CHAPTER 3

GENERALIZED $\alpha$ b-CLOSED SETS IN TOPOLOGICAL SPACES

3.1 INTRODUCTION

In 1970, Levine introduced generalized closed sets in topology. Andrijevic (1996) introduced generalized open sets in a topological space called, b-open sets. These types of sets are discussed by Ekici and Caldas (2004) under the name $\gamma$-open sets. The class of b-open set is contained in the class of semi-pre-open sets and contains all semi-open sets and pre-open sets. Since the advent of these notions, several researches have been done which produced interesting results.

The aim of the present chapter is to continue the study of generalized b-closed sets. The notion of generalized $\alpha$ b-closed sets as introduced and its various characterizations are investigated. Also, $T_{g\alpha b}$-spaces has been introduced in topological spaces and its nature, properties, theorem discussed with examples.

3.2 GENERALIZED $\alpha$ b-CLOSED SETS

The present section gives the definition of generalized $\alpha$ b-closed set and investigates some of its properties.
**Definition 3.2.1:** A subset $A$ of a topological space $X$ is called generalized $\alpha b$-closed (briefly $g \alpha b$-closed) set, if $\text{bcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is an $\alpha$-open. The collection of all the $g \alpha b$-closed sets in $X$ are denoted by $g \alpha b$-C$(X)$.

**Theorem 3.2.2:** Let $A$ be a $g \alpha b$-closed subset of $(X, \tau)$, $\text{bcl}(A) - A$ then does not contain any non-empty $\alpha$-closed sets.

**Proof:** Necessary: Suppose $F$ is a non-empty $\alpha$-closed subset of $X$ such that $F \subseteq \text{bcl}(A) - A$. Now $F \subseteq \text{bcl}(A) - A \Rightarrow F \subseteq \text{bcl}(A) \cap A^c$,

$\Rightarrow F \subseteq \text{bcl}(A)$ and $F \subseteq A^c$

$\Rightarrow A \subseteq F^c$.

Since $F^c$ is $\alpha$-open and $A$ is $g \alpha b$-closed, $\text{bcl}(A) \subseteq F^c \Rightarrow F \subseteq (\text{bcl}(A))^c$. Thus, $F \subseteq (\text{bcl}(A)) \cap (\text{bcl}(A))^c = \phi$. That is $F = \phi$. Implies $\text{bcl}(A) - A = \phi$ contains no non-empty $\alpha$-closed set.

Sufficient: Let $A \subseteq U$ and $U$ is an $\alpha$-open, then $\text{bcl}(A) \subseteq U$. Suppose that $\text{bcl}(A)$ does not contain in $U$, then $\text{bcl}(A) \cap U^c$ is a non-$\alpha$-empty closed set of $\text{bcl}(A) - A$, which is a contradiction. Therefore, $\text{bcl}(A) \subseteq U$. Hence, $A$ is $g \alpha b$-closed.

**Theorem 3.2.3:** Let $A$ be a $g \alpha b$-closed set, then $A$ is gb-closed if and only if $\text{bcl}(A) - A = \phi$ is closed.

**Proof:** Necessary: Assume that $A$ is $g \alpha b$-closed. Since $\text{bcl}(A) = A$, $\text{bcl}(A) - A = \phi$ is gb-closed and hence closed.

Sufficient: Conversely, assume that $\text{bcl}(A) - A$ be closed. By the above theorem, $\text{bcl}(A) - A$ does not contain any non-empty $\alpha$-closed set. That is $\text{bcl}(A) - A = \phi$, so $A = \text{bcl}(A)$. Therefore, $A$ is gb-closed.
Theorem 3.2.4: Let $A$ be a $g\alpha$ b-closed and suppose that $F$ is an $\alpha$-open, then $A \cap F$ is $g\alpha$ b-closed.

Proof: To show that $A \cap F$ is $g\alpha$ b-closed one has to show that $\text{bcl}(A \cap F) \subseteq U$, where $U$ is an $\alpha$-open and $A \cap F \subseteq U$. Now $\text{cl}(\text{int}(\text{cl}(A \cap F))) \subseteq A \cap F$, $\text{cl}(\text{int}(\text{cl}(A \cap F))) \subseteq A \cap F \subseteq U$. Implies that $\text{cl}(\text{int}(\text{cl}(A \cap F))) \subseteq U$. Thus, $\text{cl}(\text{int}(A \cap F)) \subseteq U$ and $\text{int}(\text{cl}(A \cap F)) \subseteq U$, as $U$ is an $\alpha$-open. Now, $\text{cl}(\text{int}(A \cap F)) \cup \text{int}(\text{cl}(A \cap F)) \subseteq U$, that is $\text{bcl}(A \cap F) \subseteq U$. Hence, proved.

