CHAPTER – III

MAGNETO-ELASTIC VIBRATIONS

PART – A

Magneto-Elastic Longitudinal Waves in a Triple Layered Cylindrical Shell*

1. INTRODUCTION

Srivastava & Verma [1] have discussed the magneto-elastic longitudinal waves in a circular cylindrical shell having Cauchy’s initial stress. Abd-alla & Abbas[2] dealt with the longitudinal waves propagation in a transversely isotropic circular cylinder. In this part, we discuss the longitudinal waves in a triple layered cylindrical shell placed in an axial magnetic field. Bessel functions have been used as a tool to solve the problem. Such type of problems are of importance in Geophysics.

2. PROBLEM, BASIC EQUATIONS, BOUNDARY AND CONTINUITY CONDITIONS

We consider a cylindrical shell of isotopic material \((b \leq r \leq a)\). The inner core \((0 \leq r \leq b)\) and the outer core \((a \leq r \leq d)\) are of visco-elastic material of general linear type. The interaction between elastic and magnetic field has been discussed by considering longitudinal waves propagation in a cylindrical shell of triple layer placed in an axial magnetic field. Making use

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of cylindrical polar co-ordinates \((r, \theta, z)\), the components of displacement for longitudinal waves are taken as (cf[3]),

\[
\begin{align*}
    u_\alpha &= U_\alpha(r)e^{i(yz+pt)} \\
    v_\alpha &= 0 \\
    w_\alpha &= W_\alpha(r)e^{i(yz+pt)} \\
    (\alpha &= 1, 2, 3) \\
\end{align*}
\]

The components of stresses in terms of displacement [see (A.2.1)] for visco-elastic material of general linear type of (cf[4]),

\[
\begin{align*}
    \sigma_{rr_n} &= R \frac{\partial U_n}{\partial r} e^{i(yz+pt)} \\
    \sigma_{\theta\theta_n} &= R \frac{U_n}{r} e^{i(yz+pt)} \\
    \sigma_{zz_n} &= R i\gamma W_n e^{i(yz+pt)} \\
    \sigma_{\theta z_n} &= 0 \\
    \sigma_{rz_n} &= R \left(i\gamma U_n + \frac{\partial W_n}{\partial r}\right) e^{i(yz+pt)}
\end{align*}
\]
\[ \sigma_{r\theta_n} = 0 \]

\[(n = 1, 3)\]

The components of stresses in terms of displacement [see (A.2.1)] for isotropic materials are (cf[3]),

\[
\sigma_{rr_\ell} = \left[ (\lambda_\ell + 2\mu_\ell) \frac{\partial U_\ell}{\partial r} + \lambda_\ell \left( \frac{U_\ell}{r} + i\gamma W_\ell \right) \right] e^{i(yz+pt)}
\]

\[
\sigma_{\theta\theta_\ell} = \left[ (\lambda_\ell + 2\mu_\ell) \frac{U_\ell}{r} + \lambda_\ell \left( \frac{\partial U_\ell}{\partial r} + i\gamma W_\ell \right) \right] e^{i(yz+pt)}
\]

\[
\sigma_{zz_\ell} = \left[ \lambda_\ell \left( \frac{\partial U_\ell}{\partial r} + \frac{U_\ell}{r} \right) + 2\mu_\ell i\gamma U_\ell \right.
\]

\[
+ \lambda_\ell i\gamma W_\ell + 2\mu_\ell \frac{\partial W_\ell}{\partial r} \right] e^{i(yz+pt)}
\]

\[\sigma_{\theta z_\ell} = 0\]

\[\sigma_{r z_\ell} = 2\mu_\ell \left[ ipU_\ell + \frac{\partial W_\ell}{\partial r} \right] e^{i(yz+pt)}\]

\[\sigma_{r\theta_\ell} = 0\]

\[(\ell = 2)\]
The second stress equation is identically satisfied, whereas first & third stress equation are (cf[3]),

\[ \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rrr} - \sigma_{r \theta \theta}}{r} + F_r = \rho_a \frac{\partial^2 u_r}{\partial t^2} \]

\[ \frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rzz}}{r} + F_z = \rho_a \frac{\partial^2 u_z}{\partial t^2} \]

\( (\alpha = 1, 2, 3) \)

The electromagnetic field equations in the body are (cf. [5]),

\[ h_\alpha = \text{rot} \left( \xi_\alpha \times H \right) \]

\[ \text{rot } h_\alpha = 4\pi j_\alpha \]

\[ E_\alpha = -\frac{1}{c} \left( \frac{\partial \xi_\alpha}{\partial t} \times H \right) \]

The electromagnetic field equations in vacuum are (cf[5]),

\[ \text{rot } E^* = -\frac{1}{c} \frac{\partial h^*}{\partial t} \]

\[ \text{rot } h^* = \frac{1}{c} \frac{\partial E^*}{\partial t} \]
\[(A.2.9) \quad \nabla^2 h^* = \frac{1}{c^2} \frac{\partial^2 h^*}{\partial t^2}\]

\[(A.2.10) \quad \nabla^2 E^* = \frac{1}{c^2} \frac{\partial^2 E^*}{\partial t^2}\]

The electromagnetic forces are (cf[5]),

\[(A.2.11) \quad F_{\alpha} \equiv [F_r, F_\theta, F_\phi] \]

\[= \frac{1}{4\pi} \left[ \text{rot rot} \left( \varepsilon_{\alpha} \times H \right) \times H \right] \]

\[(\alpha = 1, 2, 3)\]

The boundary conditions are,

\[(A.2.12) \quad \sigma_{rr_1} + T_{rr_1} = T_{rr}^*\]

\[\sigma_{rz_1} + T_{rz_1} = T_{rz}^*\]

\[E_1 = E^*\]

\[h_1 = h^*\]

for \(r = d\).

