Chapter 4

Delay Differential Equation with Nonpositive Neutral Term - II
4. Delay Differential Equation with Nonpositive Neutral Term - II

4.1 Introduction

This chapter deals with the oscillation of all solutions of second order nonlinear neutral differential equation of the form

\[
(r(t)(z'(t))^\alpha)' + q(t)f(x(\sigma(t))) = 0, \quad t \geq t_0 \geq 0
\]  

(4.1.1)

where \( z(t) = x(t) - a(t)x(\tau(t)) \), and \( \alpha > 0 \) is a ratio of odd positive integers. Throughout, we assume that the following conditions are satisfied without further mention:

\( (C_1) \) \( r, a, q \in C([t_0, \infty), \mathbb{R}) \), \( r(t) > 0 \), \( \int_{t_0}^{\infty} r^{-1/\alpha}(t)dt = \infty \), \( 0 \leq a(t) < p < 1 \), \( p \) is a constant, and \( q(t) > 0 \) for all \( t \geq t_0 \);

\( (C_2) \) \( \tau \in C([t_0, \infty), \mathbb{R}) \), \( \tau(t) \leq t \), and \( \lim_{t \to \infty} \tau(t) = \infty \);

\( (C_3) \) \( \sigma \in C([t_0, \infty), \mathbb{R}) \), \( \sigma'(t) > 0 \), \( \sigma(t) \leq t \), and \( \lim_{t \to \infty} \sigma(t) = \infty \);

\( (C_4) \) \( f \in C(\mathbb{R}, \mathbb{R}) \), \( uf(u) > 0 \) for all \( u \neq 0 \), and there exists a positive constant \( k \) such that \( \frac{f(u)}{u^\alpha} \geq k \) for all \( u \neq 0 \).

By a solution of equation (4.1.1), we mean a continuous function \( x \in ([T_x, \infty), \mathbb{R}) \), \( T_x \geq t_0 \) which has the property \( r(t)(z'(t))^\alpha \in C'([T_x, \infty), \mathbb{R}) \), and satisfies equation (4.1.1) on the interval \([T_x, \infty)\).

In [69, 93], the authors obtained several oscillation theorems for equation (4.1.1) under the assumptions that

\[
0 \leq a(t) \leq a < 1, \quad (4.1.2)
\]
\[ \tau(t) = t - \tau_0 \leq t, \quad \text{and} \quad \sigma(t) = t - \sigma_0 \leq t. \] 

(4.1.3)

Recently in [48], the authors considered the equation (4.1.1) under the conditions (4.1.2) and \( \int_{\tau_0}^{\infty} r^{-1/\alpha}(t)dt = \infty \), and established that all solutions of equation (4.1.1) are either oscillatory or tend to zero monotonically. Also, the same authors raised the question when all solutions are just oscillatory for the equation (4.1.1) when \( \int_{\tau_0}^{\infty} r^{-1/\alpha}(t)dt = \infty \).

Motivated by the above observation, in this chapter we obtain conditions for the oscillation of all solutions of equation (4.1.1). In Section 4.2, we present oscillation theorems for equation (4.1.1) and in Section 4.3, we provide some examples to illustrate the main results. Thus the results obtained in this chapter improve some of the results in [48, 69, 93].

### 4.2 Oscillation Results

In this section, we present some new oscillation results for the equation (4.1.1). Without loss of generality, we can deal only with positive solutions of equation (4.1.1) since the proof for the negative case is similar. We begin with the following lemma.

**Lemma 4.2.1.** Assume that \( x \) is a positive solution of equation (4.1.1). Then \( x \) satisfies the following two possible cases:

(I) \( z(t) > 0, \quad z'(t) > 0, \quad (r(t)(z'(t))^{\alpha})' \leq 0; \)

(II) \( z(t) < 0, \quad z'(t) > 0, \quad (r(t)(z'(t))^{\alpha})' \leq 0 \)

for \( t \geq t_1 \), where \( t_1 \geq \tau_0 \) is sufficiently large.
Proof. Suppose that there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \geq t_1$. It follows from (4.1.1) that

$$(r(t)(z'(t))^\alpha)' \leq -kq(t)x^\alpha(\sigma(t)) \leq 0.$$  

Hence, $r(t)(z'(t))^\alpha$ is nonincreasing and of one sign. That is, there exists a $t_2 \geq t_1$ such that $z'(t) > 0$ or $z'(t) < 0$ for $t \geq t_2$.

If $z'(t) > 0$ for $t \geq t_2$, then we have (I) or (II). We prove now that $z'(t) < 0$ cannot occur. If $z'(t) < 0$ for $t \geq t_2$, then

$$r(t)(z'(t))^\alpha \leq -c < 0 \quad \text{for} \quad t \geq t_2,$$

where $c = -r(t_2)(z'(t_2))^\alpha > 0$. Thus, we conclude that

$$z(t) \leq z(t_2) - c^{1/\alpha} \int_{t_2}^{r_1} r^{-1/\alpha}(s)ds.$$  

By virtue of condition $(C_1)$, $\lim_{t \to \infty} z(t) = -\infty$. We consider now the following two cases separately.

Case (I). If $x$ is unbounded, then there exists a sequence $\{t_k\}$ such that $\lim_{k \to \infty} t_k = \infty$ and $\lim_{k \to \infty} x(t_k) = \infty$, where $x(t_k) = \max\{x(s) : t_0 \leq s \leq t_k\}$. Since $\lim_{t \to \infty} \tau(t) = \infty$, $\tau(t_k) > t_0$ for all sufficiently large $k$. By $\tau(t) \leq t$,

$$x(\tau(t_k)) = \max\{x(s) : t_0 \leq s \leq \tau(t_k)\} \leq \max\{x(s) : t_0 \leq s \leq t_k\} = x(t_k).$$

Therefore, for all large $k$,

$$z(t_k) = x(t_k) - p(t_k)x(\tau(t_k)) \geq (1 - p(t_k))x(t_k) > 0,$$

which contradicts the fact that $\lim_{t \to \infty} z(t) = -\infty$.

Case (II). If $x$ is bounded, then $z$ is also bounded, which contradicts $\lim_{t \to \infty} z(t) = -\infty$. Hence, $z$ satisfies one of the Cases (I) or (II). This completes the proof. □
Lemma 4.2.2. If \( x \) is a positive solution of equation (4.1.1) such that Case(I) of Lemma 4.2.1 holds, then

\[
x(t) \geq z(t) \geq R(t)r^\frac{1}{\alpha}(t)z'(t), \quad t \geq T \geq t_0,
\]

and \( \frac{z(t)}{R(t)} \) is strictly decreasing, where \( R(t) = \int_{t_0}^{t} r^{-\frac{1}{\alpha}}(s)ds \).

**Proof.** From the definition of \( z \), we have \( x(t) = z(t) + a(t)x(\tau(t)) \) and therefore \( x(t) \geq z(t) \) for all \( t \geq T \geq t_0 \). Since \( r(t)(z'(t))^\alpha \) is nonincreasing, we have for \( t \geq T \geq t_0 \),

\[
z(t) = z(T) + \int_{T}^{t} \left( r(t)(z'(t))^\alpha \right)^\frac{1}{\alpha} \frac{ds}{r^\frac{1}{\alpha}(s)} \geq R(t)r^\frac{1}{\alpha}(t)z'(t).
\]

Now

\[
\left( \frac{z(t)}{R(t)} \right)' = \frac{r^\frac{1}{\alpha}(t)R(t)z'(t) - z(t)}{r^\frac{1}{\alpha}(t)R^2(t)} \leq 0, \quad t \geq T \geq t_0
\]

by (4.2.1). Hence \( \frac{z(t)}{R(t)} \) is nonincreasing for all \( t \geq T \geq t_0 \). This completes the proof. \( \square \)

The following Theorems 4.2.1 and 4.2.2 are improving the Theorems 4.3.1 and 4.3.2 of [48], respectively.

**Theorem 4.2.1.** Assume that \( \sigma(t) < \tau(t) \) for all \( t \geq t_0 \). If there exists a positive nondecreasing function \( \rho \in C'([t_0, \infty), \mathbb{R}) \) such that, for all sufficiently large \( T \geq t_0 \)

\[
\lim \sup_{t \to \infty} \int_{T}^{t} \left[ kp(t)q(t) \left( 1 + a(\sigma(t)) \frac{R(\tau(\sigma(t)))}{R(\sigma(t))} \right)^\alpha - \frac{\rho'(t)(\sigma'(t))^\alpha}{R^\alpha(\sigma(t))} \right] dt = \infty,
\]

and

\[
\lim \sup_{t \to \infty} \int_{\tau^{-1}(\sigma(t))}^{t} \frac{1}{r^\frac{1}{\alpha}(s)} \left( \int_{s}^{t} q(u)du \right)^\frac{1}{\alpha} ds > \frac{p}{k^{1/\alpha}},
\]

then every solution of equation (4.1.1) is oscillatory.

**Proof.** Assume that \( x(t) \) is a positive solution of equation (4.1.1). Then there exists a \( T \geq t_0 \) such that \( x(t) > 0, \ x(\tau(t)) > 0, \) and \( x(\sigma(t)) > 0 \) for all \( t \geq T \). Then by Lemma 4.2.1, \( z(t) \) satisfies one of the Cases (I) and (II) for all \( t \geq T \).
Case (I). From the definition of $z$ and $(C_2)$, we have
\[ x(t) \geq z(t) + a(t)z(\tau(t)) \geq \left(1 + a(t)\frac{R(\tau(t))}{R(t)}\right)z(t), \quad t \geq T, \quad (4.2.4) \]
where we have used $\frac{z(t)}{R(t)}$ is decreasing. Using (4.2.4) and $(C_4)$ in equation (4.1.1), we have
\[ (r(t)(z'(t))^\alpha)' + kq(t) \left(1 + a(\sigma(t))\frac{R(\tau(\sigma(t)))}{R(\sigma(t))}\right)^\alpha z^\alpha(\sigma(t)) \leq 0, \quad t \geq T. \]
Define
\[ w(t) = \rho(t)\frac{r(t)(z'(t))^\alpha}{z^\alpha(\sigma(t))}, \quad t \geq T. \]
Then $w(t) > 0$ for $t \geq T$, and
\[ w'(t) = \rho'(t)\frac{r(t)(z'(t))^\alpha}{z^\alpha(\sigma(t))} + \rho(t)\frac{r(t)(z'(t))^\alpha}{z^\alpha(\sigma(t))}' - \rho(t)\alpha\frac{r(t)(z'(t))^\alpha}{z^{\alpha+1}(\sigma(t))}z'(\sigma(t))\sigma'(t). \]
Using $r(t)(z'(t))^\alpha \leq r(\sigma(t))(z'(\sigma(t)))^\alpha$ and (4.2.1) in the last inequality, we have
\[ w'(t) \leq -kp(t)q(t) \left(1 + a(\sigma(t))\frac{R(\tau(\sigma(t)))}{R(\sigma(t))}\right)^\alpha + \rho'(t)\frac{(\sigma'(t))^\alpha}{R^\alpha(\sigma(t))}, \quad t \geq T. \]
Integrating the last inequality from $T$ to $t$, we obtain
\[ \int_T^t \left[kp(s)q(s) \left(1 + a(\sigma(s))\frac{R(\tau(\sigma(s)))}{R(\sigma(s))}\right)^\alpha - \frac{\rho'(s)(\sigma'(s))^\alpha}{R^\alpha(\sigma(s))}\right] ds \leq w(T) \]
which contradicts (4.2.2).

Case (II). From the definition of $z$ and $(C_1)$, we have
\[ x(\tau(t)) > -\frac{z(t)}{p}, \quad t \geq T \geq t_0. \quad (4.2.5) \]
Using (4.2.5) and $(C_4)$ in equation (4.1.1), we obtain
\[ (r(t)(z'(t))^\alpha)' - \frac{k}{p^\alpha}q(t)z^\alpha(\tau^{-1}(\sigma(t))) \leq 0, \quad t \geq T. \quad (4.2.6) \]
Integrating (4.2.6) from $s$ to $t$ for $t > s$, we have
\[ r(t)(z'(t))^\alpha - r(s)(z'(s))^\alpha - \frac{k}{p^\alpha} \int_s^t q(u)z^\alpha(\tau^{-1}(\sigma(u))) du \leq 0. \]
Again integrating the last inequality from $\tau^{-1}(\sigma(t))$ to $t$ for $s$, and using the fact that $z$ is negative and increasing, we have

$$z(\tau^{-1}(\sigma(t))) - z(t) \leq \frac{k^\alpha}{p} z(\tau^{-1}(\sigma(t))) \int_{\tau^{-1}(\sigma(t))}^{t} \frac{1}{r^\alpha(s)} \left( \int_s^t q(u)du \right)^{\frac{1}{\alpha}} ds,$$

or

$$\frac{p}{k^\alpha} \geq \frac{1}{r^\alpha(s)} \left( \int_s^t q(u)du \right)^{\frac{1}{\alpha}} ds$$

which contradicts (4.2.3). The proof is now completed. \qed

Let $\rho(t) = 1$. Then from Theorem 4.2.1, we obtain the following corollary.

**Corollary 4.2.1.** Let $\tau(t) < \sigma(t)$ for $t \geq t_0$. If condition (4.2.3), and

$$\int_{t_0}^{\infty} q(t) \left( 1 + a(\sigma(t)) \frac{R(\tau(\sigma(t)))}{R(\sigma(t))} \right) dt = \infty \quad (4.2.7)$$

are satisfied, then every solution of equation (4.1.1) is oscillatory.

For $\alpha > 1$, we derive the following result different from Theorem 4.2.1.

**Theorem 4.2.2.** Let $\alpha > 1$ hold, and $\sigma(t) < \tau(t)$ for $t \geq t_0$. Assume that there exists a positive nondecreasing function $\rho \in C^1([t_0, \infty), \mathbb{R})$ such that for all sufficiently large $T \geq t_0$,

$$\lim_{t \to \infty} \sup \int_T^t \left[ k(\rho(t))q(t) \left( 1 + a(\sigma(t)) \frac{R(\tau(\sigma(t)))}{R(\sigma(t))} \right)^\alpha - \frac{(\rho'(t))^2 r^\alpha(\sigma(t))}{4\alpha \rho(t)\sigma'(t)R^{\alpha-1}(\sigma(t))} \right] dt = \infty. \quad (4.2.8)$$

If condition (4.2.3) holds, then every solution of equation (4.1.1) is oscillatory.

**Proof.** As above, we assume that $x$ is a positive solution of equation (4.1.1). Then by Lemma 4.2.1, $z$ satisfies one of (I) and (II). Assume first that $z$ satisfies Case (I) of Lemma 4.2.1. Then define $w$ as in the proof of Theorem 4.2.1. Then $w > 0$, and

$$w'(t) = -k(\rho(t))q(t) \left( 1 + a(\sigma(t)) \frac{R(\tau(\sigma(t)))}{R(\sigma(t))} \right)^\alpha + \frac{\rho'(t)}{\rho(t)} w(t) - \alpha \sigma'(t) w(t) \frac{z'(\sigma(t))}{z(\sigma(t))} \quad (4.2.9)$$
Now by (4.2.1) and \( r(t)(z'(t))^\alpha \leq r(\sigma(t))(z'(\sigma(t)))^\alpha \), we have
\[
\frac{z'(\sigma(t))}{z(\sigma(t))} \geq \frac{R^{\alpha-1}(\sigma(t))(\sigma'(t))^\alpha}{\frac{1}{\alpha}(\sigma(t))\rho(t)} w(t), \quad t \geq T. \tag{4.2.10}
\]

Using (4.2.10) in (4.2.9), we obtain
\[
w'(t) \leq -k\rho(t)q(t) \left(1 + a(\sigma(t))\frac{R(\tau(\sigma(t)))}{R(\sigma(t))}\right)^\alpha + \frac{\rho'(t)}{\rho(t)} w(t) \\
- \alpha \frac{R^{\alpha-1}(\sigma(t))(\sigma'(t))^\alpha}{\frac{1}{\alpha}(\sigma(t))\rho(t)} w_2^\alpha(t),
\]
or
\[
w'(t) \leq -k\rho(t)q(t) \left(1 + a(\sigma(t))\frac{R(\tau(\sigma(t)))}{R(\sigma(t))}\right)^\alpha + \frac{1}{4\alpha} \frac{\rho'(t)^2}{\rho(t)R^{\alpha-1}(\sigma(t))(\sigma'(t))^\alpha}.
\]

Integrating the last inequality from \( T \) to \( t \), we obtain
\[
\int_T^t k\rho(s)q(s) \left(1 + a(\sigma(s))\frac{R(\tau(\sigma(s)))}{R(\sigma(s))}\right)^\alpha - \frac{\rho'(s)^2}{4\alpha\rho(s)\sigma'(s)R^{\alpha-1}(\sigma(s))} ds \leq w(T),
\]
which contradicts (4.2.8).

If \( z \) satisfies Case (II) of Lemma 4.2.1, then proceeding as in the proof of Theorem 4.2.1 (Case(II)), we obtain a contradiction with (4.2.3). The proof is now completed.

Next we consider the case \( \alpha = 1, \tau(t) = t - k, \) and \( \sigma(t) = t - \ell \) where \( k \) and \( \ell \) are positive constants with \( \ell > k \).

**Theorem 4.2.3.** Assume conditions \((C_1) - (C_4)\) hold with \( \alpha = 1, \tau(t) = t - k, \) and \( \sigma(t) = t - \ell \) where \( k \) and \( \ell \) are positive constants with \( \ell > k \). If
\[
\liminf_{t \to \infty} \int_{t-\ell}^t q(s)(R(s-\ell) + a(s-\ell)R(s-\ell-k))ds > \frac{1}{ke^t}, \tag{4.2.11}
\]
and
\[
\limsup_{t \to \infty} \int_{t-\ell+k}^t \frac{1}{\tau(s)} \left(\int_s^t q(u)du\right)ds > \frac{p}{k}, \tag{4.2.12}
\]
then every solution of equation (4.1.1) is oscillatory.
\textbf{Proof.} As above, we assume that \( z \) is a positive solution of equation (4.1.1). Then by Lemma 4.2.1, \( z \) satisfies one of (I) and (II).

\textbf{Case (I).} Using (4.2.4) and (C4) in equation (4.1.1), we have
\[
(r(t)z'(t))' + kq(t) \left(1 + a(t - \ell) \frac{R(t - \ell - k)}{R(t - \ell)}\right) z(t - \ell) \leq 0, \quad t \geq T. \tag{4.2.13}
\]

From Lemma 4.2.1, we have
\[
z(t - \ell) \geq R(t - \ell)r(t - \ell)z(t - \ell), \quad t \geq T. \tag{4.2.14}
\]

Using (4.2.14) in (4.2.13) we obtain
\[
(r(t)z'(t))' + kq(t)(R(t - \ell) + a(t - \ell)R(t - \ell - k))r(t - \ell)z(t - \ell) \leq 0.
\]

Let \( w(t) = r(t)z'(t) \). Then \( w(t) > 0 \) and
\[
w'(t) + kq(t)(R(t - \ell) + a(t - \ell)R(t - \ell - k))w(t - \ell) \leq 0. \tag{4.2.15}
\]

In view of Theorem 6.4.2 [35], the condition (4.2.11) implies that the inequality (4.2.15) has no positive solution, which is a contradiction.

\textbf{Case (II).} The proof is similar to that of Theorem 4.2.1, and hence the details are omitted. This completes the proof. \( \square \)

\section{4.3 Examples}

In this section, we present some examples to illustrate the main results obtained in the previous section.

\textbf{Example 4.3.1.} Consider a second order neutral differential equation
\[
\left((z'(t))^{\frac{1}{3}}\right)' + \frac{4}{t}\left((\frac{t}{3})\right)^{\frac{1}{3}} = 0, \quad t \geq 1, \tag{4.3.1}
\]
where \( z(t) = x(t) - \frac{1}{2}x(t/2) \). Here \( \alpha = \frac{1}{3} \), \( r(t) = 1 \), \( q(t) = \frac{4}{t} \), \( \tau(t) = \frac{t}{2} \), \( \sigma(t) = \frac{t}{3} \), and \( k = 1 \). By taking \( \rho(t) = 1 \), we see that all conditions of Corollary 4.2.1 are satisfied, and hence every solution of equation (4.3.1) is oscillatory.
Example 4.3.2. Consider a second order neutral differential equation

\[(t^2(z'(t))^3)' + \frac{10}{t^2}x^3 \left(\frac{t}{3}\right) = 0, \quad t \geq 1,\]

where \(z(t) = x(t) - \frac{1}{2}x(t/2)\). Here \(\alpha = 3\), \(\tau(t) = t^2\), \(q(t) = \frac{10}{t^2}\), \(\tau(t) = \frac{t}{2}\), \(\sigma(t) = \frac{t}{3}\), and \(k = 1\). By taking \(\rho(t) = t\), we see that all conditions of Theorem 4.2.2 are satisfied, and hence every solution of equation (4.3.2) is oscillatory.

Example 4.3.3. Consider a second order neutral differential equation

\[\left(x(t) - \frac{1}{2}x(t - \frac{\pi}{2})\right)'' + 8x(t - \pi) = 0, \quad t \geq 1.\]

Here \(r(t) = 1\), \(a(t) = \frac{1}{2}\), \(q(t) = 8\), \(\tau(t) = t - \frac{\pi}{2}\), \(\sigma(t) = t - \pi\), \(\alpha = 1\), and \(k = 1\). It is easy to see that all conditions of Theorem 4.2.3 are satisfied, and hence every solution of equation (4.3.3) is oscillatory. In fact \(x(t) = \sin 4t\) is one such oscillatory solution of this equation.

We conclude this chapter with the following remark.

Remark 4.3.1. It would be interesting to obtain conditions for the oscillation of all solutions of equation (4.1.1) under the condition \(\int_{t_0}^{t} \frac{ds}{r(t)} < \infty\) as \(t \to \infty\).