Theorem 3.2.5: Suppose that $B \subseteq A \subseteq X$, $B$ is $g\alpha$ b-closed set relative to $A$ and $A$ is $g\alpha$ b-closed set in $X$, then $B$ is $g\alpha$ b-closed set relative to $X$.

Proof: Let $B \subseteq U$ and $U$ be an $\alpha$-open set in $X$. Given that $B \subseteq A \subseteq X$, then $B \subseteq A$ and $B \subseteq U$, that is $B \subseteq A \cap U$. Since $B$ is $g\alpha$ b-closed set relative to $A$, $\text{bcl}(B) \subseteq A \cap U \subseteq U$. Therefore, $A \cup (\text{bcl}(B)) \subseteq U$.

Now $(A \cup (\text{bcl}(B))) \cap (\text{bcl}(B))^C \subseteq U \cap (\text{bcl}(B))^C$

$\Rightarrow (A \cap (\text{bcl}(B))^C) \cup ((\text{bcl}(B)) \cap (\text{bcl}(B))^C) \subseteq U \cap (\text{bcl}(B))^C$

$\Rightarrow (A \cap (\text{bcl}(B))^C) \subseteq U \cap (\text{bcl}(B))^C$

Now $A$ is $g\alpha$ b-closed set and $B \subseteq A$

$\text{bcl}(B) \subseteq \text{bcl}(A) \subseteq U \cap (\text{bcl}(B))^C$

$\Rightarrow \text{bcl}(B) \subseteq U \cap (\text{bcl}(B))^C$

$\Rightarrow \text{bcl}(B) \subseteq U$, but not contain in $(\text{bcl}(B))^C$

$\Rightarrow B$ is $g\alpha$ b-closed set relative to $X$.

Theorem 3.2.6: If a subset $A$ is $g\alpha$ b-closed and $A \subseteq B \subseteq \text{bcl}(A)$, then $B$ is $g\alpha$ b-closed set.
Proof: Let $B \subseteq U$, $U$ is an $\alpha$-open, then $A \subseteq B$ and $A \subseteq U$. Since $A$ is $g\alpha$ b-closed, $\text{bcl}(A) \subseteq U$, but $B \subseteq \text{bcl}(A)$, implies $\text{bcl}(B) \subseteq \text{bcl}(A)$. Therefore, $\text{bcl}(B) \subseteq \text{bcl}(A) \subseteq U$. Thus $\text{bcl}(B) \subseteq U$ and $U$ is an $\alpha$-open. Hence, $B$ is $g\alpha$ b-closed.

Theorem 3.2.7: Let $A \subseteq Y \subseteq X$ and suppose that $A$ is $g\alpha$ b-closed set in $X$, $A$ is then $g\alpha$ b-closed set relative to $Y$.

Proof: Given that $A \subseteq Y \subseteq X$ and $A$ is $g\alpha$ b-closed set in $X$, to show that $A$ is $g\alpha$ b-closed set relative to $Y$. Let $A \subseteq Y \cap U$, where $U$ is an $\alpha$-open in $X$, then $\text{bcl}(A) \subseteq U$ and $\text{bcl}(A) \cap Y \subseteq Y \cap U$. Therefore, $\text{bcl}(A) \cap Y$ is the b-closure of $A$ in $Y$. Thus, $A$ is $g\alpha$ b-closed set relative to $Y$.

Theorem 3.2.8: The intersection of any two subsets of $g\alpha$ b-closed sets in $X$ is $g\alpha$ b-closed.

Proof: Let $A$ and $B$ be the subsets of $g\alpha$ b-closed sets, $A \subseteq U$ and $\text{bcl}(A) \subseteq U$, $B \subseteq U$ and $\text{bcl}(B) \subseteq U$, $U$ is an $\alpha$-open. Therefore, $A \cap B \subseteq A$ and $\text{bcl}(A \cap B) \subseteq \text{bcl}(A)$, $A \cap B \subseteq B$ and $\text{bcl}(A \cap B) \subseteq \text{bcl}(B)$. Hence, $\text{bcl}(A \cap B) \subseteq U$ and $U$ is an $\alpha$-open. Thus, $A \cap B$ is $g\alpha$ b-closed set.

