, \mathcal{L}_v \Gamma^{i}_{jk} = 0, $$

where $\mathcal{L}_v$ denotes the operator of Lie-differentiation and $\Gamma^{i}_{jk}$ is the
non-symmetric connection coefficient.

By virtue of the above point transformation, the Lie-derivative of
any arbitrary tensor field $T^i_j(x, \dot{x})$ and the connection coefficient
$\Gamma^i_{jk}(x, \dot{x})(\neq \Gamma^i_{kj})$ in view of the $\Theta$- covariant derivative $[I-(11.8)]$ are
given by (Gupta [3])

$$ (2.3) \, \mathcal{L}_v \, T^i_j(x, \dot{x}) = T^i_j \, ^*|_k \, v^k + (\hat{\partial}^i_k \, T^i_j)(v^k \, ^*|_h) \dot{x}^h - \\
- T^i_j \, ^*|_k + T^i_j(v^k \, ^*|_j), $$

and

$$ (2.4) \, \mathcal{L}_v \, \Gamma^i_{jk}(x, \dot{x}) = (v^i \, ^*|_j) \, ^|_k + (\hat{\partial}^i_j \, \Gamma^i_{jk}) \, v^i \, ^*|_h \, \dot{x}^h + v^h \, R^i_{ijk}, $$

where $R^i_{ijk}$ has been given in $[I-(9.29)].$

Between the operators $\mathcal{L}_v, \hat{\partial}^i$ and $^*|_k$, we have the following
commutation formulae (Gupta [13])

$$ (2.5) \, \mathcal{L}_v \left( \hat{\partial}^i_k \, T^i_j \right) - \hat{\partial}^i_k \left( \mathcal{L}_v \, T^i_j \right) = 0, $$

$$ (2.6) \, \mathcal{L}_v \left( T^i_j \, ^|_k \right) - \left( \mathcal{L}_v \, T^i_j \right) \, ^|_k = T^i_j \, \mathcal{L}_v \, \Gamma^i_{hk} - T^i_h \, \mathcal{L}_v \, \Gamma^{i}_{jk} - \\
- (\hat{\partial}^i_h \, T^i_j)(\mathcal{L}_v \, \Gamma^{i}_{sh}) \dot{x}^s, $$

and

$$ (2.7) \, \left( \mathcal{L}_v \, \Gamma^i_{hk} \right) \, ^|_k - \left( \mathcal{L}_v \, \Gamma^i_{hk} \right) \, ^|_j = \mathcal{L}_v \, R^i_{hk} + \dot{x}^i \, \Gamma^i_{rhj} \, \mathcal{L}_v \, \Gamma^i_{jk} - \\
- \dot{x}^i \, \Gamma^i_{rhk} \, \mathcal{L}_v \, \Gamma^i_{lj} + N^i_{lj} \, \mathcal{L}_v \, \Gamma^i_{qr}, $$
where $R^t_{bh}$ and $N^t_{kj}$ have been given in [I-(9.29)].

We now have given the following definitions which we shall use in the later discussion.

**DEFINITION (2.1):**

A Finsler space $F^*_n$ equipped with non-symmetric connection

\[
\Gamma^j_{jk} \neq \Gamma^j_{kj}
\]

said to be a $R^+$-recurrent $F^*_n$ if

\[
(2.8) \quad R^t_{jkh} \cdot e = \lambda_t R^t_{jkh},
\]

where $\lambda_t(x)$ is a non-null recurrence vector.

**DEFINITION (2.2):**

An $n$-dimensional $F^*_n$ equipped with non-symmetric connection is said to be an affinely connected $R^+$-space if

\[
(2.9) \quad \hat{\delta}_t \Gamma^j_{jk} = 0
\]

holds.

