Chapter VI

ON A SPECIAL FORM OF T-TENSOR AND T^2-LIKE FINSLER SPACE
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1. INTRODUCTION:

Let \( f^n \) be an \( n \)-dimensional Finsler space equipped with the metric function \( L(x,y) \). If \( C_{ijk}(x,y) \) be the \( (h) \) hv-torsion tensor of \( F^n \) then its \( v \)-covariant differentiation denoted by \( |k \) is used in defining the tensor \( T_{hijk} \) in the form (Matsushita 1977a).

\[
T_{hijk} = L C_{hijk} + C_{ijk} I_h I_i + C_{ijk} I_j I_k + C_{hik} I_j + C_{hij} I_k
\]

Where \( I_i \) are covariant components of unit vector along the elements of support \( y^i \).

In [10], three special forms of \( T_{hijk} \) given by

(A) \[
T_{hijk} = p \left\{ h_{hi} h_{jk} + h_{bj} h_{ik} + h_{hk} h_{ij} \right\}
\]

(B) \[
T_{hijk} = h_{hi} p_{jk} + h_{hj} p_{ik} + h_{hk} p_{lj} + h_{lk} p_{lj} + h_{lj} p_{hk} + h_{jk} p_{hi}
\]

(C) \[
T_{hijk} = a_n c_{ijk} + a_n c_{hjk} + a_n c_{hij} + a_n h_{i} Q_{h} + h_{ij} Q_{ik} + h_{i} Q_{hj} + h_{hj} Q_{h} + h_{jk} Q_{hi}
\]

Where \( p \) is a certain scalar and \( h_{ij} \) is an angular metric tensor defined by

\[
h_{ij} = g_{ij} - I_i I_j
\]

\( p_{ij} \) and \( h_{ij} \) are components of certain tensor fields and \( a_n \) are components of a covariant vector. Examples of Finsler space having these forms of \( T_{hijk} \) are given in our previous paper.

In this chapter, we shall study the properties of non-Riemannian Finsler space in which \( T_{hijk} \) is of the form where \( p \) is a scalar function, positively homogeneous of degree three in \( y^i \) \( a_n \) are components of covariant vector field which is positively homogeneous of degree two in \( y^i \) and \( C_{ij} \) is torsion vector.
S2. FINSLER SPACE WITH T-TENSOR OF THE FORM (1.2):

First of all we shall discuss the two dimensional Finsler space \( f^2 \). With reference of Berwald’s frame \(( l_i, m_i)\) the angular metric tensor and \( h \) the torsion tensor are given by (Ikeda 1979)

\[
(2.1) \quad h_{\gamma \nu} = m_\gamma, \quad m_\gamma, \quad l, C_{\nu k} = l m_\nu m_j m_k
\]

where \( l \) is the principal scalar.

From (1.1) the (2.1) it follow that T-tensor of \( F^2 \) is given by

\[
LT_{\nu k} = l_\sigma m_\nu m_j m_k
\]

where

\[
(2.2) \quad l_\sigma = L \frac{\partial 1}{\partial y^k} m^k
\]

Since \( m_i = \frac{C_i}{C} \), where \( C_i = C\cdot C_i^0 g^0 \), from (2.1), it follow that \( l = l C \).

Thus, if we write

\[
(2.3) \quad l \alpha_i = \alpha_i l_i + \alpha_i m_i
\]

then from (1.2) and (2.2) it follow that

\[
(2.4) \quad \alpha_i = 0, \quad l_\sigma = L, C^4 + 4 \alpha_i C_i^3.
\]

Hence

**Theorem 2.1**: In a two dimensional Finsler Space \( F^2 \) the T-tensor \( T_{\nu k} \) can be expressed in the form (1.2) and scalar components \( \alpha_i \) and \( \alpha_2 \) satisfy the relations given in (2.4).