Remark 3.2.9: If the subsets $A$ and $B$ are $g\alpha$ b-closed sets, their union need not be $g\alpha$ b-closed set.

Example 3.2.10: Let $X = \{a, b, c\}$ with $\tau = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}\}$. In this topological space $(X, \tau)$, the subsets $\{a\}$ and $\{c\}$ are $g\alpha$ b-closed, but their union $\{a, c\}$ is not $g\alpha$ b-closed.

Theorem 3.2.11: If a subset $A$ of a topological space $X$ is nowhere dense, it is then $g\alpha$ b-closed.

Proof: Suppose a set $A$ is nowhere dense, it is then $\text{int}(\text{cl}(A)) = \emptyset$. It is obvious that $\text{bcl}(A) \subseteq \text{cl}(A)$ and also $\text{bcl}(A) \subseteq \text{int}(\text{bcl}(A)) \subseteq \text{int}(\text{cl}(A))$. 

Therefore \( \text{int} (\text{cl}(A)) = \phi \), which implies \( \text{bcl}(A) = \phi \). Thus, \( A \) is \( \alpha b \)-closed in \( X \).

**Remark 3.2.12:** The converse of the above theorem need not be true as seen from the following example.

**Example 3.2.13:** Consider \( X = \{a, b, c\} \), with a topology \( \tau = \{X, \phi, \{a\}, \{b\}, \{a,b\}\} \). In this topological space \( (X, \tau) \), the subset \( \{a\} \) is \( \alpha b \)-closed but not nowhere dense.

**Theorem 3.2.14:** If a subset \( A \) of a topological space \( X \) is \( \alpha b \)-closed, it is then \( gb \)-closed.

**Proof:** Suppose \( A \) is a \( \alpha b \)-closed set in \( X \). Since every open set is \( \alpha \)-open sets, \( U \) is an open set. Therefore, \( \text{bcl}(A) \subseteq U \) and \( U \) is an open. Thus, \( A \) is \( gb \)-closed.

**Remark 3.2.15:** The converse of the above theorem need not be true as seen from the following example.

**Example 3.2.16:** Consider \( X = \{a, b, c\} \) with topology \( \tau = \{X, \phi, \{a\}, \{a,c\}\} \). In this topological space \( (X, \tau) \), the subset \( \{a, b\} \) is \( gb \)-closed but not \( \alpha b \)-closed.

**Theorem 3.2.17:** If a subset \( A \) of a topological space \( X \) is \( \alpha b \)-closed, it is then \( gp \)-closed.

**Proof:** Suppose \( A \) is a \( \alpha b \)-closed set. Since every pre-closed set is \( b \)-closed, \( \text{bcl}(A) \subseteq \text{pcl}(A) \subseteq U \). Therefore, \( \text{pcl}(A) \subseteq U \) and \( U \) is open. Thus, \( A \) is \( gp \)-closed set in \( X \).

**Remark 3.2.18:** The converse of the above theorem need not be true as seen from the following example.
**Example 3.2.19:** Consider $X = \{a, b, c\}$ with topology $\tau = \{X, \emptyset, \{a\}\}$. In this topological space $(X, \tau)$, the subset $\{a, b\}$ is gp-closed but not $g\alpha b$-closed.

**Theorem 3.2.20:** If a subset $A$ of a topological space $X$ is $g\alpha$-closed, it is then $g\alpha b$-closed.

**Proof:** Let $A$ be a $g\alpha$-closed. Now $bcl(A) \subseteq cl_\alpha(A) \subseteq U$, implies that $bcl(A) \subseteq U$ and $U$ is an $\alpha$-open. Thus, $A$ is $g\alpha b$-closed.

**Remark 3.2.21:** The converse of the above theorem need not be true as seen from the following example.

**Example 3.2.22:** Let $X = \{a, b, c\}$ with $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. In this topological space $(X, \tau)$, the subset $\{a\}$ is $g\alpha b$-closed but not $g\alpha$-closed.