The continuity conditions are,
(A.2.13) \[ \sigma_{rr_1} = \sigma_{rr_2} \quad \text{for } r = a. \]

\[ \sigma_{rr_2} = \sigma_{rr_3} \quad \text{for } r = b \]

\[ E_1 = E_2 \quad \text{for } r = a \]

\[ E_2 = E_3 \quad \text{for } r = b \]

where,

\( \xi \) \quad \equiv \quad \text{Displacement vector having components (u, v, w).} \]

\( h \) \quad \equiv \quad \text{Perturbation vector having components (} h_r, h_\theta, h_z \text{)} \]

\( H \) \quad \equiv \quad \text{Axial Magnetic field vector.} \]

\( T_{rr}, T_{rz} \) \quad \equiv \quad \text{Maxwell’s stress tensor} \]

\( E \) \quad \equiv \quad \text{Electric intensity vector having components} \ (E_r, E_\theta, E_z) \]

\[ \frac{2\pi}{\gamma} \] \quad \equiv \quad \text{Wave length} \]

\[ \frac{v}{\gamma} \] \quad \equiv \quad \text{Phase velocity.} \]

\( c \) \quad \equiv \quad \text{Velocity of light in vacuum} \]

\( * \) \quad \equiv \quad \text{Values in vacuum} \]
\[ G \equiv \text{Elastic constant} \]
\[ \lambda, \mu \equiv \text{Lame's constants} \]
\[ D \equiv \frac{\partial}{\partial t} \]
\[ \alpha \equiv 1, 2, 3; \text{Numbers; 1, 2, 3 stand for first, second and third medium.} \]

3. GENERAL SOLUTIONS

Making use of (A.2.1) & (A.2.6), we get

(A.3.1) \[ E_{r\alpha} = E_{z\alpha} = 0 \]

and

\[ E_{\theta\alpha} = \frac{ipH}{c} U_\alpha e^{i(yz+pt)} \]

(A.3.2) \[ h_{r\alpha} = i\gamma H U_\alpha e^{i(yz+pt)} \]
\[ h_{\theta\alpha} = 0 \]
\[ h_{z\alpha} = -\frac{H}{r} \frac{\partial}{\partial r} (rU_\alpha) e^{i(yz+pt)} \]

Taking the help of (A.2.1) & (A.2.6), the electro magnetic forces (A.2.11) take the form,
\begin{align}
\tag{A.3.3} F_{\theta\alpha} &= F_{z\alpha} = 0 \\
F_{r\alpha} &= \frac{\mu^2}{4\pi} \left[ \frac{\partial^2 u_\alpha}{\partial r^2} + \frac{1}{r} \frac{\partial u_\alpha}{\partial r} - \frac{u_\alpha}{r^2} - \gamma^2 U_\alpha \right] e^{i(yz+pt)}
\end{align}

Considering the components in (A.3.1) & (A.3.2), we take,

\begin{align}
\tag{A.3.4} E_\theta^* &= E' e^{i(yz+pt)} \\
\tag{A.3.5} h_r^* &= h_r' e^{i(yz+pt)} \\
\tag{A.3.6} h_z^* &= h_z' e^{i(yz+pt)}
\end{align}

We put (3.4), (3.5) & (3.6) in (2.7)-(2.10), to get

\begin{align}
\tag{A.3.7} E_\theta^* &= \frac{c}{\rho} \left[ \gamma h_r' + i \frac{\partial h_z'}{\partial r} \right] e^{i(yz+pt)} \\
\tag{A.3.8} \frac{\partial^2 h_r'}{\partial r^2} + \frac{1}{r} \frac{\partial h_r'}{\partial r} + L^2 h_r' &= 0 \\
\tag{A.3.9} \frac{\partial^2 h_z'}{\partial r^2} + \frac{1}{r} \frac{\partial h_z'}{\partial r} + L^2 h_z' &= 0
\end{align}

Maxwell stress tensor in the body and in vacuum are (cf[5]),

\begin{align}
\tag{A.3.10} T_{rr\alpha} &= -\frac{H}{4\pi} (h_z)_{\alpha}
\end{align}
\[ T_{\tau \alpha} = \frac{H}{4\pi} (h_{\tau})_{\alpha} \]

and in vacuum are

\[ T_{\tau \tau}^* = \frac{H}{4\pi} h_{z}^' \]
\[ T_{\tau \zeta}^* = \frac{H}{4\pi} h_{r}^' \]

where,

\[ L^2 = \left( \frac{p^2}{c^2} - \gamma^2 \right) \]

and terms with * are the values in vacuum.