Using the equations (2.7) and (2.2), we get the Lie-derivative of $R^t_{jkh} (x, \dot{x})$ as

\[
(2.10) \quad \mathcal{L}_t R^t_{jkh} = 0.
\]

Taking the Lie-derivative of both sides of (2.8) and thereafter using the equations (2.2), (2.6) and (2.10), we get

\[
(2.11) \quad (\mathcal{L}_t, \lambda_t) R^t_{jkh} = 0.
\]

Consequently, if the $R^+$-recurrent $F^*_n$ is considered to be non flat, (a non flat $F^*_n$ characterised by $R^t_{jkh} \neq 0$) then we have

\[
(2.12) \quad (\mathcal{L}_t, \lambda_t) = 0.
\]

Thus, with the help of (2.12) we can say that the recurrence vector of a non flat $R^+$-recurrent $F^*_n$ must be a Lie-invariant. Next, we propose to study a $R^+$-recurrent $F^*_n$ admitting an infinitesimal point transformation (2.1) satisfying (2.12) brevity we shall now onwards call such a restricted as $R^+$-recurrent space.
3. **THE VANISHING OF $\mathcal{L}_x R_{jkh}^i (x, \dot{x})$:**

In this section, first of all we prove the following.

**Lemma (3.1):**

In a $R^+$-recurrent $F_n^{\star}$, if the recurrence vector $\lambda_{\star} (x)$ is a gradient one, we have $\lambda_{\star} v^\star = \text{constant}$.

**Proof:**

For brevity, we write $\alpha = \lambda_m v^m$, with the help of equations (2.3) and (2.12), we get

$$
(3.1) \mathcal{L}_x \lambda_m = \lambda_m + |s \ v^s + \lambda s v^s |m.
$$

If we now assume that $\lambda_m + |s = \lambda s + |m$ than if is obvious that $\alpha + |s = 0$, which will mean that $\alpha = \lambda_m v^m$ is a constant.

By virtue of (2.3), the Lie-derivative of $R_{jkh}^i (x, \dot{x})$ in view of (2.8) can be written as

$$
(3.2) \mathcal{L}_x R_{jkh}^i = \alpha R_{jkh}^i - R_{jkh}^i v^j + |p R_{pjh}^i v^p + |j R_{jpj}^i v^p + |k +
+ R_{jkp}^i v^p + |h + \left( \dot{\lambda}_p R_{jkh}^i \right) v^p + |m \dot{x}^m.
$$

The commutation formula [I-(11.1)] used for $R_{jkh}^i (x, \dot{x})$, we get

$$
(3.3) R_{jkh}^i + |m R_{jkh}^i + |mt = \left( \dot{\lambda}_p R_{jkh}^i \right) R_{qhm}^p \dot{x}^q + R_{jkh}^i R_{p/qm} -
- R_{pjh}^i R_{jkh}^i - R_{jpj}^i R_{p/km} +
+ R_{jkp}^i N_{p/qm}.
$$

(3.3) in view of the definition (2.8) given

$$
(3.4) \left( \lambda_{\star} + |m - \lambda_m + |l \right) R_{jkh}^i = \left( \dot{\lambda}_p R_{jkh}^i \right) R_{qhm}^p \dot{x}^q + R_{jkh}^i R_{p/qm} -
- R_{pjh}^i R_{jkh}^i - R_{jpj}^i R_{p/km} +
+ \lambda_p R_{jkh}^i N_{p/qm}.
$$

Now, we assume that $\alpha$ is not a constant, then from Lemma (3.2), we can find the following
(3.5) $\beta^m_l(x) = (\lambda_l^m + \lambda_m^l) \neq 0$.

Now, we consider a suitable non-symmetric tensor $\Psi^{ij}$ which satisfies that relation

(3.6) $R_{jkh}^{i} \Psi^{kh} = \psi_{i}^{|j}$. We now multiply (3.4) by $\Psi^{lm}$ and sum it over $\ell$ and $m$ and get

(3.7) $\beta_m^l \Psi^{lm} R_{jkh}^{i} = -\left( \hat{\delta}_l^p R_{jkh}^{i} \right) \psi_{i}^{|l} \hat{x}^q + R_{jkh}^{p} \psi_{i}^{|p} - R_{jkh}^{i} \psi_{i}^{|j} - R_{jkh}^{i} \psi_{i}^{|l} - R_{jkh}^{i} \psi_{i}^{|l} + dR_{jkh}^{i}$,

where

(3.8) $d = \lambda_p^m N_{ml}^p \Psi^{lm}$.