**Theorem 3.2.23:** If a subset $A$ of a topological space $X$ is $\alpha g$-closed, it is then $g\alpha b$-closed.

**Proof:** Let $A$ be $\alpha g$-closed. Since every open sets are $\alpha$-open sets and also $bcl(A) \subseteq cl_\alpha(A) \subseteq U$. Therefore, $bcl(A) \subseteq U$ and $U$ is an $\alpha$-open. Thus, $A$ is $g\alpha b$-closed.

**Remark 3.2.24:** The converse of the above theorem need not be true as seen from the following example.

**Example 3.2.25:** Let $X = \{a, b, c\}$ with $\tau = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}\}$. In this topological space $(X, \tau)$, the subset $\{a\}$ is $g\alpha b$-closed, but not $\alpha g$-closed.

**Theorem 3.2.26:** If a subset $A$ of a topological space $X$ is $g\alpha b$-closed, it is then $gs$-closed.

**Proof:** Let $A$ be a $g\alpha b$-closed and $A \subseteq U$. Since every semi-closed set is $b$-closed set, $bcl(A) \subseteq scl(A) \subseteq U$. Therefore, $scl(A) \subseteq U$ and $U$ is an open. Thus, $A$ is $gs$-closed.
Remark 3.2.27: The converse of the above theorem need not be true as seen from the following example.

Example 3.2.28: Let \( X = \{a, b, c\} \) with \( \tau = \{X, \phi, \{a\}\} \). In this topological space \( (X, \tau) \), the subset \( \{a, b\} \) is gs-closed, but not g \( \alpha \) b-closed set.

Theorem 3.2.29: If a subset \( A \) of a topological space \( X \) is sg-closed, it is then g \( \alpha \) b-closed.

Proof: Let \( A \) be a sg-closed, \( \text{scl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is semi-open. Since every semi-closed set is b-closed, \( \text{bcl}(A) \subseteq U \), \( A \subseteq U \) and \( U \) is an \( \alpha \)-open. Therefore \( A \) is g \( \alpha \) b-closed.

Remark 3.2.30: The converse of the above theorem need not be true as seen from the following example.

Example 3.2.31: Let \( X = \{a, b, c\} \) with \( \tau = \{X, \phi, \{b, c\}\} \). In this topological space \( (X, \tau) \), the subset \( \{b\} \) is g \( \alpha \) b-closed, but not sg-closed.

Theorem 3.2.32: If a subset \( A \) of a topological space \( X \) is g \( \alpha \) b-closed, it is then gpr-closed.

Proof: Let \( A \) be a g \( \alpha \) b-closed. Since every pre-closed set is b-closed, \( \text{bcl}(A) \subseteq \text{pcl}(A) \subseteq U \). Therefore, \( \text{pcl}(A) \subseteq U \) and \( U \) is regular open. Thus, \( A \) is gpr-closed.

Remark 3.2.33: The converse of the above theorem need not be true as seen from the following example.

Example 3.2.34: Let \( X = \{a, b, c\} \) with \( \tau = \{X, \phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}\} \). In this topological space \( (X, \tau) \), the subset \( \{a\} \) is gpr-closed but not g \( \alpha \) b-closed.

Theorem 3.2.35: A subset \( A \) of a topological space \( X \) is swg-closed, it is then g \( \alpha \) b-closed.
**Proof:** Let $A$ be a swg-closed set in $X$. Since every semi-closed set is $b$-closed. Therefore $\text{bcl}(A) \subseteq U$ and $U$ is an $\alpha$-open. Thus, $A$ is $g\alpha$ b-closed.

**Remark 3.2.36:** The converse of the above theorem need not be true as seen from the following example.

**Example 3.2.37:** Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{c\}, \{a, c\}\}$. In this topological space $(X, \tau)$, the subset $\{a\}$ is $g\alpha$ b-closed, but which is not swg-closed.

**Remark 3.2.38:** The following examples show that $g\alpha$ b-closed and g-closed sets are independent.

**Example 3.2.39:** Let $X = \{a, b, c\}$ with topology $\tau = \{X, \phi, \{a\}\}$. In this topological space $(X, \tau)$, the subset $\{a, b\}$ is g-closed, but not $g\alpha$ b-closed set.

**Example 3.2.40:** Let $X = \{a, b, c\}$ with topology $\tau = \{X, \phi, \{a\}, \{c\}, \{a, c\}\}$. In this topological space $(X, \tau)$, the subset $\{a\}$ is $g\alpha$ b-closed, but not g-closed.