4. SOLUTION OF THE PROBLEM

We put (A.2.1), (A.2.2), (A.2.3), (A.3.1) & (A.3.3) in (A.2.4) and (A.2.5) to get,

\[ \left[ R + \frac{H^2}{4\pi} \right] + \left[ \frac{\partial^2 U_1}{\partial r^2} + \frac{1}{r} \frac{\partial U_1}{\partial r} - \frac{U_1}{r^2} \right] \]
\[ + \left[ \rho_1 p^2 - \gamma^2 \left( R + \frac{H^2}{4\pi} \right) \right] U_1 + i\gamma R \frac{\partial w_1}{\partial r} = 0 \]

\[ \left[ R \left( \frac{\partial^2 w_1}{\partial r^2} + \frac{1}{r} \frac{\partial w_1}{\partial r} \right) \right] + \left[ \rho_1 p^2 - \gamma^2 R \right] w_1 + iR \left( \frac{\partial U_1}{\partial r} + \frac{U_1}{r} \right) = 0 \]
\[(A.4.3) \quad (\lambda_2 + 2\mu_2 + H^2) \left[ \frac{\partial^2 u_2}{\partial r^2} + \frac{1}{r} \frac{\partial u_2}{\partial r} - \frac{u_2}{r^2} \right] + \left[ \rho_2 p^2 - \gamma^2 H^2 - \mu_2 \gamma^2 \right] U_2 i(\lambda_2 + \mu_2) \left[ \frac{\partial u_2}{\partial r} + \frac{u_2}{r} \right] = 0 \]

\[(A.4.4) \quad \mu_2 \left[ \frac{\partial^2 w_2}{\partial r^2} + \frac{1}{r} \frac{\partial w_2}{\partial r} \right] + \left[ \rho_1 p^2 - (\lambda_2^2 + 2\mu_2) \gamma^2 \right] w_2 + i\gamma (\lambda_2 + 2\mu_2) \left[ \frac{\partial u_2}{\partial r} + \frac{u_2}{r} \right] = 0 \]

\[(A.4.5) \quad \left( R + \frac{H^2}{4\pi} \right) \left[ \frac{\partial^2 u_3}{\partial r^2} + \frac{1}{r} \frac{\partial u_3}{\partial r} - \frac{u_3}{r^2} \right] \]

\[+ \left[ \rho_3 p^2 - \gamma^2 \left( R + \frac{H^2}{4\pi} \right) \right] U_3 + i\gamma R \left[ \frac{\partial w_3}{\partial r} \right] = 0 \]

\[(A.4.6) \quad R \left[ \frac{\partial^2 w_3}{\partial r^2} + \frac{1}{r} \frac{\partial w_3}{\partial r} \right] + (\rho_3 p^2 - \gamma^2 R) w_3 + i\gamma R \left[ \frac{\partial u_3}{\partial r} + \frac{u_3}{r} \right] = 0 \]

As a trial solution, let us assume (cf[6]),

\[(A.4.7) \quad U_\alpha = A_\alpha J_1(\rho_\alpha r) \]

\[W_\alpha = B_\alpha J_0(\rho_\alpha r) \]

As a consequences of (A.4.7), equations (A.4.1)-(A.4.6) provide six equations. We eliminate $A_1, B_1'$ among, first two equations; $A_2, B_2'$ among
middle two equations and $A_3, B_3'$ among last two equations to get the following equations -

\[(A.4.8)\]
\[
\left[ \rho_1 p^2 - R \left( \gamma^2 + P_1^2 \right) \right] \times \left[ \rho_1 p^2 - \left( \gamma^2 + P_1^2 \right) \right]
\times \left( R + \frac{H^2}{4\pi} \right) - \gamma^2 R P_1^2 = 0
\]

\[(A.4.9)\]
\[
\left[ \rho_2 p^2 - \gamma^2 \left( \mu^2 + \frac{H^2}{4\pi} \right) - \left( \lambda_2 + 2\mu_2 + \frac{H^2}{4\pi} \right) P_2^2 \right]
\times \left[ \mu_2 P_2^2 + \left( \lambda_2^2 - 2\mu_2 \gamma^2 \right) \right]
\times \left( \lambda_2 + 2\mu_2 \right) \gamma^2 P_2^2 = 0
\]

\[(A.4.10)\]
\[
\left[ \rho_3 p^2 - R \left( \gamma + P_3^2 \right) \right] \times \left[ \rho_3 p^2 - \left( \gamma^2 + P_2^2 \right) \right]
\times \left( R + \frac{H^2}{4\pi} \right) - \gamma^2 R P_3^2 = 0
\]

If $K_\alpha^2$ & $M_\alpha^2$ ($\alpha = 1, 2, 3$) be the roots of the equation (A.4.8)-(A.4.10), the solution of (A.4.1) - (A.4.6) become,

\[(A.4.11)\]
\[
U_1 = \left[ A_1 J_1(K_1 r) + B_1 J_1(M_1 r) \right]
\]

\[(A.4.12)\]
\[
W_1 = \left[ A_1 \theta_1 J_0(K_1 r) + B_1 \phi_1 J_0(M_1 r) \right]
\]
(A.4.13) \[ U_2 = [A_2 J_1(K_2 r) + B_2 J_1(M_2 r)] \]

(A.4.14) \[ W_2 = [A_2 \theta_2 J_0(K_2 r) + B_2 \varphi_2 J_0(M_2 r)] \]

(A.4.15) \[ U_3 = [A_3 J_1(K_3 r) + B_3 J_1(M_3 r)] \]

(A.4.16) \[ W_3 = [A_3 \theta_3 J_0(K_3 r) + B_3 \varphi_3 J_0(M_3 r)] \]

Under the assumption that \( K_\alpha \) & \( M_\alpha \) (\( \alpha = 1, 2, 3 \)) take the real values only and

\[ \theta_n = \left[ \rho_n p^2 - \left\{ y^2 + k_n^2 \right\} \left( R + \frac{h^2}{4\pi} \right) \right] / i\gamma K_n R \]

\[ \varphi_n = \left[ \rho_n p^2 - \left\{ y^2 + M_n^2 \right\} \left( R + \frac{h^2}{4\pi} \right) \right] / i\gamma M_n R \]