Motsumoto (1973) [6] developed the theory of three dimensional Finsler Space referring to the orthogonal frame \( e_{(i,\alpha)} \), \( \alpha = 1,2,3^\alpha \) with reference to this frame the tensor \( C_{hij} \) is written as
\[ (2.5) \quad L_{\alpha}^j = C_{\alpha \beta \gamma} e^\alpha (\omega) i e^\beta (\tau) j \]

where the scalar components \( C_{\alpha \beta \gamma} \) are such that

\[ (2.6) \quad C_{\alpha \beta \gamma} = 0, \quad C_{I I}^{22} = H, \quad C_{I 33} = I, \quad C_{333} = J. \]

The scalars \( H, I \) and \( J \) are called main scalars and satisfy the equation

\[ (2.7) \quad H + I = L C. \]

The scalar components of \( LT_{\alpha \beta \gamma} \) are given by (Singh et al. 1982) [10].

\[ (2.8) \quad LT_{\alpha \beta \gamma} = \left[ C_{\alpha \| \alpha} + C_{\| \beta} \delta_{\| \gamma} + C_{\| \beta} \delta_{\| \gamma} + C_{\| \beta} \delta_{\| \gamma} \right] e_{\| \alpha} e_{\| \beta} e_{\| \gamma} \]

where the semicolon denotes the \( v \)-scalar derivative (Motsumoto 1973) [6]. We shall use the following relations which have been obtained by Motsumoto (1973) [6].

\[ (2.9) \quad \]

\[ \begin{align*}
C_{I I}^{222} &= -J \beta + (H - 2I) \nu \delta \\
C_{I 33} &= J \beta + 3I \nu \delta \\
C_{333} &= J \beta + 3I \nu \delta
\end{align*} \]

where \( \nu \delta \) are the scalar components of \( v \)-connection vector (Motsumoto 1973)

Let \( a_i \) be the scalar components of \( L a_i \)

\[ (2.10) \quad L a_i = a_i e^i (\omega) \]

We assume that the \( T \)-tensor of the space \( F^3 \) have the from (1.2) then in view of the relation (2.10), we have

\[ (2.11) \quad LT_{\alpha \beta \gamma} = C^3 \left[ C L_\rho \delta_{2 \alpha} \delta_{2 \beta} \delta_{2 \gamma} + a_i \delta_{2 \beta} \delta_{2 \gamma} \right] e_{\| \alpha} e_{\| \beta} e_{\| \gamma} \]

\[ + \alpha_{\| \beta} \delta_{2 \alpha} \delta_{2 \gamma} \delta_{2 \delta} + \alpha_{\| \gamma} \delta_{2 \alpha} \delta_{2 \beta} \delta_{2 \delta} \]

\[ + \alpha_{\| \delta} \delta_{2 \alpha} \delta_{2 \beta} \delta_{2 \gamma} \] \( e_{\| \alpha} e_{\| \beta} e_{\| \gamma} e_{\| \delta} \)

Comparing (2.8) with (2.11), we get
of the form (1.2), the v-connection vector vanishes if the scalar component $a_3$ of $L_{ai}$ vanishes.

Now we shall give examples of n-dimensional Finsler space $(n > 2)$ whose T-tensor is of the form (1.2).

A $C^2$-like Finsler space is defined as an $n$ ($n > 2$) dimensional Finsler space in which $(h)$ hv-torsion tensor is of the (Motsumoto and Numata 1980)

\begin{equation}
C_{hij} = \frac{1}{C^2} C^h C^i C^j
\end{equation}

Since $C_{hij} \mid _k - C^i \mid _h = 0$, from (2.15), we have

\begin{equation}
C^h C^j \Gamma^i _{ik} + C^h C^i \Gamma^j _{i} - C^h C^j \Gamma^h _{i} - C^k C^i \Gamma^h _{i} = 0
\end{equation}

where

\begin{equation}
\Gamma^i _{ik} = C^i \mid _k - \frac{1}{2C^4} C^2 \mid _k C^i.
\end{equation}

Since $2 C^i \mid _k C^i = C^2 \mid _k$, from (2.17), we have

\begin{equation}
\Gamma^i _{ik} C^i = 0 \text{ and } \Gamma^i _{ik} C^k = \frac{1}{2} \left( C^2 \mid _i - \frac{1}{C^2} C^2 \mid _k C^k C^i \right).
\end{equation}