**Remark 3.2.41:** The following examples show that $g\alpha$ b-closed and wg-closed sets are independent.

**Example 3.2.42:** Let $X = \{a, b, c\}$ with topology $\tau = \{X, \phi, \{a\}\}$. In this topological space $(X, \tau)$, the subset $\{a, b\}$ is wg-closed, but not $g\alpha$ b-closed set.

**Example 3.2.43:** Let $X = \{a, b, c\}$ with topology $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. In this topological space $(X, \tau)$, the subset $\{a\}$ is $g\alpha$ b-closed, but not wg-closed set.
Remark 3.2.44: Figure 3.1 gives the implication relationship of \( g \alpha \) b-closed sets based on the above results.

\[
\begin{align*}
\text{swg-closed set} & \quad \text{ag-closed set} & \quad \text{gs-closed set} \\
\text{g-closed set} & \quad \text{\( g \alpha \) b-closed set} & \quad \text{wg-closed set} \\
\text{\( g \alpha \) -closed set} & \quad \text{gb-closed set} \\
\text{sg-closed set} & \quad \text{gp-closed set} & \quad \text{gpr-closed set}
\end{align*}
\]

\textbf{Figure 3.1 Implication of \( g \alpha \) b- closed set}

where \( \rightarrow \) represent A implies B.

\( \rightarrow \) represent A does not implies B.

\( \leftrightarrow \) represent B does not implies A.

3.3 GENERALIZED \( \alpha \) b-OPEN SETS AND GENERALIZED \( \alpha \) b-NEIGHBOURHOODS

The present section introduces the concept of generalized \( \alpha \) b-open sets in topological space and studies some of their properties.

\textbf{Definition 3.3.1:} A subset \( A \) of a topological space \( X \) is called a generalized \( \alpha \) b-open (briefly g \( \alpha \) b-open) set, if its complement \( A^c \) is g \( \alpha \) b-closed.

The collection of all the g \( \alpha \) b-open sets in \( X \) is denoted by g \( \alpha \) b-O(\( X \)).

\textbf{Theorem 3.3.2:} If a subset \( A \) of a topological space \( X \) is g \( \alpha \) b-open if and only if \( U \subseteq \text{bint}(A) \), whenever \( U \) is an \( \alpha \)-closed and \( U \subseteq A \).
**Proof:** Assume that $A$ is $g\alpha$-b-open, then $A^c$ is $g\alpha$-b-closed. Let $U$ be $\alpha$-closed set in $X$ contained in $A$, $U^c$ is then $\alpha$-open in $X$ and containing $A^c$. Since $A^c$ is $g\alpha$-b-closed, $\text{bcl}(A^c) \subseteq U^c$. Therefore, $U \subseteq \text{bint}(A)$.

Conversely, $U \subseteq \text{bint}(A)$ whenever $U \subseteq A$ and $U$ is an $\alpha$-closed in $X$. Let $G$ be an $\alpha$-open set containing $A^c$, then $G^c \subseteq \text{bint}(A)$ taking complement on both sides $\text{bcl}(A^c) \subseteq G$, hence $A^c$ is $g\alpha$-b-closed. Therefore, $A$ is $g\alpha$-b-open.

**Definition 3.3.3:** A subset $N$ of a topological space $X$ is said to be $g\alpha$-b-neighbourhood of $x \in X$, if there exists a $g\alpha$-b-open set $G$ such that $x \in G \subset N$.

**Definition 3.3.4:** A subset $N$ of a topological space $X$ is called a $g\alpha$-b-neighbourhood of $A \subset X$, if there exists a $g\alpha$-b-open set $G$ such that $A \subset G \subset N$.

**Remark 3.3.5:** A $g\alpha$-b-neighbourhood $N$ of $x \in X$ need not be $g\alpha$-b-open in $X$, as seen from the following examples.

**Example 3.3.6:** Let $X = \{a, b, c, d\}$ with topology $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. In this topological space $(X, \tau)$, $g\alpha$-b-$O(X) = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, c, d\}\}$. The set $\{b, c\}$ is not $g\alpha$-b-open, but it is $g\alpha$-b-neighbourhood of $\{b\}$, as $\{b\}$ is $g\alpha$-b-open set such that $b \in \{b\} \subset \{b, c\}$.