(n = 1, 3)

\[ \theta_\ell = \left\{ \rho_\ell p^2 - y^2 \left( \frac{h^2}{4\pi} + \mu_\ell \right) \right\} \]

\[ \quad \quad \quad - \left\{ \lambda_\ell + 2\mu_\ell + \frac{h^2}{4\pi} K_\ell^2 \right\} / [i(\lambda_\ell + 2\mu_\ell)\gamma K_\ell^2] \]

\[ \varphi_\ell = \left\{ \rho_\ell p^2 - y^2 \left( \frac{h^2}{4\pi} + \mu_\ell \right) \right\} \]
\[-\left(\lambda_\ell + 2\mu_\ell + \frac{H^2}{4\pi} K_\ell^2\right) \left[\frac{i(\lambda_\ell + 2\mu_\ell)yM_\ell^2}{\ell}\right]\]

($\ell = 2$)

where, $A_\alpha$ & $B_\alpha$ are constants, such that $\alpha = 1, 2, 3$.

For finite cylindrical shell, the solution of (A.3.8) and (A.3.9) with the help of (A.3.5) & (A.3.6) become,

(A.1.17) \hspace{1cm} h_r' = D_1J_0(Lr)

(A.1.18) \hspace{1cm} h_z' = D_2J_0(Mr)

In our case, the boundary and continuity conditions are,

(A.4.19) \hspace{1cm} \sigma_{rr_1} + T_{rr_1} = T_{rr}^* \hspace{1cm} \text{for} \ r = d

(A.4.20) \hspace{1cm} \sigma_{rz_1} + T_{rz_1} = T_{rz}^* \hspace{1cm} \text{for} \ r = d

(A.4.21) \hspace{1cm} E_{\theta_1} = E_{\theta}^* \hspace{1cm} \text{for} \ r = d

(A.4.22) \hspace{1cm} H_{Z_1} = H_{Z}^* \hspace{1cm} \text{for} \ r = d

(A.4.23) \hspace{1cm} \sigma_{rr_1} = \sigma_{rr_2} \hspace{1cm} \text{for} \ r = b
(A.4.24) \[ \sigma_{rr_2} = \sigma_{rr_3} \quad \text{for } r = a \]

(A.4.25) \[ E_{\theta_1} = E_{\theta_2} \quad \text{for } r = b \]

(A.4.26) \[ E_{\theta_2} = E_{\theta_3} \quad \text{for } r = a \]

Substituting the respective values in (A.4.19)-(A.4.26), we get the following equations:

(A.4.27) \[ A_1 Y_{11} + B_1 Y_{12} + D_2 Y_{18} = 0 \]

(A.4.28) \[ A_1 Y_{21} + B_1 Y_{22} + D_1 Y_{27} = 0 \]

(A.4.29) \[ A_1 Y_{31} + B_1 Y_{32} + D_1 Y_{37} + D_2 Y_{38} = 0 \]

(A.4.30) \[ A_1 Y_{41} + B_1 Y_{42} + D_2 Y_{48} = 0 \]

(A.4.31) \[ A_1 Y_{51} + B_1 Y_{52} + A_2 Y_{53} + B_2 Y_{54} = 0 \]

(A.4.32) \[ A_2 Y_{63} + B_2 Y_{42} + A_3 Y_{65} + B_3 Y_{66} = 0 \]

(A.4.33) \[ A_1 Y_{71} + B_1 Y_{72} + A_2 Y_{73} + B_2 Y_{74} = 0 \]

(A.4.32) \[ A_2 Y_{83} + B_2 Y_{84} + A_3 Y_{85} + B_3 Y_{86} = 0 \]
Eliminating $A_1, B_1, A_2, B_2, A_3, B_3, D_1$ & $D_2$ between (A.4.27)-(A.4.34), we get the following frequency equation:

$$
\begin{vmatrix}
Y_{11} & Y_{12} & 0 & 0 & 0 & 0 & Y_{18} \\
Y_{21} & Y_{22} & 0 & 0 & 0 & 0 & Y_{27} \\
Y_{31} & Y_{32} & 0 & 0 & 0 & Y_{37} & Y_{38} \\
Y_{41} & Y_{42} & 0 & 0 & 0 & 0 & Y_{48} \\
Y_{51} & Y_{52} & Y_{53} & Y_{54} & 0 & 0 & 0 \\
0 & 0 & Y_{63} & Y_{64} & Y_{65} & Y_{66} & 0 \\
Y_{71} & Y_{72} & Y_{73} & Y_{74} & 0 & 0 & 0 \\
0 & 0 & Y_{83} & Y_{84} & Y_{85} & Y_{86} & 0 \\
\end{vmatrix} = 0
$$