Comparing (3.7) with (3.2), we get

(3.9) $\xi \cdot R_{jkh}^{i} = \left( \alpha - \beta_m^l \Psi^{lm} + d \right) R_{jkh}^{i}$.

Form (3.9) it is almost clear that $\xi \cdot R_{jkh}^{i}$ vanishes if and only if

$\alpha = \beta_m^l \Psi^{lm} - d$.

With the help of (3.2) and (3.4), we can construct the following identity

(3.10) $\beta_m^l \xi \cdot R_{jkh}^{i} = R_{jkh}^{i} \left( \alpha R_{p(lm}^{j} - \beta_m^l \psi_{i}^{|p} \right) - R_{jkh}^{i} \left( \alpha R_{j(lm}^{k} - \beta_m^l \psi_{i}^{|k} \right) - R_{jkh}^{i} \left( \alpha R_{k(lm}^{p} - \beta_m^l \psi_{i}^{|p} \right) - R_{jkh}^{i} \left( \alpha R_{m(lm}^{p} - \beta_m^l \psi_{i}^{|p} \right) - \left( \hat{\delta}_l^p R_{jkh}^{i} \right) \left( \alpha R_{q(lm}^{p} - \beta_m^l \psi_{i}^{|q} \right) \hat{x}^q + \alpha \lambda_p^m R_{jkh}^{i} N_{ml}^p$.

Form (3.10) we conclude that if the skew part $N_{ml}^p$ of the non-symmetric connection $\Gamma_{jk}^i$ vanishes then $\xi \cdot R_{jkh}^{i} = 0$ if

(3.11) $\alpha R_{jkh}^{i} = \beta_{kh}^l \psi_{i}^{|l}$.
Whereas, if the skew part of the non-symmetric connection does not vanish then $\mathbf{R}_{jkh}^{i} = 0$ if

$$
(3.12) \quad (a) \quad \alpha R_{jkh}^{i} = \beta_{kh} v^{i} \mid_{j} \\

and \quad (b) \quad \alpha \lambda_{p} R_{jkh}^{i} N_{m}^{p} = 0,
$$

where $v^{i}$ does not mean a parallel vector.

We now give the following definition.

**DEFINITION (3.1):**

A $R^{+}$-recurrent $P_{n}^{*}$ satisfying $\lambda_{m} v^{m} \neq$ constant is called a special one of the first kind.

Now, we again come back to the case $\lambda_{m} v^{m} =$ constant of the foregoing Lemma (3.1) then (3.4) is replaced by

$$
(3.13) \quad -\left(\hat{\partial}_{\mu} R_{jkh}^{i}\right) R_{p}^{p} \sqrt{q_{(m}} R_{i}^{i} R_{f}^{p} - R_{p k h}^{i} R_{j h}^{f} R_{i}^{p} - R_{j k h}^{i} R_{f}^{p} R_{k}^{m} - \\
- R_{j k h}^{i} R_{n}^{p} + \lambda_{p} R_{j k h}^{i} N_{n}^{p} = 0.
$$

In view of the equation (3.6), multiplying (3.13) by $\Psi_{n}^{0}$ and summing over $\ell$ and $m$, we get

$$
(3.14) \quad -\left(\hat{\partial}_{\mu} R_{jkh}^{i}\right) v^{p} \mid_{q} \sqrt{q} + R_{j k h}^{i} v^{p} \mid_{p} - R_{p k h}^{i} v^{p} \mid_{j} - R_{j m}^{i} v^{p} \mid_{h} - \\
- R_{j k h}^{i} v^{p} \mid_{k} + d R_{j k h}^{i} = 0,
$$

where $d$ has been given by (3.8).

We now introduce (3.14) into the right hand side of (3.2) and get

$$
(3.15) \quad \mathbf{R}_{jkh}^{i} = (\alpha + d) R_{jkh}^{i},
$$

where $\alpha = \lambda_{m} v^{m}$ and $d$ has been given by (3.8).

