Chapter 1

Introduction
1. Introduction

1.1 Differential Equations

Differential equations play an important role in modelling the fundamentally describing continuous systems in science and technology and also in the diverse fields such as statistics, economics, medicine, biology, etc. Many important phenomena in these fields are governed by various types of nonlinear functional differential equations. Differential equations can be classified into ordinary, delay, advanced and neutral type differential equations, etc.

A differential equation of the form

\[ x'(t) + q(t)f(x(t - \tau)) = 0, \]  \hspace{1cm} (1.1.1)

where \( \tau > 0 \) is constant, is called delay differential equation, and is called advanced type differential equation if \( \tau < 0 \).

A differential equation of the form

\[ (x(t) + a(t)x(t - \tau))' + q(t)f(x(t - \sigma)) = 0, \]  \hspace{1cm} (1.1.2)

where \( \tau > 0 \) and \( \sigma > 0 \) are constants, is called neutral type differential equation.

A differential equation of the form

\[ (x(t) + a(t)x(t - \tau_1) + b(t)x(t + \tau_2))' + q_1(t)f(x(t - \sigma_1)) + q_2(t)g(x(t + \sigma_2)) = 0, \]  \hspace{1cm} (1.1.3)

where \( \tau_1 > 0, \tau_2 > 0, \sigma_1 > 0, \) and \( \sigma_2 > 0 \) are constants, is called a mixed type differential equation.

Differential equations with delayed and advanced argument (also called mixed differential equations or equations with mixed arguments) occur in many problems
of economy, biology and physics (see for example [2, 13, 35, 36]) because differential equations with mixed arguments are much more suitable than delay differential equations for an adequate treatment of dynamic phenomena.

Most of the differential equations appearing in real world problem are nonlinear in nature, and it is a well known fact that these equations cannot be solvable to get a closed form solutions. In the absence of closed form solutions, a rewarding alternative is to study the qualitative behavior of solutions of these equations. In the qualitative theory of differential equations oscillatory behavior of solutions play an important role. A nontrivial solution of a differential equation is said to be oscillatory if it is neither eventually positive nor eventually negative and nonoscillatory otherwise. These types of solutions occur in many physical phenomena such as, vibrating mechanical systems, electrical circuits, and in population dynamics.

1.2 Motivation

The development of oscillation theory for ordinary differential equations began in the 1836’s when the classical work of Sturm [78] appeared, in which theorems of oscillation and comparison of the solutions of second order linear homogenous ordinary differential equations were proved. The first oscillation results for differential equation with a translated argument were obtained by Fite [25] in 1921. The neutral differential equations, enjoy specific properties which make their study difficult both in aspects of ideas and techniques. Because of these difficulties less number of works devoted to the investigation of the oscillatory properties of the neutral differential equations. The first work in which a criterion for oscillation of the solutions of neutral equations was proved by Zahariev and Bainov [100] in 1980. Afterwards many researchers have done extensive work on this topic, and substantially contributed for the development of the oscillation theory of neutral type differential equations, see
for example Agarwal et. al [1–6], Ayanlar et. al [7], Baculikova et. al [8–12], Bainov et.al [13], Berezansky et. al [14], Dong [15], Driver [16], Dzurina et. al [17–22], Erbe et. al [23, 24], Grace et. al [26–30], Grammatikopouls et.al [31–34], Gyori et.al [35], Hale [36], Han et. al [37–40], Hasanbulli et. al [42, 43], Jiang et. al [44], Karpuz et. al [45], Lackova [46], Q.Li et. al [48], T.Li et. al [49–60], L.Liu et. al [61], X.Liu et. al [62], Manojlović et. al [63], Padhi [64, 65], Philos [66], Pinelas [67, 68], Qin et. al [69], Rogovchenko [70, 71], Ruan [72], Sahina [73], Saker [74], Stavroulakis [75–77], Sun et. al [79, 80], Tang [81], Thandapani et. al [82–89], Tiryaki et. al [90, 91], Wang et. al [92], Wong et. al [93], Xu et. al [94–96], J.Yang [97], Q.Yang et. al [98], Ye et. al [99], and Zhang et. al [101, 102].

Keeping in view, the importance of the subject, and in the light of the above trend, we obtained some significant results on the following topics:

1. Quasilinear differential equation with several neutral terms;

2. Delay differential equation with nonpositive neutral term - I;

3. Delay differential equation with nonpositive neutral term - II;

4. Neutral differential equation with mixed neutral term - I;

5. Neutral differential equation with mixed neutral term - II.

1.3 Plan of the Thesis

This thesis consists of six chapters including this introductory chapter, which provides necessary introduction and motivation for the present study.

In Chapter 2, we investigate the oscillatory behavior of solutions of a second order quasilinear neutral differential equation of the form

\[(r(t)(z'(t))^\alpha)' + q(t)f(x(\sigma(t))) = 0, \quad t \geq t_0 \geq 0,\]  

(1.3.1)
where \( z(t) = x(t) + \sum_{i=1}^{m} p_i(t)x(\tau_i(t)) \), \( m > 0 \) is an integer, \( q \in C([t_0, \infty), p_i, \tau_i, \sigma \in C([t_0, \infty), r(t) > 0, q(t) > 0, 0 \leq p_i(t) \leq b_i < \infty, \lim_{t \to \infty} \sigma(t) = \infty, \tau_i \circ \sigma = \sigma \circ \tau_i, \tau_i'(t) \geq \lambda_i > 0 \) for \( i = 1, 2, \ldots, m \), \( f \in C(\mathbb{R}, \mathbb{R}) \) is nondecreasing with \( u f(u) > 0 \) and \( \frac{f(u)}{u^\alpha} \geq M > 0 \), \( \alpha \) and \( \beta \) are ratio of odd positive integers, and \( R(t) = \int_{t_0}^{t} \frac{ds}{r(s)^{1/\alpha(s)}} \to \infty \) as \( t \to \infty \). Section 2.1 presents necessary introduction and motivation. In Section 2.2, we present sufficient conditions for the oscillation of all solutions of equation (1.3.1), and in Section 2.3, we provide some examples to illustrate the main results. The results established in this chapter generalize, and complement to those given in [10, 91, 102].

In Chapter 3, we are concerned with the oscillatory behavior of second order neutral delay differential equation of the form

\[
(r(t)z'(t))' + q(t)f(x(\sigma(t))) = 0, \quad t \geq t_0 \geq 0,
\]

where \( z(t) = x(t) - p(t)x(\tau(t)), r, p, q \in C([t_0, \infty), \mathbb{R}), r(t) > 0, 0 \leq p(t) \leq p_0 < 1, q(t) > 0 \) for all \( t \geq t_0, \tau \in C([t_0, \infty), \mathbb{R}), \tau(t) \leq t, \lim_{t \to \infty} \tau(t) = \infty, \sigma \in C([t_0, \infty), \mathbb{R}), \sigma'(t) > 0, \sigma(t) \leq t, \lim_{t \to \infty} \sigma(t) = \infty, f \in C(\mathbb{R}, \mathbb{R}), u f(u) > 0 \) for all \( u \neq 0 \), and there exists a positive constant \( M \) such that \( \frac{f(u)}{u^\alpha} \geq M \) for all \( u \neq 0 \). Section 3.1, presents necessary introduction and motivation. In Section 3.2, we establish some sufficient conditions for the oscillation of all solutions of equation (1.3.2), and some examples are given in Section 3.3 to illustrate the main results. The results obtained in this chapter complement, and generalize the results established in [13, 48, 93, 98].

Chapter 4 deals with the oscillation of all solutions of second order nonlinear neutral differential equation of the form

\[
(r(t)(z'(t))^\alpha)' + q(t)f(x(\sigma(t))) = 0, \quad t \geq t_0 \geq 0,
\]

where \( z(t) = x(t) - a(t)x(\tau(t)), \) and \( \alpha > 0 \) is a ratio of odd positive integers,
\( r, a, q \in C([t_0, \infty), \mathbb{R}) \), \( r(t) > 0 \), \( \int_{t_0}^{\infty} r^{-1/p}(t) dt = \infty \), \( 0 \leq a(t) < p < 1 \), \( p \) is a constant, \( q(t) > 0 \) for all \( t \geq t_0 \), \( r \in C([t_0, \infty), \mathbb{R}) \), \( \tau(t) \leq t \), \( \lim_{t \to \infty} \tau(t) = \infty \), \( \sigma \in C([t_0, \infty), \mathbb{R}) \), \( \sigma'(t) > 0 \), \( \sigma(t) \leq t \), \( \lim_{t \to \infty} \sigma(t) = \infty \), \( f \in C(\mathbb{R}, \mathbb{R}) \), \( uf(u) > 0 \) for all \( u \neq 0 \), and there exists a positive constant \( k \) such that \( f(u) \geq k \) for all \( u \neq 0 \).

In Section 4.1, necessary introduction and motivation are provided. In Section 4.2, we present oscillation theorems for equation (1.3.3) and in Section 4.3, we provide some examples to illustrate the main results. Thus the results obtained in this chapter improve some of the results in [48, 69, 93].

In Chapter 5, we study the oscillatory behavior of solution of the second order neutral differential equation of the form

\[
(r(t)z'(t))' + q(t)x(\sigma(t)) = 0, \quad t \geq t_0 \geq 0,
\]  

(1.3.4)

where \( z(t) = x(t) + a(t)x(t-\tau) + b(t)x(t+\delta) \), \( a, b, q \in C([t_0, \infty), \mathbb{R}) \), \( 0 \leq a(t) \leq a < \infty \), \( 0 \leq b(t) \leq b < \infty \), \( q(t) > 0 \), \( r \in C([t_0, \infty), \mathbb{R}) \), \( r(t) > 0 \), \( \int_{t_0}^{\infty} \frac{1}{r(t)} dt = \infty \), \( \tau, \delta \) are nonnegative constants, \( \sigma \in C([t_0, \infty), \mathbb{R}) \), \( \sigma'(t) > 0 \), \( \lim_{t \to \infty} \sigma(t) = \infty \), and \( \sigma(t \pm \delta) = \sigma(t) \pm \delta \). In Section 5.1, necessary introduction and motivation are provided and in Section 5.2, we establish some sufficient conditions for the oscillation of all solutions of equation (1.3.4). Examples are given in Section 5.3 to illustrate the main results. The results presented in this chapter generalize, and improve those obtained in [20, 29, 30, 32, 61, 95].

Finally, in Chapter 6 we investigate the oscillatory behavior of solutions of the second order nonlinear neutral differential equation of the form

\[
(r(t)z'(t))' + q(t)f(x(\sigma(t))) = 0, \quad t \geq t_0 \geq 0,
\]  

(1.3.5)

where \( z(t) = x(t) + a(t)x(t-\tau) + b(t)x(t+\delta) \), \( a, b, q \in C([t_0, \infty), \mathbb{R}) \), \( 0 \leq a(t) \leq a < \infty \), \( 0 \leq b(t) \leq b < \infty \), \( q(t) > 0 \) for all \( t \geq t_0 \), \( r \in C([t_0, \infty), \mathbb{R}) \), \( r(t) > 0 \),
\[ \int_{t_0}^{\infty} \frac{1}{r(t)} dt < \infty, \quad \tau, \quad \delta \text{ are nonnegative constants, } \sigma \in C([t_0, \infty), \mathbb{R}), \quad \sigma'(t) > 0, \quad \sigma(t) \leq t, \quad \lim_{t \to \infty} \sigma(t) = \infty, \text{ and } \sigma(t \pm \alpha) = \sigma(t) \pm \alpha \text{ for any } \alpha > 0, \quad \text{and } f \in C(\mathbb{R}, \mathbb{R}), \quad f_u(t) \geq k > 0 \text{ for } u \neq 0, \quad k \text{ is a constant.} \] Section 6.1 provides necessary introduction and motivation. In Section 6.2, we use Riccati transformation technique to obtain some sufficient conditions for the oscillation of all solutions of equation (1.3.5). Examples are provided in Section 6.3 to illustrate the main results. The results presented in this chapter generalize, and improve those obtained in [38, 95, 99].

Thus, we have obtained some new results, improve, generalize, and extended some of the existing results on the oscillatory and asymptotic behavior of delay and neutral differential equations. Further, examples are presented in each chapter to illustrate the importance of the results.