(A.4.35)

where,

$$
Y_{11} = \left[ \left( R + \frac{H^2}{4\pi} \right) K_1 J_0(K_1d) - \frac{R}{d} J_1(K_1d) \right]
$$

$$
Y_{12} = \left[ \left( R + \frac{H^2}{4\pi} \right) M_1 J_0(M_1d) - \frac{R}{d} J_1(M_1d) \right]
$$

$$
Y_{18} = \left[ \left( -\frac{H^2}{4\pi} \right) J_0(M_1d) \right],
$$

$$
Y_{25} = \left[ \left( -\frac{H^2}{4\pi} \right) J_0(Ld) \right]
$$

$$
Y_{21} = \left[ i\gamma \left( R + \frac{H^2}{4\pi} \right) J_1(K_1d) - RK_1\theta_1 \right],
$$
\[ Y_{31} = \left( \frac{ipH}{c} \right) J_1(K_1d), \]

\[ Y_{32} = \left( R + \frac{\mu^2}{4\pi} \right) J_1(M_1d) - R\Phi_1M_1, \]

\[ Y_{37} = \left[ -\frac{c}{p} \gamma J_0(Ld) \right], \]

\[ Y_{38} = [MJ_1(Md)] \]

\[ Y_{41} = [i\gamma HJ_1(K_1d)], \]

\[ Y_{42} = [i\gamma HJ_1(M_1d)], \]

\[ Y_{47} = [J_0(Ld)], \]

\[ Y_{51} = \left[ RK_1J_0(K_1b) - \frac{1}{b} J_1(K_1b) \right], \]

\[ Y_{52} = \left[ RM_1J_0(M_1b) - \frac{1}{b} J_1(M_1b) \right], \]

\[ Y_{53} = \left[ - (\lambda_2 + 2\mu_2) K_2J_1(K_2b) + \lambda_2 \left( \theta_2 i\gamma + \frac{1}{b} \right) J_0(K_2b) \right], \]

\[ Y_{54} = \left[ - (\lambda_2 + 2\mu_2) M_2J_1(M_2b) + \lambda_2 \left( \Phi_2 i\gamma + \frac{1}{b} \right) J_0(M_2b) \right], \]

\[ Y_{63} = \left[ - (\lambda_2 + 2\mu_2) K_2J_1(K_2a) + \lambda_2 \left( \theta_2 i\gamma + \frac{1}{a} \right) J_0(K_2a) \right], \]

\[ Y_{64} = \left[ - (\lambda_2 + 2\mu_2) M_2J_1(M_2a) + \lambda_2 \left( \Phi_2 i\gamma + \frac{1}{a} \right) J_0(M_2a) \right], \]

\[ Y_{65} = \left[ RK_1J_0(K_1a) - \frac{1}{a} J_1(K_1a) \right]. \]
\[ Y_{66} = \left[ RM_1 J_0(M_1 a) - \frac{1}{a} J_1(M_1 a) \right] \]

\[ Y_{71} = [J_1(K_1 b)] , \quad Y_{72} = [J_1(M_1 b)] , \]

\[ Y_{73} = [-\theta_1 J_0(K_1 b)] , \quad Y_{74} = [-\theta_1 J_0(M_1 b)] \]

\[ Y_{83} = [J_1(K_1 a)] , \quad Y_{84} = [J_1(M_1 a)] \]

\[ Y_{85} = [\theta_2 J_0(K_2 a)] , \quad Y_{86} = [\theta_2 J_0(M_2 a)] \]

For a finite cylindrical shell, we neglect the higher powers of radii which reduces the frequency equation into a simplified form. Now we may plot a graph showing the variation of stresses with time for specific materials.
PART – B

Magneto-Elastic Radial Vibrations in a Composite Shape
Non-Homogeneous Visco-Elastic Cylinder

1. INTRODUCTION

Abd-alla [7] investigated magneto-elastic radial vibrations of a transversely isotropic hollow cylinder. Gourakashwar [15] discussed propagation of waves in a magneto-elastic media. Problems of magneto-elastic vibrations have been dealt by other researchers also ([9]-[14]). In this part, we consider radial vibrations in a non-homogeneous composite cylinder placed in an axial magnetic field. Bessel functions have been used to solve the problem.

2. BASIC EQUATIONS, BOUNDARY AND CONTINUITY CONDITIONS

Making use of cylindrical polar co-ordinates \((r, \theta, z)\), the components of displacement for radial vibrations are taken as (cf[3]),

\[
(B.2.1) \quad u_{r\alpha} = U_{\alpha}(r)e^{ipt} \\

u_{\theta\alpha} = u_{z\alpha} = 0 ,
\]

\((\alpha = 1, 2)\)
The variation of densities is characterised by,

\[
\rho_\alpha = \rho_\alpha' r^{-n_\alpha}, \quad (\alpha = 1, 2)
\]

where, $\rho_1'$ & $\rho_2'$ are density constants & $n$ is any integer.

We consider a composite shape cylinder, constituted of Achenbech and Chao type visco-elastic material ($b \leq r \leq a$) and general linear type visco-elastic material ($0 \leq r \leq b$), executing radial vibrations in presence of magnetic field.

The non-vanishing stress components in terms of displacement for visco-elastic material of Achenbech and Chao type are (cf[8]),

\[
\sigma_{rr_1} = Q_1 \frac{\partial u_1}{\partial r} e^{ipt}
\]

\[
\sigma_{\theta r_1} = Q_1 \frac{u_1}{r} e^{ipt}
\]

The non-vanishing components of stresses in terms of displacement for visco-elastic material of general linear type are (cf[4]),

\[
\sigma_{rr_2} = Q_2 \frac{\partial u_2}{\partial r} e^{ipt}
\]

\[
\sigma_{\theta r_2} = Q_2 \frac{u_2}{r} e^{ipt}
\]
The only non-vanishing stress equations are (cf[3]),

\[
\rho \frac{\partial^2 u_\alpha}{\partial t^2} = \frac{\partial \sigma_{rr\alpha}}{\partial r} + \frac{(\sigma_{rr\alpha} - \sigma_{\theta\theta\alpha})}{\alpha} + F_r
\]