**Theorem 3.3.7:** Every neighbourhood $N$ of $x \in X$ is $g\alpha$-b-neighbourhood of $X$.

**Proof:** Let $N$ be the neighbourhood of a point $x \in X$, to prove that $N$ is a $g\alpha$-b-neighbourhood of $x$. By definition of neighbourhood, there exists an open set $G$ such that $x \in G \subset N$. As every open set $G$ is $g\alpha$-b-open set, such that $x \in G \subset N$. Hence, $N$ is $g\alpha$-b-neighbourhood of $x$. 


Remark 3.3.8: In general, \( g\alpha \-b\)-neighbourhood \( N \) of \( x \in X \) need not be a neighbourhood of \( x \) in \( X \), as seen from the following example.

Example 3.3.9: Let \( X = \{a, b, c, d\} \) with topology \( \tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\} \). In this topological space \((X, \tau)\), \( g\alpha \-b\-O(X) = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\} \). The set \( \{b, d\} \) is the neighbourhood of \( \{b\} \), since \( \{b\} \) is \( g\alpha \-b\)-open set, such that \( b \in \{b\} \subset \{b, d\} \). However, the set \( \{b, d\} \) is not a neighbourhood of the point \( \{b\} \) such that \( b \in G \subset \{b, d\} \).

Theorem 3.3.10: If a subset \( N \) of a space \( X \) is \( g\alpha \-b\)-closed, then \( N \) is \( g\alpha \-b\)-neighbourhood of each of its points.

Proof: Suppose \( N \) is \( g\alpha \-b\)-closed subset of topological space \((X, \tau)\). Let \( x \in N \), it can be claimed that \( N \) is \( g\alpha \-b\)-neighbourhood of \( x \). \( N \) is \( g\alpha \-b\)-closed set such that \( x \in N \subset N \), since \( x \) is an arbitrary point of \( N \) it follows that \( N \) is \( g\alpha \-b\)-neighbourhood of each of its points.

Remark 3.3.11: The converse of the above theorem need not be true as seen from the following example.

Example 3.3.12: Let \( X = \{a, b, c, d\} \) with topology \( \tau = \{X, \emptyset, \{a, b\}\} \). In this topological space \((X, \tau)\), \( g\alpha \-b\-cl(X) = \{X, \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\} \) the set \( \{a, b, d\} \) is the neighbourhood of \( \{a, b\} \) and \( \{b, d\} \), since \( a, b \in \{a, b\} \subset \{a, b, d\} \) and \( b, d \in \{b, d\} \subset \{a, b, d\} \) that is \( \{a, b, d\} \) is the \( g\alpha \-b\)-neighbourhood of each of its points. However, \( \{a, b, d\} \) is not \( g\alpha \-b\)-closed in \( X \).

3.4 GENERALIZED \( \alpha \-b\- SPACES \)

This section introduces a new space \( T_{gab} \)-spaces in topology and studies some of their properties.
**Definition 3.4.1:** A topological space $X$ is said to be $T_{gab}$-space, if every $g \alpha b$-closed subset of $X$ is an $\alpha$-closed in $X$.

**Theorem 3.4.2:** Every $T_{gab}$-space is $T_{1/2}$-space.

**Proof:** It is assumed that $(X, \tau)$ is a $T_{gab}$-space. Let $A$ be a $g \alpha b$-closed. As every $g \alpha b$-closed set is $g$-closed and $X$ is $T_{gab}$-space, $A$ is closed. Therefore, $X$ is $T_{1/2}$-space.

**Remark 3.4.3:** The converse of the above theorem need not be true as seen from the following example.

**Example 3.4.4:** Let $X = \{a, b, c\}$ with $\tau = \{x, \phi, \{a\}, \{c\}, \{a, c\}\}$. In this topological space $(X, \tau)$, the subset $\{a\}$ is $g \alpha b$-closed, but not $\alpha$-closed.

**Theorem 3.4.5:** Every semi-$T_{1/2}$-space is $T_{gab}$-space.

**Proof:** It is assumed that $(X, \tau)$ is a semi-$T_{1/2}$-space. Let $A$ be a sg-closed. As every sg-closed set is $g \alpha b$-closed and $X$ is semi-$T_{1/2}$-space, $A$ is an $\alpha$-closed. Therefore, $X$ is $T_{gab}$-space.