Contracting (2.16) with $g^{ih}$ and using (2.18) we get

\begin{equation}
C^2 \Gamma^i _{ik} = \alpha C^2 C^i \mid C^k + \frac{1}{2} C^k \mid C^2 \mid _i
\end{equation}

where

\begin{equation}
\alpha C^2 = \Gamma^i _{ik} g^{ik} - \frac{1}{2C^2} C^2 \mid j C^i.
\end{equation}

By virtue of equs. (2.17) and (2.19), we have the following:

**Lemma 2.1** — In a $C^2$-like Finsler space there exists a scalar $\alpha$ such that

\begin{equation}
C^i \mid j = \alpha C^i \mid C^j + \frac{1}{C} \left( C^j \mid C^i + C^j \mid C^i \right).
\end{equation}
\[ (2.12) \quad C_{\alpha\beta\gamma\delta} + C_{\beta\gamma\delta} \delta_{1\alpha} + C_{\alpha\gamma\delta} \delta_{1\beta} + C_{\alpha\beta\delta} \delta_{1\gamma} = C^4 \delta_{2\alpha} \delta_{2\beta} \delta_{2\gamma} \delta_{2\delta} + C^3 (\alpha_{\alpha} \delta_{2\beta} \delta_{2\gamma} \delta_{2\delta} + \delta_{2\alpha} \delta_{2\beta} \delta_{2\gamma} \delta_{2\delta}) + \alpha_{\gamma} \delta_{2\alpha} \delta_{2\beta} \delta_{2\delta} + \alpha_{\delta} \delta_{2\alpha} \delta_{2\beta} \delta_{2\gamma} \delta_{2\delta}. \]

Since \( T_{hijk} \) is an indicatory tensor, (1.2) it follow that \( \alpha_{i} y^{i} = 0 \), which in view of (2.10) gives \( \alpha_{1} = 0 \). Thus equations (2.6), (2.9) and (2.12) yield

\[ (2.13) \quad \begin{align*}
(\text{a}) & \quad \mathcal{H}_{\rho} + 3J_{\nu^{\rho}} = C^3 (L_{\rho} C + 3 \alpha_{2}) \delta_{2\alpha} + C^3 \alpha_{\delta} \\
(\text{b}) & \quad -J_{\nu^{\delta}} + (H - 2I)_{\nu^{\delta}} = C^3 \alpha_{3} \delta_{2\delta} \\
(\text{c}) & \quad I_{\nu^{\delta}} - 3J_{\nu^{\delta}} = 0 \\
(\text{d}) & \quad J_{\nu^{\delta}} + 3I_{\nu^{\delta}} = 0.
\end{align*} \]

These equations give

\[ \alpha_{1} = 0, \quad \alpha_{2} = 1/4 \left( \frac{L}{C^3} \right. C_{i2} \omega - L C_{\rho}, \right) \]

\[ (2.14) \quad \alpha_{3} = \frac{L}{C^2} v_{2} \text{ and } v_{3} = 0. \]

Hence we have the following:

**Theorem 2.2** — In a three dimensional Finsler space if the \( T \)-tensor is

of the form (1.2) then scalar component \( v_{i} \) of \( \nu \)-connection vector vanishes and

the scalar components \( a_{\alpha} \) of \( L_{ai} \) are given by (2.14).

Since in any three dimensional Finsler space \( v_{i} = 0 \) (Motsumoto 1973)[6].

Theorem 2.2 gives the following:

**Theorem 2.3** — In a three dimensional Finsler space with the \( T \)-tensor
Now we are in position to prove the following

**Theorem 2.4** — The T-tensor of a C2-like Finsler space is of the form (1.2).

**Proof:** The v-covariant differentiation of (2.15) gives

\[ C_{hij|k} = \frac{1}{C^2} \left( C_{h|k} C_i C_j + C_{i|k} C_h C_j + C_{i|k} C_h C_i \right) \]

\[ - \frac{1}{C^3} C^2 |k C_h C_i C_j \]

which in view of Lemma 2.1, gives

\[ (2.21) \quad C_{hij|k} = \frac{3\alpha}{C^2} C_h C_i C_j C_k + \frac{1}{C^3} \left( C_{|h} C_i C_j C_k + C_{|i} C_h C_j C_k \right) \]

\[ + C_{|k} C_h C_i C_j + C_{|k} C_h C_i C_j \]