(\(\alpha = 1, 2\))

The electromagnetic field equations for infinitely conducting media and vacuum are (cf[5]),

\[
h_\alpha = \text{curl} \left( \xi_\alpha \times H \right)
\]

\[
\text{curl} \; h_\alpha = 4\pi j_\alpha
\]

\[
E_\alpha = -\frac{1}{c} \left( \frac{\partial \xi_\alpha \times H}{\partial t} \right)
\]

and

\[
\text{curl} \; E^* = -\frac{1}{c} \frac{\partial h^*}{\partial t}
\]

\[
\text{curl} \; h^* = \frac{1}{c} \frac{\partial E^*}{\partial t}
\]

\[
\nabla^2 E^* = \frac{1}{c^2} \frac{\partial^2 E^*}{\partial t^2}
\]
(B.2.11) \[ \nabla^2 \hat{h}^* = \frac{1}{c^2} \frac{\partial^2 \hat{h}^*}{\partial t^2} \]

The boundary conditions on the surface are,

(B.2.12) \[ \sigma_{rr_1} + M_{rr_1} = M_{rr}^* \quad \text{for } r = a \]

\[ E = E^* \quad \text{for } r = a \]

\[ h = h^* \quad \text{for } r = a \]

and continuity conditions are,

(B.2.13) \[ \sigma_{rr_1} = \sigma_{rr_2} \quad \text{for } r = b \]

\[ E_1 = E_2 \quad \text{for } r = b \]

\[ h_1 = h_2 \quad \text{for } r = b \]

The electromagnetic forces are given by

(B.2.14) \[ \mathbf{F} \equiv (F_r, F_\theta, F_\phi) \]

\[ \equiv \frac{1}{4\pi} \left[ \text{curl} \; \text{curl} \; \left( \xi \times H \right) \times H \right] \]
where,

\[ \alpha \equiv 1, 2; \] subscripts 1 & 2 stand for the values in outer and inner medium.

\[ * \equiv \text{values in vacuum}. \]

\[ K', E', s \equiv \text{Material constants for outer medium}. \]

\[ Q_1 \equiv E' \frac{(s+ip)^2}{(K'+ip)^2} \]

\[ G \equiv \text{Elastic constant for outer medium}. \]

\[ \theta_1, \theta_2 \equiv \text{visco-elastic constant for outer medium}. \]

\[ D \equiv \frac{\partial}{\partial \tau} \]

\[ Q_2 \equiv \frac{G(1+ip\theta_2)}{(1+ip\theta_1)} \]

\[ H \equiv \text{Magnetic field vector having components} (0, 0, H). \]

\[ h \equiv \text{Perturbation vector having components} (h_r, h_\theta, h_z). \]
\[ E \equiv \text{Electric intensity vector having components} \ (E_r, E_\theta, E_z). \]

\[ j \equiv \text{current density vector.} \]

\[ \sigma_{rr, \ldots} \text{ etc.} \equiv \text{Stress components.} \]

\[ \xi \equiv \text{Displacement components having components} \ (u_r, u_\theta, u_z) \]

\[ M_{rr} \equiv \text{Maxwell stress tensor.} \]

\[ c \equiv \text{velocity of light in vacumm.} \]

3. GENERAL SOLUTIONS

We substitute (B.2.1) in (B.2.5) and (B.2.6) to get,

\[
\begin{align*}
(h_r)_\alpha &= (h_\theta)_\alpha = 0 \\
(h_z)_\alpha &= -H \left[ \frac{\partial u_\alpha}{\partial r} + \frac{v_\alpha}{r} \right] e^{ipt}
\end{align*}
\]

and

\[
\begin{align*}
(E_r)_\alpha &= (E_z)_\alpha = 0
\end{align*}
\]
\[(E_\theta)_\alpha = \frac{i\hbar p}{c} U_\alpha e^{ipt}\]

Solving (B.2.14) with the help of (B.2.1), (B.2.6) & (B.2.7), we get,

\[(B.3.3) \quad F_\theta = F_z = 0\]

\[F_r = \frac{\hbar^2}{4\pi} \left[ \frac{\partial^2 U_\alpha}{\partial r^2} + \frac{1}{r} \frac{\partial U_\alpha}{\partial r} \right]\]

Also, we have

\[(B.3.4) \quad M_{rr} = - \frac{\hbar}{4\pi} h_{z\alpha}\]

\[M_{rr}^* = - \frac{\hbar}{4\pi} h^*_{z\alpha}\]

We put (B.3.3), (B.2.3), (B.2.4), (B.2.2) in (B.2.5) and use the transformation

\[(B.3.5) \quad U_\alpha = \psi_\alpha\]

\[(B.3.6) \quad Z_\alpha = \frac{2}{(m_\alpha + 2)} \cdot r \left( \frac{m_\alpha + 2}{2} \right) \quad (\alpha = 1, 2)\]

to get,

\[(B.3.7) \quad \frac{\partial^2 \psi_1}{\partial z_1^2} + \frac{1}{z_1} \frac{\partial \psi_1}{\partial z_1} + \left( \mu_1^2 - \frac{\beta_1^2}{z_1^2} \right) \psi_1 = 0\]
and

\[(B.3.8) \quad \frac{\partial^2 \psi_2}{\partial z_2^2} + \frac{1}{z_2} \frac{\partial \psi_2}{\partial z_2} + \left( \mu_2^2 - \frac{\beta_2^2}{z_2^2} \right) \psi_2 = 0\]

The solution of (B.3.7) & (B.3.8) become,

\[(B.3.9) \quad \psi_1 = [A_1 J_{\beta_1}(\mu_1 z_1) + B_1 Y_{\beta_1}(\mu_1 z_2)]\]

\[(B.3.10) \quad \psi_2 = [A_2 J_{\beta_2}(\mu_2 z_2) + B_2 Y_{\beta_2}(\mu_2 z_2)]\]

Keeping in view the relations (B.3.1) & (B.3.2), we put

\[(B.3.11) \quad h_\theta^* = V(r)e^{ipt}\]

\[(B.3.12) \quad E_\theta^* = W(r)e^{ipt}\]

in (B.2.11) & (B.2.10), to get

\[(B.3.13) \quad \frac{\partial^2 V}{\partial r^2} + \frac{1}{r} \frac{\partial V}{\partial r} + \frac{p^2}{c^2} V = 0\]

\[(B.3.14) \quad \frac{\partial^2 W}{\partial r^2} + \frac{1}{r} \frac{\partial W}{\partial r} + \frac{p^2}{c^2} W = 0\]
The solution of (B.3.13) becomes,

(B.3.15) \[ V = \left[ C_1 I_0 \left( \frac{pr}{c} \right) + D_1 K_0 \left( \frac{pr}{c} \right) \right] \]

Making use of (B.2.8) & (B.2.9), together with (B.3.11), we get

(B.3.16) \[ W = i \frac{c}{p} \frac{\partial V}{\partial r} \]

4. SOLUTION OF THE PROBLEM

The displacements \( u_{r_1} \) & \( u_{r_2} \) become,

(B.4.1) \[ u_1 = \left[ A_1 J \beta_1 \left( \frac{2 \mu_1}{m_1 + 2} \frac{r^{m_1 + 2}}{2} \right) + B_1 Y \xi_1 \left( \frac{2 \mu_1}{m_1 + 2} \frac{r^{m_1 + 2}}{2} \right) \right] e^{ipt} \]

(B.4.2) \[ u_2 = \left[ A_2 J \beta_2 \left( \frac{2 \mu_2}{m_2 + 2} \frac{r^{m_2 + 2}}{2} \right) + B_2 Y \xi_2 \left( \frac{2 \mu_2}{m_2 + 2} \frac{r^{m_2 + 2}}{2} \right) \right] e^{ipt} \]

(B.4.3) \[ h_0^* = \left[ C_1 I_0 \left( \frac{pr}{c} \right) + D_1 \left( \frac{pr}{c} \right) \right] e^{ipt} \]

(B.4.4) \[ E_0^* = i \frac{c}{p} \frac{\partial V}{\partial r} \]

\[ = i \frac{c}{p} \frac{\partial}{\partial r} \left[ C_1 I_0 \left( \frac{pr}{c} \right) + D_1 K_0 \left( \frac{pr}{c} \right) \right] e^{ipt} \]
where,

\[
\begin{align*}
A_1, B_1; \quad & \equiv \text{Constants} \\
A_2, B_2; \quad & \equiv \text{Bessel functions of order } \beta_{\alpha} \\
C_1, D_1; \quad & \equiv \text{(} \alpha = 1, 2) \text{ and kind first and second} \\
\end{align*}
\]

respectively.

\[
I_0, K_0 \quad \equiv \text{Modified Bessel functions of order zero and kind first and second}
\]
respectively.

In our case, the boundary and continuity conditions become,

\[
(B.4.5) \quad \sigma_{rr_1} + M_{rr_1} = M_{rr}^*
\]

\[
E_{\theta_1} = E_{\theta}^*
\]

\[
h_{z1} = h_{z}^*
\]

for \( r = a \) and

\[
(B.4.6) \quad \sigma_{rr_1} = \sigma_{rr_2}
\]
\[ E_{\theta_1} = E_{\theta_2} \]

\[ h_{z_1} = h_{z_2} \]

for \( r = b \)

On substituting the respective values in (B.4.5) & (B.4.6), we get,

(B.4.7) \[ A_1 Y_{11} + B_1 Y_{12} + C_1 Y_{15} + D_1 Y_{16} = 0 \]

(B.4.8) \[ A_1 Y_{21} + B_1 Y_{22} + C_1 Y_{25} + D_1 Y_{26} = 0 \]

(B.4.9) \[ A_1 Y_{31} + B_1 Y_{32} + C_1 Y_{35} + D_1 Y_{36} = 0 \]

(B.4.10) \[ A_1 Y_{41} + B_1 Y_{42} + A_2 Y_{43} + B_2 Y_{44} = 0 \]

(B.4.11) \[ A_1 Y_{51} + B_1 Y_{52} + A_2 Y_{53} + B_2 Y_{54} = 0 \]

(B.4.12) \[ A_1 Y_{61} + B_1 Y_{62} + A_2 Y_{63} + B_2 Y_{64} = 0 \]