**Remark 3.4.6:** The converse of the above theorem need not be true as seen from the following example.

**Example 3.4.7:** Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a, b\}\}$. In this topological space $(X, \tau)$, the subset $\{a, c\}$ is sg-closed, but not semi-closed.
Theorem 3.4.8: Every $T_{gab}$-space is pre-$T_{1/2}$-space.

Proof: It is assumed that $(X, \tau)$ is a $T_{gab}$-space. Let $A$ be a $g \alpha b$-closed. As every $g \alpha b$-closed set is gp-closed set and $X$ is $T_{gab}$-space, $A$ is pre-closed. Therefore, $X$ is pre-$T_{1/2}$-space.

Remark 3.4.9: The converse of the above theorem need not be true as seen from the following example.

Example 3.4.10: Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{b\},\{a, b\}\}$. In this topological space $(X, \tau)$, the subset $\{a\}$ is $g \alpha b$-closed, but not $\alpha$-closed.

Theorem 3.4.11: Every $\alpha T_d$-space is $T_{gab}$-space.

Proof: It is assumed that $(X, \tau)$ is a $\alpha T_d$-space. Let $A$ be an $\alpha g$-closed. Every $\alpha g$-closed set is $g \alpha b$-closed and $X$ is $\alpha T_d$-space, $A$ is an $\alpha$-closed. Therefore, $X$ is $T_{gab}$-space.

Remark 3.4.12: The converse of the above theorem need not be true as seen from the following example.

Example 3.4.13: Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{c\}, \{b\}, \{a, b\}\}$. In this topological space $(X, \tau)$, the subset $\{b\}$ is $\alpha g$-closed, but not $g$-closed.

Theorem 3.4.14: Every $T_{gab}$-space is pre-regular-$T_{1/2}$-space.

Proof: It is assumed that $(X, \tau)$ is a $T_{gab}$-space. Let $A$ be a $g \alpha b$-closed. Every $g \alpha b$-closed set is gpr-closed set and $X$ is $T_{gab}$-space, $A$ is pre-closed. Therefore, $X$ is pre-regular-$T_{1/2}$-space.
**Remark 3.4.15:** The converse of the above theorem need not be true as seen from the following example.

**Example 3.4.16:** Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{b, c\}\}$. In this topological space $(X, \tau)$, the subset $\{c\}$ is $g\alpha$ b-closed, but not $\alpha$-closed.

**Theorem 3.4.17:** Every $T_{ag}$-space is $T_{gab}$-space.

**Proof:** It is assumed that $(X, \tau)$ is a $T_{ag}$-space. Let $A$ be a $\alpha$ g-closed. Every $\alpha$ g-closed set is $g\alpha$ b-closed and $X$ is $T_{ag}$-space, $A$ is an $\alpha$-closed. Therefore, $X$ is $T_{gab}$-space.

**Remark 3.4.18:** The converse of the above theorem need not be true as seen from the following example.

**Example 3.4.19:** Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{b\}, \{a, b\}\}$. In this topological space $(X, \tau)$, the subset $\{b, c\}$ is $\alpha$ g-closed, but not $g\alpha$-closed.

**Theorem 3.4.20:** Every $T_{gab}$-space is $T_{gs}$-space.

**Proof:** It is assumed that $(X, \tau)$ is a $T_{gab}$-space. Let $A$ be a $g\alpha$ b-closed. As every $g\alpha$ b-closed set is gs-closed and $X$ is $T_{gab}$-space, $A$ is sg -closed. Therefore, $X$ is $T_{gs}$-space.

**Remark 3.4.21:** The converse of the above theorem need not be true as seen from the following example.

**Example 3.4.22:** Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{c\}, \{a, c\}\}$. In this topological space $(X, \tau)$, the subset $\{a\}$ is $g\alpha$ b-closed, but not $\alpha$-closed.
**Remark 3.4.23:** By the above theorem and results the following relationship has been obtained.

![Diagram showing relationships between different types of spaces](image)

**Figure 3.2 Separation axioms on \( T_{gab} \) - Spaces**

where
- \( \rightarrow \) B represent A implies B.
- \( \nrightarrow \) B represent A does not implies B.
- \( \iff \) B represent B does not implies A.

**Conclusion 3.4.24:** The present chapter has introduced a new concept called generalized \( \alpha \) b-closed set in topological spaces. It also analyzed some of the properties. The implication shows the relationship between the generalized \( \alpha \) b-closed sets and the other existing sets.