Equation (1.1), (2.15) and (2.21) give the form (1.2), where

\[ \rho = \frac{3\alpha L}{C^2}, \quad \alpha = \left( \frac{LC_i}{C^3} + \frac{L_i}{C^3} \right). \]

There are other examples of a Finsler space whose T-tensor is of the form (2.1). If L is a metric function of a two dimensional Finsler space $F^2$ and $L^*$ is the metric function of $(n - 2)$ dimensional Riemannian space $R^{n-2}$ then the Finsler space $F^2 \times R^{n-2}$ with metric $\sqrt{L + L^*}$ is C2-like Finsler space (Motsumoto & Numata 1980). [9] Hence by virtue of Theorem 2.4 the T-tensor of Finsler space $F^2 \times R^{n-2}$ is of the form (1.2).

§ 3.2 Like Finsler Space

In this section we shall deal the particular form of (1.2) in which $a_h$ is a null vector. Firstly we shall prove the following.

**Theorem 3.1** — If the T-tensor $T_{hijk}$ is written in decomposed form
\[ T_{hijk} = C_{i} \ T_{ijk} \]

with \( C_{i} \neq 0 \) then Thijk is written in the form

\[ (3.1) \quad T_{hijk} = \rho \ C_{i} \ C_{j} \ C_{k} \]

**Proof:** Since \( T_{hijk} \) is symmetric in all its four indices, \( T_{ijk} \) is a symmetric tensor in all its indices. Since \( C_{i} \) is non-vanishing vector, we may suppose \( C_{1} \neq 0 \), then \( T_{ijk} = T_{i,jk} \) implies that \( C_{1} \ T_{ijk} = C_{1} \ T_{i,jk} \), which gives

\[ (3.2) \quad T_{ijk} = C_{i} \ T_{jk} \]

where

\[ T_{jk} = T_{i,jk} / C_{1} \]

In view of relation (3.2) the identity \( T_{ijk} = T_{i,jk} \) gives \( C_{1} \ T_{jk} = C_{j} \ T_{ik} \), which gives \( T_{jk} = C_{j} \ T_{i,k} \), where \( T_{k} = T_{ij} / C_{1} \).

Since \( T_{jk} \) is symmetric tensor, we have

\[ T_{ik} - T_{ki} = C_{1} \ T_{k} - C_{k} \ T_{1} = 0. \]

Thus

\[ T_{k} = \rho \ C_{k} \]

where \( \rho = T_{1} / C_{1} \).

Hence we get eqn. (3.1)

By virtue of eqn. (2.2) we have the following

**Theorem 3.2** — In any non-Riemannian Finsler space \( F^{2} \) the \( T \)-tensor is of the form (3.1) where \( \rho = T_{1} / C_{1} \).

In view of this theorem we give the following definition:

**Definition** — A non-Riemannian Finsler space \( F^{n} \) (\( n > 2 \)) is called T2-like Finsler space, if the \( T \)-tensor \( T_{hijk} \) is written in the form (3.1).

In order to discuss the properties of three-dimensional T2-like Finsler space, we shall use the equation (2.13). Since \( \alpha_{h} \) is a null vector its scalar components \( a_{u} \) vanishes and the equation (2.13) yield.