With the help of (3.15), we conclude that if $\alpha + d = 0$ then

$$
\mathbf{R}_{jkh}^{i} = 0
$$
holds. we now give the following definition
DEFINITION (3.2):

When \( \alpha = \dot{\lambda}_m v^m = \text{constant holds good} \), a \( R^* \)-recurrent \( F^*_n \) is said to be a special one \( F^*_n \) of the second kind.

We now summaries all these results in the form of following theorems.

THEOREM (3.1):

In a special recurrent \( F^*_n \) of the first kind, if \( F^*_n \) satisfies (3.12a) and (3.12b) both then \( \mathcal{L}_v R^i_{jkh} = 0 \) holds good.

THEOREM (3.2):

In a special \( R^* \)-recurrent \( F^*_n \) of the first kind, if \( F^*_n \) satisfies (3.11) then \( \mathcal{L}_v R^i_{jkh} = 0 \) holds good only when the skew symmetric part \( N^p_{mci} \) of the symmetric connection \( \Gamma^p_{mci} \) vanishes.

THEOREM (3.3):

In a special \( R^* \)-recurrent \( F^*_n \) of the second kind, if \( \alpha + d = 0 \) then \( \mathcal{L}_v R^i_{jkh} = 0 \) hold identically where \( \alpha = \dot{\lambda}_m v^m \) and \( d \) has been given by (3.8).

THEOREM (3.4):

In a symmetry \( R^* \)-recurrent \( F^*_n \) of the second kind \( \mathcal{L}_v R^i_{jkh} = 0 \) holds if \( d = 0 \).

4. COMPLETE CONDITION:

We shall here find a necessary and sufficient condition for (3.11). In view of the assumption (2.12), we have

\[
(4.1) \quad \mathcal{L}_v \lambda_m = \lambda_m + \dot{v}^i v^i + \left( \dot{\lambda}_i v^i \right) - \lambda_4 v^i = 0.
\]

(4.1) by virtue of (3.5) reduces to

\[
(4.2) \quad \alpha_m v^i + \beta_m v^i = 0.
\]
In view of (2.3), the Lie-derivative of $\beta_{lm}(x)$ is given by

\[(4.3) \quad \mathcal{L}_p B_{lm} (x) = B_{lm} \mid_p v^p + B_{pm} v^p \mid_p + B_{lp} v^p \mid_m.
\]

(4.3) by virtue of (2.6), (2.12) and (3.5) reduce into the following form

\[(4.4) \quad \mathcal{L}_p B_{lm} = 0.
\]

Differentiating (3.4) covariantly with respect to $x^n$ and using (2.8) and (3.4), we get

\[(4.5) \quad B_{lm} \mid_n R_{jkh}^i = \lambda_n \left[ R_{jkh}^i R_{p/m}^p - R_{p/kh}^i R_{j/l}^l - R_{j/kh}^i R_{m/l}^l - R_{j/kh}^i R_{p/m}^p - \right.
\]

\[\left. - \left( \partial_p R_{j/kh}^i \right) R_{q/m}^l \dot{X}^q \right] + \lambda_p \mid_n R_{j/kh}^i N_{m/l}^p + \lambda_{p} R_{j/kh}^i N_{m/l}^p,
\]

If we now assume that the recurrence vector and skew symmetric part of the non-symmetric connection is also 1-recurrent with the same recurrence vector then from (4.5) we get

\[(4.6) \quad \beta_{lm} \mid_n R_{jkh}^i = \lambda_n \left[ R_{jkh}^i R_{p/m}^p - R_{p/kh}^i R_{j/l}^l - R_{j/kh}^i R_{m/l}^l - R_{j/kh}^i R_{p/m}^p - \right.
\]

\[\left. - \left( \partial_p R_{j/kh}^i \right) R_{q/m}^l \dot{X}^q + \lambda_p R_{j/kh}^i N_{m/l}^p + \lambda_{p} R_{j/kh}^i N_{m/l}^p \right].
\]