Eliminating \( A_1, B_1, A_2, B_2, C_1, D_1 \), we get,
\[
\begin{bmatrix}
Y_{11} & Y_{12} & 0 & 0 & Y_{15} & Y_{16} \\
Y_{21} & Y_{22} & 0 & 0 & Y_{25} & Y_{26} \\
Y_{31} & Y_{32} & 0 & 0 & Y_{35} & Y_{36} \\
Y_{41} & Y_{42} & Y_{43} & Y_{44} & 0 & 0 \\
Y_{51} & Y_{52} & Y_{53} & Y_{54} & 0 & 0 \\
Y_{61} & Y_{62} & Y_{63} & Y_{64} & 0 & 0 \\
\end{bmatrix} = 0
\]

where,

\[(Y_{11})_{r=a} = (Y_{12})_{r=b} = \left[\left(Q_1 + \frac{H^2}{4\pi}\right) \right. \]

\[\left\{2\mu_1 r^{-\frac{m_1}{r}} I_{\beta_1-1}\left(\frac{2\mu_1}{m_1+2} r^{-\frac{m_1+2}{2}}\right)\right\} \]

\[+ \left(Q_1 + \frac{H^2}{4\pi} + \frac{H^2}{4\pi} \right) \left\{\frac{1}{r} J_{\beta_1}\left(\frac{2\mu_1}{m_1+2} r^{-\frac{m_1+2}{2}}\right)\right\} \]

\[(Y_{15})_{r=a} = \left[ -\frac{H}{4\pi} I_0 \left(\frac{pr}{c}\right) \right] , \]

\[(Y_{16})_{r=a} = \left[ -\frac{H}{4\pi} K_0 \left(\frac{pr}{c}\right) \right] , \]

\[(Y_{12})_{r=a} = (Y_{42})_{r=b} = \left[\left(Q_1 + \frac{H^2}{4\pi}\right) \right. \]

\[2\mu_1 r^{-\frac{m_1}{r}} Y_{\beta_1-1}\left(\frac{2\mu_1}{m_1+2} r^{-\frac{m_1+2}{2}}\right) \]

\[+ \left(Q_1 + \frac{H^2}{4\pi} + \frac{H^2}{4\pi} \right) \left\{\frac{1}{r} J_{\beta_1}\left(\frac{2\mu_1}{m_1+2} r^{-\frac{m_1+2}{2}}\right)\right\} \]
\[(Y_{21})_{r=a} = \left[ J_{\beta_1} \left( \frac{2\mu_1}{m_{1+2}} \frac{r}{2} \right) \right] ; \]

\[(Y_{22})_{r=a} = \left[ Y_{\beta_1} \left( \frac{2\mu_1}{m_{1+2}} \frac{r}{2} \right) \right] \]

\[(Y_{25})_{r=a} = \left[ \frac{c}{\rho H} \frac{L_1}{1} \left( \frac{pr}{c} \right) \right] , \quad Y_{26} = \left[ \frac{c}{\rho H} K_1 \left( \frac{pr}{c} \right) \right] \]

\[\frac{(Y_{61})_{r=a}}{\alpha=1} = \frac{(Y_{31})_{r=a}}{\alpha=1} = - \left[ H \left( \frac{2\mu_\alpha}{r^2} \frac{m_\alpha}{m_{\alpha+2}} \right) \frac{2\mu_\alpha}{r} \frac{m_\alpha+2}{2} \right] \]

\[\frac{1}{r} J_{\beta_\alpha} \left( \frac{2\mu_\alpha}{m_{\alpha+2}} \frac{m_\alpha+2}{2} \right) \]

\[\frac{(Y_{62})_{r=b}}{\alpha=2} = \frac{(Y_{32})_{r=a}}{\alpha=2} = - \left[ H \left( \frac{2\mu_\alpha}{r^2} \frac{m_\alpha}{m_{\alpha+2}} \right) \frac{2\mu_\alpha}{r} \frac{m_\alpha+2}{2} \right] \]

\[\frac{1}{r} Y_{\beta_\alpha} \left( \frac{2\mu_\alpha}{m_{\alpha+2}} \frac{m_\alpha+2}{2} \right) \]

\[\frac{(Y_{51})_{r=b}}{\alpha=1} = \frac{(Y_{53})_{r=b}}{\alpha=2} = \left[ \frac{H}{c} \frac{ip}{\rho} \left( J_{\beta_\alpha} \left( \frac{2\mu_\alpha}{m_{\alpha+2}} \frac{m_\alpha+2}{2} \right) \right) \right] \]

\[\frac{(Y_{52})_{r=b}}{\alpha=1} = \frac{(Y_{54})_{r=b}}{\alpha=2} = \left[ \frac{H}{c} \frac{ip}{\rho} \left( J_{\beta_\alpha} \left( \frac{2\mu_\alpha}{m_{\alpha+2}} \frac{m_\alpha+2}{2} \right) \right) \right] \]
\( (Y_{61})_{r=b} = -H \left[ 2\mu_2 \frac{m_2}{r^2} J_{\beta_2-1} \left\{ \frac{2\mu_2}{m_2+2} \frac{m_2+2}{r^2} \right\} \right] \)

\[ -\frac{1}{r} J_{\beta_2} \left\{ \frac{2\mu_2}{m_2+2} \frac{m_2+2}{r^2} \right\} \]

\( (Y_{62})_{r=b} = -H \left[ 2\mu_2 \frac{m_2}{r^2} Y_{\beta_2-1} \left\{ \frac{2\mu_2}{m_2+2} \frac{m_2+2}{r^2} \right\} \right] \)

\[ -\frac{1}{r} Y_{\beta_2} \left\{ \frac{2\mu_2}{m_2+2} \frac{m_2+2}{r^2} \right\} \]

After certain approximations, we may plot a graph showing the variation of stress component \( \sigma_{rr} \) with time for a specific set of materials.
REFERENCES