(a) \[ H_{\delta} + 3J_{\nu \delta} = L_{\rho} \ C^{4} \delta_{\nu \delta} \]
(b) $- J_{,\delta} + (H - 2I)_{,\delta} = 0$

(c) $I_{,\delta} - 3J_{\nu,\delta} = 0$

(d) $J_{,\delta} + 3J_{\nu,\delta} = 0$

These equations give

$v_\delta = 0, \; I_{;\delta} = 0, \; J_{;\delta} = 0$

and

(3.4) $H_{,\delta} = L_C^{i} C_{,\delta}^{i}$

**Theorem 3.3** — In a three dimensional T2-like Finsler space the v-connection vector vanishes identically and the main scalar I, J are functions of position (co-ordinate) only. The main scalar H satisfies the equation

$H_{,\delta} = L_C^{i} C_{,\delta}^{i}$

where

$$\delta_{2\delta} = \begin{cases} 
0 & \text{when } \delta \neq 2 \\
1 & \text{when } \delta = 2 
\end{cases}$$

If $\rho = 0$, then from (3.1), we get $T_{,ijk} = 0$ i.e. the Finsler space satisfies T-condition (Motsumoto 1975). [8] Thus by virtue of Theorem 3.3

**Corollary 3.1** — The T-condition, for a 3-dimensional non-Riemannian T2-like Finsler space, is equivalent to the fact that the v-connection vector $v_j$ vanishes identically and the main scalar H, I and J are functions of position only.

Now, we shall discuss the properties of n-dimensional $(n > 3)$ T2-like Finsler space.

Contracting (3.1) with $g^{ik}$ and using (1.1), we get

(3.5) $L_C^{i} C_{,ij} = \rho \; C^2 \; C_i C_j - I_i C_j - I_j C_i$
Again contracting (3.5) with $C^i$ and using the relation $2C^i_{[j} C^j = C^2_{[j}$, we get

$$C_{[j} = \lambda C_j + \mu I_j$$

where

$$\lambda = \rho \frac{C^3}{L} \quad \text{and} \quad \mu = -\frac{C}{L}$$

Hence we have the following

**Theorem 3.4** - In a T2-like Finsler space relation (3.5) holds and $C_{[j}$ is a linear combination of $C_j$ and $I_j$.

For the T2-likeness of C2-like Finsler space, we have the following:

**Theorem 3.5** - C2-like Finsler space is T2-like if condition (3.5) holds.

**Proof**: The necessary part of the theorem follows from the theorem (3.4)

Conversely, if (3.5) holds and $F^a$ is C2-like then $\nu$-covariant differentiation of (2.15) gives

$$C_{hijk} = \frac{1}{C^2} \left( C_{hikj} C_j C_i + C_{ijk} C_h C_j + C_{hjk} C_h C_j \right)$$

$$- \frac{1}{C^4} C_{hijk} C_h C_i C_j$$

Contracting (3.5) with $C^i$ and using the relation $2C^i_{[j} C^j = C^2_{[j}$, we get

$$LC^2_{[j} = 2C^2 \left( \rho C^2 C_j - I_j \right).$$

Substituting (2.15), (2.19) and (3.6) and (3.7) in (1.1), we get

$$T_{hijk} = \rho C_h C_i C_j C_k$$
Hence $F^n$ is T2-like Finsler space.

**Theorem 3.6** - A C2-like Finsler space is T2-like if $C_j|_k$ is a linear combination of $I_k$ and $C_k$.

**Proof**: The necessary part follows from Theorem 3.4.

Conversely, if $C_j|_i$ is a linear combination of $C_i$ and $I_j$, then $C_j|_i = \lambda C_j + \mu I_j$ for some scalar $\lambda$ and $\mu$.

Since $C$ is positively homogeneous of degree $-1$ in $y^i$ contracting the above equation with $y^i$ we get $\mu = \frac{C}{L}$. Thus we have

\[ (3.8) \quad C_j|_i - \lambda C_j = \frac{C}{L} I_i. \]

Substituting (2.15), (2.20), (3.6) and (3.8) in (1.1), we get

\[ T_{hijk} = \rho C_h C_i C_j C_k \]

where

\[ \rho = \frac{3L\alpha}{C^2} + \frac{4\alpha L}{C^3} \]

**Theorem 3.7** - If a T2-like Finsler space is e-reducible then it satisfies T-condition.

**Proof**: A C-reducible Finsler space $F^n$ is a non-Riemannian Finsler space in which (h) hv-torsion tensor is of the form (Motsumoto 1972 b)\[5\]

\[ C_{hik} = \frac{1}{(n+1)} \left( C_h h_{ij} + C_i h_{ki} + C_j h_{ki} \right). \]