We now use (3.13) in (4.6) and get

\[(4.7) \quad B_{lm} \mid_n = \lambda_n \lambda_p N_{m/l}^p.
\]

Now after making use of (4.3), (4.4) and (4.7), we get

\[(4.8) \quad \alpha \lambda R_{m/l} N_{m/l}^p + B_{pm} v^p \mid_l + B_{lp} v^p \mid_m = 0.
\]

From (4.2), we have

\[(4.9) \quad \alpha \mid_{mn} = - \left( \beta_{ms} v^s \right) \mid_n.
\]

Interchanging the indices $m$ and $n$ in (4.9) and subtracting the equation thus obtained from (4.9) itself, we get

\[(4.10) \quad \alpha \mid_{mn} - \alpha \mid_{nm} = \left( \beta_{ms} v^s \right) \mid_m - \left( \beta_{ms} v^s \right) \mid_n.
\]

$\alpha$ - being a non-constant scalar function, therefore under this assumption (4.10) gives
\[(4.11) \quad B_{mn} + l_m v^s + B_{ms} + l_n v^s = B_{ns} v^s + l_m - B_{ms} v^s + l_n,\]

where we have written \(B_{ns} = -B_{sn}\).

Using (4.11) into the left hand side of (4.8) and then again using (4.7) in the result thus obtained, we get

\[(4.12) \quad v^\mu \left( \alpha_m N_{\alpha}^m + \lambda_i N_{\alpha}^m \right) - \alpha N_{\alpha}^m = 0.\]

Thus, we can state

**THEOREM (4.1):**

In a \(R^+\)-recurrent \(F^*_n\) the necessary and sufficient condition for the existence of (3.11) is given by (4.12).

5. CONFORMAL MOTION IN A \(R^+\)-RECURRENT \(F^*_n\):

If the infinitesimal point transformation (2.1) implies that the magnitude of the vectors defined in the same tangent space are proportional and the angle between the two directions is also the same with respect to the metrics then it will be called a conformal motion in \(F^*_n\). The variation of the non-symmetric connection \(\Gamma^i_{jk}(x, \dot{x}) \neq \Gamma^i_{jk}(x, \dot{x})\) under the infinitesimal point transformation (2.1) is \(\xi_i \Gamma^i_{jk}\) and that under the conformal change is \(\tilde{\Gamma}^i_{jk}\). The two transformations will coincide if the corresponding variations are the same.

The necessary and sufficient condition in order that the infinitesimal point transformation (2.1) may define of \(R^+\)-conformal motion is given by

\[(5.1) \quad \xi_i \Gamma^i_{jk} = \delta^i_j \epsilon_k + \delta^i_k \epsilon_j - \epsilon^i g_{jk},\]

where \(\epsilon^i(x)\) is a non-zero contravariant vector field and if satisfies the following relations

\[(5.2) \quad \epsilon^i = g^{jk} \epsilon_k,\]
(b) \( \dot{\epsilon}_k \dot{x}^k = 0 \).

Now, we shall consider the conditions under which a \( R^+ \)-conformal motion becomes \( R^+ \)-curvature collineation. In view of (5.1), the commutation formula (2.7) gives

\[
\begin{align*}
\mathcal{L}_x R^i_{jk} &= \delta^i_h \left( \epsilon_{j,|k} - \epsilon_{k,|j} \right) + \epsilon^i_j \left( g_{hk} + |j \right) - g_{hk} - \left( \Gamma^r_{ij} + \Gamma^r_{hk} \epsilon_j \right) \dot{x}^r + \\
&\quad - \delta^i_{k,|j} - \epsilon^i_{k,|j} g_{kj} + \epsilon^i_{|j} g_{hk} - \left( \Gamma^r_{ij} + \Gamma^r_{hk} \epsilon_j \right) \dot{x}^r + \\
&\quad + \dot{x}^i \left( \Gamma^r_{ijk} g_{ij} + \Gamma^r_{ijk} g_{ik} \right) N^r_{kj} \epsilon_r - \delta^i_{h,|j} - N^i_{kj} \epsilon_r + \epsilon^i g_{hr} N^r_{kj},
\end{align*}
\]

where we have made use of (5.2b).

At this stage, we assume that the space is affinely connected then (5.3) with the help of (2.9) gives

\[
\begin{align*}
\mathcal{L}_x R^i_{jk} &= \delta^i_h \left( \epsilon_{j,|k} - \epsilon_{k,|j} \right) + \epsilon^i_j \left( g_{hk} + |j \right) - g_{hk} + \epsilon^i_{|j} g_{hk} - \\
&\quad + \epsilon^i_\epsilon r g_{hr} N^r_{kj},
\end{align*}
\]

We now give the following definitions

**DEFINITION (5.1):**

The infinitesimal point transformation (2.1) defines a \( R^+ \)-curvature collineation provided that the space \( F_{n^*} \) admits a vector field \( \nu^i(x) \) such that

\[
\mathcal{L}_x R^i_{jk} = 0.
\]

**DEFINITION (5.2):**

The infinitesimal transformation (2.1) defines a \( R^+ \)-Ricci collineation if these exists a vector field \( \nu^i(x) \) such that

\[
\mathcal{L}_x R_{jk} = 0 \quad \text{where} \quad R_{jk} = R^i_{jk}.
\]

We now make use of (5.5) in (5.4) and get
(5.7) $\delta^i_j(\epsilon_{j+1} \epsilon_{k+1} + \epsilon_{j'}(g_{hk} + g_{hj} + k) + \delta^i_j \epsilon_{j+k} -$
$- \delta^i_k \epsilon_{j+k} + \epsilon_{j+k} g_{hj} + \epsilon_{j+k} g_{hk} - N'_{kj} \epsilon_r \delta^i_r - N'_{i} \epsilon_h +$
$+ \epsilon^i g_{hr} N'_{ij} = 0.$

We now contract (5.7) with respect to the indices $i$ and $h$ and get

(5.8) $(n+1)(\epsilon_{j+k} - \epsilon_{j+k}) - (n+1) N'_{ij} \epsilon_r +$
$+ \epsilon^i (g_{ik} + g_{ij} + k) - \epsilon_{j+k} g_{ij} + \epsilon_{j+k} g_{ik} +$
$+ \epsilon^i g_{ir} N'_{ij} = 0.$

Therefore, we can state

**THEOREM (5.1):**

In an affinely connected $R^r - F_r^r$, in order that $R^r$-conformal motion be a $R^r$-curvature collineation (5.8) should necessarily hold.

We now contract the equation (5.3) with respect to the indices and thereafter use (5.6) and get

(5.9) $\mathcal{L}_v R_{ij} = \epsilon_{j+k} - n \epsilon_{k+1} + \epsilon^i (g_{hk} + g_{hj} + k) -$
$- \epsilon_{j+k} g_{ij} + \epsilon_{j+k} g_{ik} - A_{ij} - \Gamma^r_{irh} \epsilon_r \Gamma^r +$
$+ \left( B_{ij} + C_{ij} - D_{ij} - E_{ih} + F_{ij} \right).$

where

(5.10) (a) $A_{ij} = \Gamma^r_{ijh} \epsilon_r \epsilon^r,$
(b) $B_{ij} = \Gamma^r_{hij} g_{ij} \epsilon^r \epsilon^r,$
(c) $C_{ij} = \Gamma^r_{hij} g_{ij} \epsilon^r \epsilon^r,$
(d) $D_{ij} = N'_{ij} \epsilon_r,$
(e) $E_{ij} = N'_{ij} \epsilon_h,$
(f) $F_{ij} = \epsilon^r g_{hr} N'_{ij}.$
We now use (5.6) in (5.9) and get

\[(5.10) \epsilon_{\dot{e}_{ij}} - n \epsilon_{\dot{e}_{ij}} + \epsilon' \left( g_{\dot{e}_{ji}} - g_{\dot{e}_{ij}} \right) - \epsilon' \epsilon_{\dot{e}_{ij}} g_{ij} + \epsilon' \epsilon_{\dot{e}_{ij}} g_{ij} - \left( A_{\dot{e}_{ij}} - B_{\dot{e}_{ij}} - C_{\dot{e}_{ij}} - F_{\dot{e}_{ij}} + D_{\dot{e}_{ij}} + E_{\dot{e}_{ij}} \right) - \Gamma_{\dot{e}_{ij}}^{\ell} \epsilon_{\dot{e}_{ij}} \dot{x} = 0.\]

At this stage, we now assume that the space under consideration is affinely connected, therefore under this assumption, we get

\[(5.11) C_{\dot{e}_{ij}} = A_{\dot{e}_{ij}} = B_{\dot{e}_{ij}} = 0.\]

Using (5.11) in (5.10), we get

\[(5.12) \epsilon_{\dot{e}_{ij}} - n \epsilon_{\dot{e}_{ij}} + \epsilon' \left( g_{\dot{e}_{ji}} - g_{\dot{e}_{ij}} \right) - \epsilon' \epsilon_{\dot{e}_{ij}} g_{ij} + \epsilon' \epsilon_{\dot{e}_{ij}} g_{ij} - \left( D_{\dot{e}_{ij}} + E_{\dot{e}_{ij}} - F_{\dot{e}_{ij}} \right) - \Gamma_{\dot{e}_{ij}}^{\ell} \epsilon_{\dot{e}_{ij}} \dot{x} = 0.\]

Hence, we can state

**THEOREM (5.2):**

In an affinely connected $R^+-F^+_n$, in order that the $R^+$-conformal motion be $R^+$-Ricci collineation, the equation (5.12) must necessarily hold.

**6. SPECIAL PROJECTIVE MOTION IN A $R^+-F^+_n$:**

If the infinitesimal point transformation (2.1) leaves invariant the system of geodesics into the same system then it is called a $R^+$-projective transformation and the transformation (2.1) is said to define a $R^+$-special projective motion iff the Lie-derivative of the non-symmetric connection coefficient $\Gamma'_{\dot{e}_{ij}}(x,\dot{x})$ satisfies

\[(6.1) \mathcal{L}\Gamma'_{\dot{e}_{ij}} = \delta'_{\dot{e}_{ij}} = \delta'_{\dot{e}_{ij}} + \epsilon' \lambda_{\dot{e}_{ij}},\]

where $\lambda$ is an arbitrarily chosen positively homogeneous scalar function of degree one in $\dot{x}$'s and satisfies the following relations:

\[(6.2) \begin{align*}
(a) \quad & \lambda_k = \dot{\lambda}_k, \\
(b) \quad & \lambda_{\dot{e}_{jk}} = \dot{\lambda}_{\dot{e}_{jk}} \\
(c) \quad & \lambda_k \dot{x}^k = \lambda \\
(d) \quad & \lambda_{\dot{e}_{jk}} \dot{x}^k = 0.
\end{align*}\]
Using (6.1) into the commutation formula (2.7), we get
\begin{align}
(6.3) \, \mathcal{L}_h R^i_{hk} &= \delta^i_\lambda_j + \delta^i_\lambda_h + \delta^i_{\lambda_k} + \delta^i_{\lambda_k} - \delta^i_\lambda_k + \delta^i_\lambda_h + \delta^i_\lambda_k + \lambda_i \dot{x}^i \\
&+ \lambda_i \dot{x}^i - \Gamma^i_{rhk} \lambda_i \dot{x}^r - \lambda \Gamma^i_{kjh} + \Gamma^i_{rhk} \lambda_i \dot{x}^r + \lambda \Gamma^i_{jkb} - \\
&- N^i_{kj} \delta^i_\lambda_j - N^i_{kj} \lambda_i - N^i_{kj} \lambda_i \dot{x}^i,
\end{align}
where we have taken into account (6.2) and the fact that the covariant derivatives of \( \delta^i_\lambda \) and \( \dot{x}^i \) vanish identically. At this stage, we assume that the space \( F_n^* \) admits \( R^* \)-curvature collineation, then using (5.5) in (6.3), we get
\begin{align}
(6.4) \, \delta^i_\lambda_j + \delta^i_\lambda_h + \delta^i_{\lambda_k} + \lambda_i \dot{x}^i + \delta^i_\lambda_k + \delta^i_\lambda_h + \lambda_i \dot{x}^i \\
&+ \lambda_i \dot{x}^i - \Gamma^i_{rhk} \lambda_i \dot{x}^r - \lambda \Gamma^i_{kjh} + \Gamma^i_{rhk} \lambda_i \dot{x}^r + \lambda \Gamma^i_{jkb} - \\
&- N^i_{kj} \delta^i_\lambda_j - N^i_{kj} \lambda_i - N^i_{kj} \lambda_i \dot{x}^i = 0.
\end{align}
Contracting (6.4) with respect to the indices \( i \) and \( j \), we get
\begin{align}
(6.5) \, n \lambda_i + \lambda_k + \lambda_{hk} + \lambda_i \dot{x}^i - \lambda \left( \Gamma^i_{khi} - \Gamma^i_{ijk} \right) - \\
- \lambda \left( \Gamma^i_{khr} \lambda_i - \Gamma^i_{brk} \lambda_i \right) - N^i_{kr} \lambda_i - N^i_{ki} \lambda_i - N^i_{hi} \lambda_i \dot{x}^i = 0.
\end{align}
Therefore, we can state:

**THEOREM (6.1):**

In a \( R^* - F_n^* \), in order that \( R^* \)-special projective motion be \( R^* \)-curvature collineation (6.5) must necessarily hold.

We now contract (6.3) with respect to the indices \( i \) and \( k \), we get
\begin{align}
(6.6) \, \mathcal{L}_v R^i_{hk} &= \lambda_j + \lambda_h + \lambda_k + \lambda_i \dot{x}^i - \lambda \left( \Gamma^i_{khi} - \Gamma^i_{ijk} \right) + \\
&+ \Gamma^i_{rhk} \lambda_i \dot{x}^r - N^i_{kj} \lambda_i - N^i_{kj} \lambda_i \dot{x}^i.
\end{align}
At this stage, we suppose that the infinitesimal transformation (2.1) defines a \( R^* \)-Ricci collineation characterised by (5.6), then from (6.6), we get

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\[(6.7) \quad \lambda_j + h - n \lambda_h + l - \Gamma_{rh}^i \lambda_r \lambda^r - \lambda \left( \Gamma_{khi} - \Gamma_{jhi} \right) + \]
\[+ \Gamma_{rh}^i \lambda^r \lambda_j - N_{r'}^i \lambda_r - N_{r'}^i \lambda_h - N_{r'}^i \lambda_{hr} \lambda^i = 0. \]

**THEOREM (6.2):**

In a $R^+ - F_n^*$, in order that $R^+$- special projective motion be $R^+$-Ricci collineation (6.7) must necessarily hold.

At the stage, if we assume that the space under consideration is affinely then under this assumption (6.5) and (6.7) assume the following alternative form

\[(6.8) \quad n \lambda_h + l - \lambda_k + l - \lambda_{hk} + l - \lambda^r - N_{kh}^r \lambda_r - N_{r'}^r \lambda_h - N_{r'}^r \lambda_{hr} \lambda^i = 0, \]
and

\[(6.9) \quad \lambda_j + h - n \lambda_h + l - N_{r'}^i \lambda_r - N_{r'}^i \lambda_h - N_{r'}^i \lambda_{hr} \lambda^i = 0. \]

Thus, we can state:

**THEOREM (6.3):**

In an affinely connected $R^+ - F_n^*$ in order that $R^+$-special projective motion be $R^+$-curvature collineation (6.8) must necessarily hold.

**THEOREM (6.4):**

In an affinely connected $R^+ - F_n^*$, in order that $R^+$-special projective motion be $R^+$-Ricci collineation (6.9) must necessarily hold.
# REFERENCES

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