In a C-reducible Finsler space there exists scalar $M$ such that (Motsumoto 1974) \[7\]

\[ (3.9) \quad T_{hijk} = M (h_{ij} h_{hk} + h_{hi} h_{ik} + h_{hj} h_{hk}). \]
By virtue of eqns. (3.1) and (3.9), we get

\[ M ( h_{ij} h_{ik} + h_{ji} h_{jk} + h_{ki} h_{ik} ) = \rho C_i C_j C_k C_k \]

which after contraction with \( g_{hk} \), gives

\[ M (n + 1) h_{ij} = \rho C_i C_j \]

Since the rank of \( h_{ij} \) is \((n - 1)\) and the rank of \( C_i C_j \) is 1, therefore for \( n > 3 \), eqn. (3.10) is valid only when \( \rho = M = 0 \).

This proves the theorem.

The v-curvature tensor \( S_{hijk} \) is given by

\[ S_{hijk} = C_{hkr} C_{ij}^r - C_{hjr} C_{ik}^r. \]

The v-covariant differentiation of above equation and the application of equation (1.1) and (3.1), give

\[ \begin{align*}
L S_{hijk} \mid^1 &= (C_{ij} C_h C_k + C_{hijk} C_i C_j - C_h C_j - C_{hj} C_i C_k) C_i \\
- 2 l_i S_{hijk} - l_i S_{lijk} - l_i S_{hijk} - l_j S_{hijk} - l_k S_{hijk}
\end{align*} \]

where the dot denotes the contraction with \( C^r \). The indicatorised tensor \( T_{hijkl} \) of \( S_{hijk} \mid^1 \) is defined as

\[ T_{hijkl} = L S_{hijk} \mid^1 + 2 l_i S_{hijk} + l_h S_{lijk} + l_i S_{hijk} + l_j S_{hijk} + l_k S_{hijkl}. \]

Equation (3.11) gives the following:

**Theorem 3.8** — The indicatorised tensor \( T_{hijk} \) of \( L S_{hijk} \mid^1 \) for a T2-like Finsler space is of the form.

\[ T_{hijkl} = (C_{ij} C_h C_k + C_{hkk} C_i C_j - C_{ik} C_h C_j - C_{hj} C_i C_k) C_l. \]
Contracting (3.11) with $g^{hk}$ we get

\[(3.12) \quad L S_{hij} = \rho [ C^2 (C_{ij} + C_i C_j) - (C_i \cdots C_{hijk} C_j \cdots C_i) C_j \cdots \]

\[-2L_{ij} S_{il} - L_{ij} S_{il} - L_{ij} S_{il} \]

where

\[ S_{ij} = S_{h ij} g^{hk}. \]

Again contracting (a) with $g^{ij}$ we get

\[(3.13) \quad L S_{ij} = 2\rho (C^4 - C_i) C_i - 2L_{ij} S \]

where

\[ S = S_{ij} g^{ij}. \]

This gives the following

**Theorem 3.9**: In a T2-like Finsler space $F^n$ if the v-scalar $S$ vanishes identically and $C^4 \neq C$., then $F^n$ satisfies T-condition.

An S3-like Finsler space characterised by the relation (Fukui and Yamada 1979) [1].

\[(3.14) \quad S_{hijk} = \frac{S}{(n-1)(n-2)} (h_{hk} h_{ij} - h_{ij} h_{ik}) \]

For an S3-like Finsler space the function $L^2 S$ is a function of co-ordinates only (Fukui and Yamada 1979) [1]. Hence $(L^2 S)_{ij} = 0$. This result and (3.12) give

\[ \rho (C^4 C \cdots C_i = 0. \]

Therefore, we get:

**Theorem 3.10**: If a T2-like finsler space $F^n$ is S3-like and $C^4 \neq C$, then $F^n$ satisfies T-condition.

Since in every three dimensional Finsler space (3.13) holds, therefore, we get
Corollary 3.2: A T2-like 3-dimensional Finsler space with $C^1 \subset C^2$ is a Finsler space satisfying T-condition.

84. REFERENCES: