Chapter 5. An Inventory Model for deteriorating items with price dependent demand under stochastic inflation process

5.1 Introduction

This chapter considers a periodic review inventory policy for deteriorating items over a finite planning horizon taking into account the effect of inflation rate. The movement in inflation rate over time is assumed to be described by the Vasicek model and Cox-Ingersoll-Ross (CIR) model. Demand is assumed to be linearly decreasing in the inflation rate, and the proportion of demand backlogged during shortage is a decreasing function of the waiting time for replenishment. The optimal number of reorder cycles, which may be unequal in length, and their lengths are determined so as to maximize the total expected profit over the planning period. The optimal policy is numerically studied for given sets of model parameters and the sensitivity analysis of the policy to change in these parameters is also carried out.

5.2 An Inventory Model for Deteriorating Items with Price Dependent Demand under Vasicek Model

Inventory managers are faced with the problem of keeping control over the total cost associatied with inventory. In this regard, proper decisions pertaining to 'when to order' and 'how much to order' help in reducing the total expected cost. However, in most studies these decisions are taken assuming that the costs remain constant over time, that is inflation does not have a significant role in the decision making. For low inflation rate, this may be acceptable. But when the inflation rate is high, it can significantly raise the different cost components, and the above assumption can lead to poor inventory decisions. It is, therefore, important to investigate how time-value of money influences various policies. Studies in this direction have been reported by Buzacott (1975), who considered EOQ model with inflation, subject to different types of pricing policies. Misra (1979), who developed a discounted-cost model and included internal (company) and external (general economy) inflation rates for various costs associated with an inventory system, Sarker and Pan (1994), who surveyed the effects of inflation and the time value of money on order quantity with finite replenishment rate, among others. Some studies have also been conducted with variable demand, by Uthayakumar and

Geetha (2009), Maity (2010), Vrat and Padmanabhan (1990), Datta and Pal (1991), Hariga (1995), Hariga and Ben-Daya (1996) and Chung (2003), to name a few.

Maity and Pal (2015a, 2015b) investigated the effect of inflation and the time value of money on an inventory policy model with iso-elastic and hybrid demands subject to permissible delay in payment.

In this section, we consider a periodic review inventory model for deteriorating items allowing shortages, when demand is a linearly decreasing function of inflation rate. This stems from the common observation that for many items when the customers expect the price to rise in future, they tend to increase their demand for the same, while if they expect the price to go down, they buy less of it. To describe the movement of inflation rate, we use the Visacek model. The model is generally used to describe the movement of an interest rate as a factor of market risk, time and equilibrium value that the rate tends to revert towards. However, the model is also able to capture the mean reversion property of the inflation rate (cf. Jensen, 2009) that sets it apart from other financial prices. We assume that there is only one source of uncertainty in the inflation rate. During shortage, the willingness of a customer to backlog his demand is likely to decline with the length of waiting time. We have, therefore, assumed the backlogging rate to be a decreasing function of the waiting time. We further consider that the inventory manager is allowed a grace period to pay his dues, which is a common phenomenon in real life. This basically amounts to giving the manager a loan without interest for a certain period, beyond which he has to pay interest.

5.2.1 The Vasicek Model

The Vasicek model is generally used to describe the movement of an interest rate as a factor of market risk, time and equilibrium value that the rate tends to revert towards. However, the model is also able to capture the mean reversion property of the inflation rate (cf. Jensen, 2009) that sets it apart from other financial prices. We, therefore, use the model for the valuation of inflation rate futures.

We assume that there is only one source of uncertainty in the inflation rate. Then, if r_t denotes the spot rate at time t, r_t satisfies the equation

$$dr_t = m(r_t)dt + \sigma(r_t)dw_t,$$

where w_t is Wiener process and $m(r_t)$ is the model used.

Vasicek model gives

$$m(r_t) = \alpha - \beta r_t, \text{ and } \sigma(r_t) = \sigma,$$

where $\alpha,\ \beta,\ \sigma > 0.$

Hence, we get

$$dr_t = (\alpha - \beta r_t)dt + \sigma dw_t \tag{5.2.1}$$

Now, $dr_t = -\beta r_t dt$ gives $r_t = r_0 e^{-\beta t}$. So, writing $r_t = V(t)e^{-\beta t}$, where $r_0 = V(0)$, we have

$$dr_t = -\beta r_t dt + e^{-\beta t} dV(t)$$

or, $$e^{-\beta t} dV(t) = dr_t + \beta r_t dt.$$

Using this in (5.2.1) we get

$$e^{-\beta t} dV(t) = \alpha dt + \sigma dw_t$$

or, $$dV(t) = \alpha e^{\beta t} dt + \sigma e^{\beta t} dw_t\ . \tag{5.2.2}$$

Integrating both sides of (5.2.2) we have

$$V(t) = V(0) + \alpha \int_0^t e^{\beta s} ds + \sigma \int_0^t e^{\beta s} dw_s\ ,$$

which gives

$$r_t = r_0 e^{-\beta t} + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + \sigma e^{-\beta t} \int_0^t e^{\beta s} dw_s\ . \tag{5.2.3}$$

Solution (5.2.3) is a strong solution of (5.2.1) (cf. Reiß, 2007, page 26, Section 2.4.1) for which the solution exists and is unique as both $m(x_t) = \alpha - \beta x_t$ and $\sigma(x_t) = \sigma$ are locally Lipschitz in *x* and uniformly continuous in *t*.

The process r_t is Gaussian with mean and variance

$$E(r_t) = r_0 e^{-\beta t} + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right)$$

$$\begin{aligned} Var(r_t) &= \sigma^2 e^{-2\beta t} Var\left(\int_0^t e^{\beta s} dw_s \right) \\ &= \sigma^2 e^{-2\beta t} \int_0^t e^{2\beta s} dw_s dw_s \\ &= \sigma^2 e^{-2\beta t} \int_0^t e^{2\beta s} ds, \text{ since } dw_s dw_s = ds, \\ &= \sigma^2 \left(1 - e^{-2\beta t}\right) / 2\beta. \end{aligned}$$

5.2.2 Notations

$[0, H]$ = planning horizon

O = ordering cost per order at time $t = 0$

h = unit inventory holding cost per unit time at time $t = 0$

c = purchasing cost per unit of item at time $t = 0$

θ = constant fraction of the on-hand inventory deteriorating per unit time

I_e = interest that can be earned per rupee during the planning horizon

I_p = interest paid per rupee investment in stocks during the planning horizon

p_t = nominal selling price per item in inventory at time t

r_t = inflation rate at time t

s = backorder cost at time $t = 0$

s_L = lost sale cost at time $t = 0$

5.2.3 Assumptions

The following assumptions are made for the model:

(i) The planning horizon (0, H) is finite with n (integer) replenishment periods

(ii) The reorder points are $0 = t_0, t_1, t_2, ..., t_{n-1}$, such that $0 = t_0 < t_1 < t_2 < ... < t_{n-1} < H$. These points may not be equidistant.

(iii) In the i-th reorder interval $[t_{i-1}, t_i]$, the stock height comes down to the zero level at s_i, where $t_{i-1} < s_i < t_i$, $1 \le i \le n-1$. and $s_n = t_n = H$ .However, no backlogging is allowed in the last interval $[t_{n-1}, H]$.

(iii) The demand rate at t is linearly decreasing in the inflation rate r_t at t, and is given by

$$D(t)=\begin{cases} k(A-r_t), & I(t)\geq 0 \\ R, & I(t)<0 \end{cases}, \quad \text{where} \quad k, A \geq 0.$$

(iv) During shortage, since a customer's willingness to wait is likely to decline with the length of waiting time, we take the backlogging rate at *t* to be

$$B(t)=\frac{1}{1+\delta(T-t)}, \quad \delta \geq 0.$$

(v) Items in stock deteriorate at a constant proportion θ per unit time.

5.2.4 Mathematical Model

We consider a periodic review model allowing backlogging, where orders are placed at fixed points of time on the planning horizon. The order quantity at each reorder point is just sufficient to meet the backlog in the previous period and take up the stock to some maximum height. However, in the last reorder interval, no backlogging is allowed, and the order quantity at the beginning of the interval is such that the stock height reduces to zero at the end of the period.

We define $T_j = t_j - t_{j-1}, \; 1 \leq j \leq n$, where t_n = *H*. Further, we denote by $I_i(t)$ the stock on hand at time *t* in the *i*-th replenishment period.

Now, in any replenishment period when there is stock on hand, the stock level diminishes owing to demand and deterioration, while it diminishes only due to demand during the shortage period. Hence, the differential equations defining the variation in the stock level with respect to time are given by

$$\frac{dI_i(t)}{dt}+\theta I_i(t)=k(A-r_t), \qquad t_{i-1} \leq t \leq s_i \quad i=1(1)(n-1)$$

$$\frac{dI_n(t)}{dt}+\theta I_n(t)=k(A-r_t), \qquad t_{n-1} \leq t \leq t_n$$

$$\frac{dI_i(t)}{dt}=-\frac{R}{1+\delta(t_i-t)}, \qquad s_i \leq t \leq t_i \quad i=1(1)(n-1).$$

The boundary conditions are $I_i(s_i)=0, \; 1 \leq i \leq (n-1)$, and $I_n(t_n)=0$.

Hence,

$$I_i(t) = ke^{-\theta t}\left(\int_{t_{i-1}}^{s_i} e^{\theta s}(A - r_s)ds - \int_{t_{i-1}}^{t} e^{\theta s}(A - r_s)ds\right), \text{ for } t_{i-1} \le t \le s_i, \quad i = 1(1)(n-1)$$

$$= ke^{-\theta t}\left(\int_{t_{n-1}}^{t_n} e^{\theta s}(A - r_s)ds - \int_{t_{n-1}}^{t} e^{\theta s}(A - r_s)ds\right), \text{ for } i = n$$

$$= -\int_{t_i}^{t} \frac{R}{1+\delta(t_i - t)}dt, \qquad \text{for } s_i \le t \le t_i,\ i = 1(1)(n-1).$$

Using the distribution of r_t, the expectation and variance of the stock level come out to be

$$E(I_i(t)) = E\left(ke^{-\theta t}\left(\int_{t_{i-1}}^{s_i} e^{\theta s}(A - r_s)ds - \int_{t_{i-1}}^{t} e^{\theta s}(A - r_s)ds\right)\right)$$
$$= k\left(A - \frac{\alpha}{\beta}\right)\left(\frac{e^{\theta(s_i - t)} - 1}{\theta}\right) - k\left(r_0 - \frac{\alpha}{\beta}\right)\left(\frac{e^{(\theta-\beta)s_i - \theta t} - e^{-\beta t}}{\theta - \beta}\right)$$

$$Var(I_i(t)) = k^2 e^{-2\theta t}\int_{t_{i-1}}^{t} e^{2\theta s} Var(r_s) ds$$

$$= k^2 e^{-2\theta t}\int_{t_{i-1}}^{t} e^{2\theta s}\frac{\sigma^2}{2\beta}\left(1 - e^{-2\beta s}\right)ds$$

$$= \frac{k^2\sigma^2}{2\beta}\left(\frac{1 - e^{2\theta(t_{i-1} - t)}}{2\theta} - \frac{e^{-2\beta t} - e^{2(\theta-\beta)t_{i-1} - 2\theta t}}{2(\theta - \beta)}\right).$$

5.2.5 Profit Functions

The different components of the profit function and their expressions are as follows.

(a) Inventory holding cost:

$$HC_i = he^{t_{i-1}E\left(r_{t_{i-1}}\right)}\int_{t_{i-1}}^{s_i}\left(k\left(A - \frac{\alpha}{\beta}\right)\left(\frac{e^{\theta(s_i - t)} - 1}{\theta}\right) - k\left(r_0 - \frac{\alpha}{\beta}\right)\left(\frac{e^{(\theta-\beta)s_i - \theta t} - e^{-\beta t}}{\theta - \beta}\right)\right)dt$$

$$= he^{t_{i-1}E\left(r_{t_{i-1}}\right)}\left(\frac{k}{\theta}\left(A - \frac{\alpha}{\beta}\right)\left(\frac{e^{\theta(s_i - t_{i-1})} - 1}{\theta} - (s_i - t_{i-1})\right) - \frac{k}{\theta - \beta}\left(r_0 - \frac{\alpha}{\beta}\right)\left(e^{-\beta s_i}\frac{e^{\theta(s_i - t_{i-1})} - 1}{\theta} + \frac{e^{-\beta s_i} - e^{-\beta t_{i-1}}}{\beta}\right)\right)$$

(b) Deteriorating Cost:

$$DC_i = \theta c e^{t_{i-1}E\left(r_{t_{i-1}}\right)}\int_{t_{i-1}}^{s_i}\left(k\left(A - \frac{\alpha}{\beta}\right)\left(\frac{e^{\theta(s_i - t)} - 1}{\theta}\right) - k\left(r_0 - \frac{\alpha}{\beta}\right)\left(\frac{e^{(\theta-\beta)s_i - \theta t} - e^{-\beta t}}{\theta - \beta}\right)\right)dt$$

$$=\theta c e^{t_{i-1}E\left(r_{t_{i-1}}\right)}\left(\frac{k}{\theta}\left(A-\frac{\alpha}{\beta}\right)\left(\frac{e^{\theta(s_i-t_{i-1})}-1}{\theta}-(s_i-t_{i-1})\right)-\frac{k}{\theta-\beta}\left(r_0-\frac{\alpha}{\beta}\right)\left(e^{-\beta s_i}\frac{e^{\theta(s_i-t_{i-1})}-1}{\theta}+\frac{e^{-\beta s_i}-e^{-\beta t_{i-1}}}{\beta}\right)\right)$$

(c) Backorder Cost:

$$SC_i = s e^{t_{i-1}E\left(r_{t_{i-1}}\right)}\int_{s_i}^{t_i}\int_{s_i}^{z}\frac{R}{1+\delta(t_i-t)}dtdz=\frac{Rs}{\delta^2}e^{t_{i-1}E\left(r_{t_{i-1}}\right)}\left(\delta(t_i-s_i)-\log\left(1+\delta(t_i-s_i)\right)\right)$$

(d) Lost sale cost:

$$LS_i = s_L e^{t_{i-1}E\left(r_{t_{i-1}}\right)}\int_{s_i}^{t_i}R\left(1-\frac{1}{1+\delta(t_i-t)}\right)dt \quad =\frac{Rs_L}{\delta}e^{t_{i-1}E\left(r_{t_{i-1}}\right)}\left(\delta(t_i-s_i)-\log\left(1+\delta(t_i-s_i)\right)\right)$$

(e) Average purchasing cost:

$$PC_i = ce^{t_{i-1}E\left(r_{t_{i-1}}\right)}I_i(t_{i-1})=ce^{t_{i-1}E\left(r_{t_{i-1}}\right)}\left(k\left(A-\frac{\alpha}{\beta}\right)\left(\frac{e^{\theta(s_i-t_{i-1})}-1}{\theta}\right)-k\left(r_0-\frac{\alpha}{\beta}\right)\left(\frac{e^{(\theta-\beta)s_i-\theta t_{i-1}}-e^{-\beta t_{i-1}}}{\theta-\beta}\right)\right)$$

(f) Average selling price:

$$SP_i = pe^{t_{i-1}E\left(r_{t_{i-1}}\right)}\int_{t_{i-1}}^{s_i}k\left(A-r_0e^{-\beta t}-\frac{\alpha}{\beta}\left(1-e^{-\beta t}\right)\right)dt$$

$$=pe^{t_{i-1}E\left(r_{t_{i-1}}\right)}k\left(\left(A-\frac{\alpha}{\beta}\right)(s_i-t_{i-1})+\left(r_0-\frac{\alpha}{\beta}\right)\frac{e^{-\beta s_i}-e^{-\beta t_{i-1}}}{\beta}\right)$$

(g) Ordering Cost: $O_i = Oe^{t_{i-1}E\left(r_{t_{i-1}}\right)}$

The total expected revenue is, therefore, given by

$$P(n,\{s_i\},\{t_i\})=\sum_{i=1}^{n-1}\left(SP_i-\left(PC_i+HC_i+DC_i+O_i+LS_i+SC_i\right)\right)+SP_n-\left(PC_n+HC_n+DC_n+O_n\right)$$

$$=\sum_{i=1}^{n}e^{t_{i-1}E\left(r_{t_{i-1}}\right)}[pk\{(A-\frac{\alpha}{\beta})(s_i-t_{i-1})+(r_0-\frac{\alpha}{\beta})\frac{e^{-\beta s_i}-e^{-\beta t_{i-1}}}{\beta}\}$$

$$-ck\{(A-\frac{\alpha}{\beta})(\frac{e^{\theta(s_i-t_{i-1})}-1}{\theta})-(r_0-\frac{\alpha}{\beta})(\frac{e^{(\theta-\beta)s_i-\theta t_{i-1}}-e^{-\beta t_{i-1}}}{\theta-\beta})\}$$

$$-O-(h+c\theta)\{\frac{k}{\theta}(A-\frac{\alpha}{\beta})(\frac{e^{\theta(s_i-t_{i-1})}-1}{\theta}$$

$$-(s_i-t_{i-1}))-\frac{k}{\theta-\beta}(r_0-\frac{\alpha}{\beta})(e^{-\beta s_i}\frac{e^{\theta(s_i-t_{i-1})}-1}{\theta}+\frac{e^{-\beta s_i}-e^{-\beta t_{i-1}}}{\beta})\}]$$

$$-\sum_{i=1}^{n-1}e^{t_{i-1}E\left(r_{t_{i-1}}\right)}\frac{R(s+\delta s_L)}{\delta^2}[\delta(t_i-s_i)-\log\{1+\delta(t_i-s_i)\}].$$

The total risk defines the overall potential for financial loss due to a particular course of action. Inflation is the most significant risk because it lowers the real return. We take the "risk" measure to be the variance of the on-hand inventory, which is given by

$$TR(n,\{s_i\},\{t_i\})=\sum_{i=1}^{n}\frac{k^2\sigma^2}{2\beta}\left(\frac{1-e^{2\theta(t_{i-1}-s_i)}}{2\theta}-\frac{e^{-2\beta s_i}-e^{2(\theta-\beta)t_{i-1}-2\theta s_i}}{2(\theta-\beta)}\right).$$

We aim at finding the optimum ordering policy that maximizes the expected profit subject to the total risk not exceeding a specified value v_0, say.

If the first (n-1) reorder intervals be of equal lengths T, with $s_i-t_{i-1}=T_1(<T)$ for $1\le i\le n$, so that $(n-1)T+T_1=H$, we have the following theorems.

5.2.6 Some Results

Theorem 5.2.1: If $t_i-t_{i-1}=T$, for $i=1(1)n\text{-}1$, $s_i-t_{i-1}=T_1$, for $i=1(1)n$, and $r_0, A\ge\dfrac{\alpha}{\beta}$, $\theta\ge\beta$. Then $P(T_1,T)$ is concave in T_1 for fixed T.

Proof: We have,

$$P(T_1,T)=pk\left(A-\frac{\alpha}{\beta}\right)T_1G_1+pk\left(r_0-\frac{\alpha}{\beta}\right)\frac{1-e^{-\beta T_1}}{\beta}G_2-\frac{ck}{\theta}\left(A-\frac{\alpha}{\beta}\right)\left(e^{\theta T_1}-1\right)G_1-\frac{ck}{\theta-\beta}\left(r_0-\frac{\alpha}{\beta}\right)\left(e^{(\theta-\beta)T_1}-1\right)G_2$$

$$-(h+c\theta)\left(\frac{k}{\theta}\left(A-\frac{\alpha}{\beta}\right)\left(\frac{e^{\theta T_1}-1}{\theta}-T_1\right)G_1+\frac{k}{\theta-\beta}\left(r_0-\frac{\alpha}{\beta}\right)\left(\frac{e^{(\theta-\beta)T_1}-e^{-\beta T_1}}{\theta}-\frac{1-e^{-\beta T_1}}{\beta}\right)G_2\right)-OG_1$$

$$-\frac{R(s+\delta s_L)}{\delta^2}\frac{n(n-1)}{2}\left(\delta(T-T_1)-\log(1+\delta(T-T_1))\right)$$

$$+pe^{(n-1)TE(r_{(n-1)T})}k\left(\left(A-\frac{\alpha}{\beta}\right)T+\left(r_0-\frac{\alpha}{\beta}\right)\frac{e^{-\beta H}-e^{-\beta(n-1)T}}{\beta}\right)$$

$$-ce^{(n-1)TE(r_{(n-1)T})}\left(\frac{k}{\theta}\left(A-\frac{\alpha}{\beta}\right)\left(e^{\theta T}-1\right)+\frac{k}{\theta-\beta}\left(r_0-\frac{\alpha}{\beta}\right)\left(\left(e^{\theta T-\beta H}-e^{-\beta(n-1)T}\right)\right)\right)$$

$$-(h+c\theta)e^{(n-1)TE(r_{(n-1)T})}\left(\frac{k}{\theta}\left(A-\frac{\alpha}{\beta}\right)\left(\frac{e^{\theta T}-1}{\theta}-T\right)+\frac{k}{\theta-\beta}\left(r_0-\frac{\alpha}{\beta}\right)\left(\left(\frac{e^{\theta T-\beta H}-e^{-\beta H}}{\theta}-\frac{e^{-\beta((n-1)T)}-e^{-\beta H}}{\beta}\right)\right)\right)$$

$$-Oe^{(n-1)TE(r_{(n-1)T})},$$

where $G_1=\sum_{i=1}^{n-1}e^{(i-1)T\left(\frac{\alpha}{\beta}+\left(r_0-\frac{\alpha}{\beta}\right)e^{-(i-1)\beta T}\right)}$, $G_2=\sum_{i=1}^{n-1}e^{(i-1)T\left(\frac{\alpha}{\beta}+\left(r_0-\frac{\alpha}{\beta}\right)e^{-(i-1)\beta T}\right)-\beta(i-1)T}$.

Then,

$$\frac{dP(T_1,T)}{dT_1} = pk\left(A-\frac{\alpha}{\beta}\right)G_1 + pk\left(r_0-\frac{\alpha}{\beta}\right)e^{-\beta T_1}G_2 - ck\left(A-\frac{\alpha}{\beta}\right)e^{\theta T_1}G_1 - ck\left(r_0-\frac{\alpha}{\beta}\right)e^{(\theta-\beta)T_1}G_2$$
$$-(h+c\theta)\left(\frac{k}{\theta}\left(A-\frac{\alpha}{\beta}\right)\left(e^{\theta T_1}-1\right)G_1+\frac{k}{\theta-\beta}\left(r_0-\frac{\alpha}{\beta}\right)\left(\frac{(\theta-\beta)e^{(\theta-\beta)T_1}+\beta e^{-\beta T_1}}{\theta}-e^{-\beta T_1}\right)G_2\right)$$
$$-\frac{R(s+\delta s_L)}{\delta^2}\frac{n(n-1)}{2}\left(-\delta+\frac{\delta}{1+\delta(T-T_1)}\right)$$

$$\frac{d^2P(T_1,T)}{dT_1^2} = k\left(r_0-\frac{\alpha}{\beta}\right)e^{-\beta T_1}G_2\left(-p\beta-c(\theta-\beta)e^{\theta T_1}-(h+c\theta)\frac{(\theta-\beta)e^{\theta T_1}+\beta}{\theta}\right)$$
$$-ck\theta\left(A-\frac{\alpha}{\beta}\right)e^{\theta T_1}G_1-k(h+c\theta)\left(A-\frac{\alpha}{\beta}\right)e^{\theta T_1}G_1-\frac{R(s+\delta s_L)n(n-1)}{2}\frac{1}{(1+\delta(T-T_1))^2}\le 0.$$

Provided $r_0, A \ge \frac{\alpha}{\beta}$ *and* $\theta \ge \beta$.

Thus, the total profit function is concave in T_1 □

Theorem 5.2.2: When $t_i - t_{i-1} = T$, for $i = 1(1)n\text{-}1$, and $s_i - t_{i-1} = T_1$, for $i = 1(1)n$, TR(T_1,T) is convex in T_1 for fixed T.

Proof: We have, the total risk function over time horizon H

$$TR((T_1,T) = \frac{k^2\sigma^2}{2\beta}\left(\frac{1-e^{-2\theta T_1}}{2\theta}\frac{n(n-1)}{2}-\frac{e^{-2\beta T_1}-e^{-2\theta T_1}}{2(\theta-\beta)}\sum_{i=1}^{n-1}e^{-2\beta(i-1)T}\right)$$
$$+\frac{k^2\sigma^2}{2\beta}\left(\frac{1-e^{-2\theta T}}{2\theta}-\frac{e^{-2\beta H}-e^{-2\beta(H-T)-2\theta T}}{2(\theta-\beta)}\right).$$

Hence,

$$\frac{dTR((T_1,T)}{dT_1} = \frac{k^2\sigma^2}{2\beta}\left[\frac{n(n-1)}{2}\left\{e^{-2\theta T_1}-\frac{2\theta e^{-2\theta T_1}}{2(\theta-\beta)}\right\}-\frac{2\beta e^{-2\beta(T-T_1)}}{2(\theta-\beta)}\frac{e^{-2\beta T}-e^{-2\beta H}}{1-e^{-2\beta T}}\right]$$

and

$$\frac{d^2TR((T_1,T)}{dT_1^2} = \frac{k^2\sigma^2}{2\beta}\left[\frac{n(n-1)}{2}\left\{-2\theta e^{-2\theta T_1}+\frac{2\theta^2 e^{-2\theta T_1}}{(\theta-\beta)}\right\}+\frac{2\beta^2 e^{-2\beta(T-T_1)}}{(\theta-\beta)}\frac{e^{-2\beta T}-e^{-2\beta H}}{1-e^{-2\beta T}}\right]$$

$$=\frac{k^2\sigma^2}{2\beta}[\frac{n(n-1)}{(\theta-\beta)}\theta\beta e^{-2\theta T_1}+\frac{2\beta^2 e^{-2\beta(T-T_1)}}{(\theta-\beta)}\frac{e^{-2\beta T}-e^{-2\beta H}}{1-e^{-2\beta T}}]\geq 0.$$

Thus, the total risk function is convex in T_1 □

5.2.7 Numerical Example

Table 5.2.1: 3 years Inflation rate in India (starting from January 2012 source: MOSPI)

Month	Inflation Rate	Month	Inflation Rate
0	7.65	19	9.52
1	8.83	20	9.84
2	9.47	21	10.17
3	10.26	22	11.16
4	10.36	23	9.87
5	9.93	24	8.79
6	9.86	25	8.03
7	10.03	26	8.31
8	9.73	27	8.59
9	9.75	28	8.28
10	9.9	29	7.46
11	10.56	30	7.96
12	10.79	31	7.73
13	10.91	32	6.46
14	10.39	33	5.52
15	9.39	34	4.38
16	9.31	35	5
17	9.87	36	5.11
18	9.64		

Now $r_t = r_{t-1}e^{-\beta\Delta} + \frac{\alpha}{\beta}\left(1-e^{-\beta\Delta}\right) + \sigma\sqrt{\frac{1-e^{-2\beta\Delta}}{2\beta}}N_{0,1}$

Where $\Delta = 1/12\, years$ $N_{0,1} \sim N(0,1)$

Hence $r_{t+1} = ar_t + b + \varepsilon$

The conditional probability density of an observation r_{t+1} given a previous observation r_t (with a Δ time step between them) is given by

$$f\left(r_{t+1} \mid r_t;\alpha,\beta,\hat{\sigma}\right) = \frac{1}{\sqrt{2\pi}\hat{\sigma}} e^{\left[-\frac{\left(r_t - r_{t-1}e^{-\beta\Delta} - \frac{\alpha}{\beta}\left(1-e^{-\beta\Delta}\right)\right)^2}{2\hat{\sigma}^2}\right]}$$

With $\hat{\sigma} = \sigma\sqrt{\frac{1-e^{-2\beta\Delta}}{2\beta}}$

The log-likelihood function of a set of observation $r_0, r_1, ..., r_n$ can be derived from the conditional density function

$$L(\alpha,\beta,\hat{\sigma}) = \sum_{t=1}^{n}\log f\left(r_t \mid r_{t-1};\alpha,\beta,\hat{\sigma}\right)$$

$$= -\frac{n}{2}\log(2\pi) - n\log(\hat{\sigma}) - \frac{1}{2\hat{\sigma}^2}\sum_{t=1}^{n}\left(r_t - r_{t-1}e^{-\beta\Delta} - \frac{\alpha}{\beta}\left(1-e^{-\beta\Delta}\right)\right)^2$$

By solving $\frac{\partial L(\alpha,\beta,\hat{\sigma})}{\partial \alpha} = 0$, $\frac{\partial L(\alpha,\beta,\hat{\sigma})}{\partial \beta} = 0$, $\frac{\partial L(\alpha,\beta,\hat{\sigma})}{\partial \hat{\sigma}} = 0$ we get,

$$\frac{\alpha}{\beta} = \frac{R_y R_{xx} - R_x R_{xy}}{n\left(R_{xx} - R_{xy}\right) - \left(R_x^2 - R_x R_y\right)} = \mu \qquad \beta = -\frac{1}{\Delta}\log\frac{R_{xy} - \mu R_x - \mu R_y + n\mu^2}{R_{xx} - 2\mu R_x + n\mu^2}$$

$$\hat{\sigma}^2 = \frac{1}{n}\left[R_{yy} - 2e^{-\beta\Delta}R_{xy} + e^{-2\beta\Delta}R_{xx} - 2\mu\left(1-e^{-\beta\Delta}\right)\left(R_y - e^{-\beta\Delta}R_x\right) + n\mu^2\left(1-e^{-\beta\Delta}\right)^2\right]$$

$$\sigma^2 = \hat{\sigma}^2\,\frac{2\beta}{1-e^{-2\beta\Delta}},$$

where $R_x = \sum_{t=1}^{n} r_{t-1}$, $R_y = \sum_{t=1}^{n} r_t$, $R_{xx} = \sum_{t=1}^{n}\left(r_{t-1}\right)^2$, $R_{xy} = \sum_{t=1}^{n} r_t r_{t-1}$, $R_{yy} = \sum_{t=1}^{n}\left(r_t\right)^2$

From the above data we get $\alpha = 0.0012$, $\beta = 0.1083$, $\sigma = 0.0223$

Example 5.2.1: Suppose k=100, A=50, α=0.0012, β =0.1083, r_0=0.08, σ=0.0223, θ=0.08, R=4000, c=80, p=100, S_L=2, s=3, h=5, δ=3, H=10 Years, O=200, Permissible Variance= 150.

If the first (n-1) replenishment cycles are assumed to be equal and $s_i - t_{i-1} = T_1$, for $i = 1(1)n$, the optimum policy is to place 7 orders at time 0 and thereafter at time points each of which is at a distance of T = 1.45475 units from the previous ordering point, and the order quantity in any reorder interval is just sufficient to meet the backorders in the previous period as well as the demand during the first 1.2715 units of time in that interval. The last replenishment cycle is of length 1.2715 units of time and the stock height reduces to zero at the end of the cycle. The total expected profit for the policy is Rs. 591724.47.

On the other hand, if the replenishment cycles are allowed to be of different lengths, the optimum number of cycles is 14, and the ordering policy is as given in Table 5.2.2. The total expected profit comes out to be Rs. 731696.70.

Table 5.2.2: Optimum replenishment policy

Cycle No.	Start Point	End Point	Cycle Length(T_i=t_{i+1}-t_i)	Shortage period(=t_i-s_i)
1	0	0.877074	0.877074253	0.178571429
2	0.877074	1.668645	0.791571104	0.103098262
3	1.668645	2.427041	0.758395313	0.079859571
4	2.427041	3.164415	0.737374013	0.067493656
5	3.164415	3.886418	0.722002966	0.05952381
6	3.886418	4.596441	0.710023302	0.053841312
7	4.596441	5.296799	0.700358411	0.049526803
8	5.296799	5.989196	0.692396556	0.046106945
9	5.989196	6.674942	0.685746391	0.043309933
10	6.674942	7.355081	0.680138249	0.040967095
11	7.355081	8.030454	0.67537365	0.03896748
12	8.030454	8.701758	0.671303503	0.037234717
13	8.701758	9.369568	0.667810675	0.035714286
14	9.369568	10	0.666145899	0

5.2.8 Sensitivity Analysis

In the following tables we examine how sensitive the optimum policy is to a change in the parameter values in **Example 5.2.1**.

Table 5.2.3: Changes in the values of the decision variables with change in *c*, and the corresponding % change in the expected profit from that when *c* = 80

c	*n*	Profit	% Change
70	15	1274937	74.24391
75	15	1011955	38.3025
80	14	731696.7	0
85	14	469711.5	-35.8052
90	14	215869.1	-70.4975

Table 5.2.4: Changes in the values of the decision variables with change in *p*, and the corresponding % change in the expected profit from that when *p* = 100

p	*n*	Profit	% Change
90	12	197342.5	-73.0295
95	13	460995.9	-36.9963
100	14	731696.7	0
105	15	1000839	36.78331
110	16	1268062	73.30433

Table 5.2.5: Changes in the values of the decision variables with change in *k*, and the corresponding % change in the expected profit from that when *k* = 100

k	*n*	Profit	% Change
90	11	649068.6	-11.2927
95	13	715250	-2.24774
100	14	731696.7	0
105	17	767106.5	4.839415
110	19	775367.8	5.968471

Table 5.2.6: Changes in the values of the decision variables with change in *A*, and the corresponding % change in the expected profit from that when $A = 50$

A	n	Profit	% Change
30	13	423092.6	-42.1765
40	14	583495	-20.2545
50	14	731696.7	0
60	14	879975.9	20.26512
70	15	1053259	43.94753

Table 5.2.7: Changes in the values of the decision variables with change in *h*, and the corresponding % change in the expected profit from that when $h = 5$

h	n	Profit	% Change
1	12	756637.5	3.408624
3	13	743126.7	1.562129
5	14	731696.7	0
9	15	694371.3	-5.10121
13	16	662025.3	-9.52189

Table 5.2.8: Changes in the values of the decision variables with change in *s*, and the corresponding % change in the expected profit from that when $s = 3$

s	n	Profit	% Change
1	15	751494.2	2.705696
3	14	731696.7	0
5	13	709252.4	-3.06742
9	16	681689.8	-6.83437
13	12	678254.8	-7.30383

Table 5.2.9: Changes in the values of the decision variables with change in s_L, and the corresponding % change in the expected profit from that when $s_L = 2$

s_L	n	Profit	% Change
2	15	731696.7	0
4	15	726208.3	-0.75009
6	15	721141.6	-1.44255
8	15	716305.1	-2.10355
10	15	711618.6	-2.74405

Table 5.2.10: Changes in the values of the decision variables with change in r_0, and the corresponding % change in the expected profit from that when $r_0 = 0.08$

r_0	n	Profit	% Change
0.04	12	647237.9	-11.5429
0.06	14	697883.6	-4.62119
0.08	14	731696.7	0
0.1	15	786167.7	7.444484
0.12	16	842129.7	15.09273

Table 5.2.11: Changes in the values of the decision variables with change in θ, and the corresponding % change in the expected profit from that when $\theta = 0.08$

θ	n	Profit	% Change
0.02	16	871585.9	19.11847
0.06	15	788979.3	7.828738
0.08	14	731696.7	0
0.12	13	662728.2	-9.42582
0.16	13	510789.8	-30.191

Table 5.2.12: Changes in the values of the decision variables with change in δ, and the corresponding % change in the expected profit from that when δ= 3

δ	n	Profit	% Change
1	15	733786.9	0.285667
3	14	731696.7	0
6	13	729912.3	-0.24387
11	13	728350.1	-0.45736
16	16	727508.4	-0.57241

On the basis of the results of Table 5.3.3-12, the following observations can be made.

i) Optimal profit is slightly sensitive to changes in the values of parameters δ, s_L, s and h.

ii) It is moderately sensitive to changes in parameters r_0, θ, and k.

iii) And the optimal profit is highly sensitive to changes in A, p, and c.

5.2.9 Discussion

In this sub-section, a single item inventory model with constant deterioration rate, linear demand rate, finite planning horizon, partial backordered rate, and stochastic inflation rate is developed. Here we used Vasicek model to describe fluctuations in inflation rate. The demand rate is assumed to be linearly dependent on inflation rate. The objective is to find the optimal policy that maximizes the present worth of the total system profit. The total profit function is developed using five general costs: order cost, purchasing cost, holding cost, back order cost and shortage cost with sales revenue. Each associated cost is dependent on the inflation rate in that replenishment cycle and hence is exponentially increasing with respect to time. We consider the practical situation where an inventory manager offers compensation so as to not lose the sale and establish an appropriate model for a retailer to determine the replenishment number and schedule when the backlogging rate is hyper geometric. The proposed model assumes a constant deterioration rate. In the numerical examples, it is found that the optimum average profit increases as inflation rate increases.

5.3 An Inventory Model for Deteriorating Items with price Dependent Demand under CIR model

5.3.1 Notations

The notations used are the same as in section 5.2.2 .

5.3.2 Assumptions

The assumptions used are the same as in section 5.2.3 except for the demand rate. Here the demand rate is iso-elastic and at time point t it is given by ap_t^{-b}, where $a>0, b\geq 1,$ and p_t is the price of the item at time t. Since the price is affected by inflation, the demand rate at t becomes

$$D(t)=ap_0^{-b}e^{-btr_t}\approx ap_0^{-b}\left(1-btr_t+\frac{b^2t^2r_t^2}{2}\right)=ap_0^{-b}\left(1-btr_t+\frac{b^2t^2r_t^2}{2}\right).$$

We assume the demand rate to be constant during stock-out. Hence, the demand rate at t is

$$D(t)=\begin{cases} ap_0^{-b}\left(1-btr_t+\frac{b^2t^2r_t^2}{2}\right), & I(t)\geq 0\\ R, & I(t)<0\end{cases}$$

5.3.3 The Cox-Ingersoll-Ross Model (CIR Model)

In mathematical finance, Cox-Ingersoll-Ross (CIR) model describes the evolution of inflation rate. (Brigo, D. and Fabio, M. 2001b)

Hence the inflation rate satisfies the following the stochastic difference equation:

$$dr_t=\alpha(\beta-r_t)dt+\sigma\sqrt{r_t}\,dW_t,\quad \text{where } W_t \text{ is a Wiener process.}$$

If α, β, σ are all positive and $2\alpha\beta\geq\sigma^2$ holds, the CIR process is well defined and the process is an ergodic process and possesses a stationary distribution.

r_t follows non-central Chi-square distribution with $\frac{4\alpha\beta}{\sigma^2}$ degrees of freedom and non-centrality parameter $\frac{r_0 e^{-\alpha t}}{c}$, $c = \frac{\left(1-e^{-\alpha t}\right)}{4\alpha}\sigma^2$

Given r_t at time *t*, the density of $r_{t+\Delta t}$ at time $t+\Delta t$ is

$$f\left(r_{t+\Delta t} \mid r_t ;\alpha,\beta,\sigma,\Delta t\right) = ke^{-u-v}\left(\frac{v}{u}\right)^{\frac{q}{2}} I_q\left(2\sqrt{uv}\right)$$

where $k = \frac{2\alpha}{\sigma^2\left(1-e^{-\alpha\Delta t}\right)}$, $u = kr_t e^{-\alpha\Delta t}$, $v = kr_{t+\Delta t}$, $q = \frac{2\alpha\beta}{\sigma^2} - 1$,

and $I_q\left(2\sqrt{uv}\right)$ is modified Bessel function of first kind and is of order *q*.

Then, the conditional expectation is given by

$$E\left(r_t / r_0\right) = r_0 e^{-\alpha t} + \beta\left(1-e^{-\alpha t}\right)$$

$$Var\left(r_t / r_0\right) = \frac{r_0\sigma^2}{\alpha}\left(e^{-\alpha t} - e^{-2\alpha t}\right) + \frac{\beta\sigma^2}{2\alpha}\left(1-e^{-\alpha t}\right)^2$$

5.3.4 Mathematical Model

We consider a periodic review model allowing backlogging, where orders are placed at fixed points of time on the planning horizon. The order quantity at each reorder point is just sufficient to meet the backlog in the previous period and take up the stock to some maximum height. However, in the last reorder interval, no backlogging is allowed, and the order quantity at the beginning of the interval is such that the stock height reduces to zero at the end of the period.

We define $T_j = t_j - t_{j-1}$, $1 \le j \le n$, where $t_n = H$. Further, we denote by $I_i(t)$ the stock on hand at time *t* in the *i*-th replenishment period.

Now, in any replenishment period when there is stock on hand, the stock level diminishes owing to demand and deterioration, while it diminishes only due to demand during the shortage period. Hence, the differential equations defining the variation in the stock level with respect to time are given by

$$\frac{dI_i(t)}{dt}+\theta I_i(t)=-ap_0^{-b}\left(1-btr_t+\frac{b^2t^2r_t^2}{2}\right), \qquad t_{i-1}\le t\le s_i \quad i=1(1)(n-1)$$

$$\frac{dI_n(t)}{dt}+\theta I_n(t)=-ap_0^{-b}\left(1-btr_t+\frac{b^2t^2r_t^2}{2}\right), \qquad t_{n-1}\le t\le t_n$$

$$\frac{dI_i(t)}{dt}=-\frac{R}{1+\delta(t_i-t)}, \qquad s_i\le t\le t_i \quad i=1(1)(n-1).$$

The boundary conditions are $I_i(s_i)=0$, $1\le i\le(n-1)$, and $I_n(t_n)=0$.

Hence,

$$I_i(t)=-ap_0^{-b}e^{-\theta t}\left(\int_{t_{i-1}}^{s_i}e^{\theta s}\left(1-btr_s+\frac{b^2t^2r_s^2}{2}\right)ds-\int_{t_{i-1}}^{t}e^{\theta s}\left(1-btr_s+\frac{b^2t^2r_s^2}{2}\right)ds\right),$$

for $t_{i-1}\le t\le s_i$, $\quad i=1(1)(n-1)$

$$=-ap_0^{-b}e^{-\theta t}\left(\int_{t_{n-1}}^{t_n}e^{\theta s}\left(1-btr_s+\frac{b^2t^2r_s^2}{2}\right)ds-\int_{t_{n-1}}^{t}e^{\theta s}\left(1-btr_s+\frac{b^2t^2r_s^2}{2}\right)ds\right), \text{ for } i=n$$

$$=-\int_{t_i}^{t}\frac{R}{1+\delta(t_i-t)}dt, \qquad \text{for } s_i\le t\le t_i,\ i=1(1)(n-1).$$

Using the distribution of r_t, the expectation and variance of the stock level come out to be

$$E(I_i(t))=E\left(-ap_0^{-b}e^{-\theta t}\left(\int_{t_{i-1}}^{s_i}e^{\theta s}\left(1-bsr_s+\frac{b^2s^2r_s^2}{2}\right)ds-\int_{t_{i-1}}^{t}e^{\theta s}\left(1-bsr_s+\frac{b^2s^2r_s^2}{2}\right)ds\right)\right)$$

and,

$$Var(I_i(t))=a^2p_0^{-2b}e^{-2\theta t}\int_{t_{i-1}}^{t}e^{2\theta s}Var(r_s)d$$

For finding the expressions of expected stock level and its variance, we have to find the following expressions,

$$E(r_t/r_0)=r_0e^{-\alpha t}+\beta\left(1-e^{-\alpha t}\right)$$

$$Var(r_t/r_0) = \frac{r_0\sigma^2}{\alpha}\left(e^{-\alpha t} - e^{-2\alpha t}\right) + \frac{\beta\sigma^2}{2\alpha}\left(1-e^{-\alpha t}\right)^2$$

$$E\left(r_t^2/r_0\right) = \left(r_0 e^{-\alpha t} + \beta\left(1-e^{-\alpha t}\right)\right)^2 + \frac{r_0\sigma^2}{\alpha}\left(e^{-\alpha t} - e^{-2\alpha t}\right) + \frac{\beta\sigma^2}{2\alpha}\left(1-e^{-\alpha t}\right)^2$$

$$= 2r_0\beta + e^{-2\alpha t}\left(r_0^2 - \frac{r_0\sigma^2}{\alpha}\right) + e^{-\alpha t}\left(\frac{r_0\sigma^2}{\alpha} - 2r_0\beta\right) + \frac{\beta\sigma^2 + 2\alpha\beta^2}{2\alpha}\left(1-e^{-\alpha t}\right)^2$$

$$= z_1 + z_2 e^{-\alpha t} + z_3 e^{-2\alpha t}$$

Where

$$z_1 = \beta^2 + \beta\frac{\sigma^2}{2\alpha},\ z_2 = 2r_0\beta - 2\beta^2 + \frac{\sigma^2(r_0-\beta)}{\alpha},\ z_3 = r_0^2 - 2r_0\beta + \beta^2 - \frac{\sigma^2(2r_0-\beta)}{2\alpha}$$

$$E(I(t)) = ap_0^{-b}e^{-\theta t}\left(\int_0^{T_1} E\left(1 - bsr_s + \frac{b^2s^2r_s^2}{2}\right)e^{\theta s}\mathrm{ds} - \int_0^{t} E\left(1 - bsr_s + \frac{b^2s^2r_s^2}{2}\right)e^{\theta s}\mathrm{ds}\right)$$

By Fubinis theorem,

$$E\left(\int_T X_t\,d\mu(t)\right) = \int_T E(X_t)\,d\mu(t)$$

For CIR model, $r_t > 0$.

$$E(I_i(t)) = E\left(-ap_0^{-b}e^{-\theta t}\left(\int_{t_{i-1}}^{s_i} e^{\theta s}\left(1 - bsr_s + \frac{b^2s^2r_s^2}{2}\right)ds - \int_{t_{i-1}}^{t} e^{\theta s}\left(1 - bsr_s + \frac{b^2s^2r_s^2}{2}\right)ds\right)\right)$$

$$E(I_i(t)) = -ap_0^{-b}e^{-\theta t}\left(\int_{t_{i-1}}^{s_i}\left(1 - bsE(r_s) + \frac{b^2s^2E\left(r_s^2\right)}{2}\right)e^{\theta s}\mathrm{ds} - \int_{t_{i-1}}^{t}\left(1 - bsE(r_s) + \frac{b^2s^2E\left(r_s^2\right)}{2}\right)e^{\theta s}\mathrm{ds}\right)$$

Now,

$$\int_{t_{i-1}}^{s_i} E(r_s)e^{\theta s}\,ds = \int_{t_{i-1}}^{s_i}\left(r_0 e^{(\theta-\alpha)s} + \beta\left(e^{\theta s} - e^{(\theta-\alpha)s}\right)\right)ds$$

$$= \frac{r_0-\beta}{(\theta-\alpha)}\left(e^{(\theta-\alpha)s_i} - e^{(\theta-\alpha)t_{i-1}}\right) + \beta\frac{\boldsymbol{e}^{\theta s_i} - e^{(\theta-\alpha)t_{i-1}}}{\theta}$$

$$\int_{t_{i-1}}^{t} E(r_s) e^{\theta s} ds = \int_{t_{i-1}}^{t} \left(r_0 e^{(\theta-\alpha)s} + \beta \left(e^{\theta s} - e^{(\theta-\alpha)s} \right) \right) ds = \frac{r_0 - \beta}{(\theta-\alpha)} \left(e^{(\theta-\alpha)t} - e^{(\theta-\alpha)t_{i-1}} \right) + \beta \frac{e^{\theta t} - e^{\theta t_{i-1}}}{\theta}$$

$$\int_{t_{i-1}}^{s_i} E\left(r_s^2\right) e^{\theta s} ds = \int_{t_{i-1}}^{s_i} E\left(r_s^2\right) e^{\theta s} ds = \int_{t_{i-1}}^{s_i} \left(z_1 e^{\theta s} + z_2 e^{(\theta-\alpha)s} + z_3 e^{(\theta-2\alpha)s} \right) ds$$

$$= \frac{z_1}{\theta} \left(e^{\theta s_i} - e^{\theta t_{i-1}} \right) + \frac{z_2}{(\theta-\alpha)} \left(e^{(\theta-\alpha)s_i} - e^{(\theta-\alpha)t_{i-1}} \right) + \frac{z_3}{(\theta-2\alpha)} \left(e^{(\theta-2\alpha)s_i} - e^{(\theta-2\alpha)t_{i-1}} \right)$$

$$\int_{t_{i-1}}^{t} E\left(r_s^2\right) e^{\theta s} ds = \int_{t_{i-1}}^{t} E\left(r_s^2\right) e^{\theta s} ds = \int_{t_{i-1}}^{t} \left(z_1 e^{\theta s} + z_2 e^{(\theta-\alpha)s} + z_3 e^{(\theta-2\alpha)s} \right) ds$$

$$= \frac{z_1}{\theta} \left(e^{\theta t} - e^{\theta t_{i-1}} \right) + \frac{z_2}{(\theta-\alpha)} \left(e^{(\theta-\alpha)t} - e^{(\theta-\alpha)t_{i-1}} \right) + \frac{z_3}{(\theta-2\alpha)} \left(e^{(\theta-2\alpha)t} - e^{(\theta-2\alpha)t_{i-1}} \right)$$

Hence, the expected stock level can be written as

$$E(I_i(t)) = -ap_0^{-b} e^{-\theta t} \left(\int_{t_{i-1}}^{s_i} \left(1 - bsE(r_s) + \frac{b^2 s^2 E\left(r_s^2\right)}{2} \right) e^{\theta s} \mathrm{ds} - \int_{t_{i-1}}^{t} \left(1 - bsE(r_s) + \frac{b^2 s^2 E\left(r_s^2\right)}{2} \right) e^{\theta s} \mathrm{ds} \right)$$

$$= -ap_0^{-b} e^{-\theta t} [(s_i - t_{i-1}) - b\{ \frac{\beta}{\theta} \left(s_i e^{\theta s_i} - t_{i-1} e^{\theta t_{i-1}} \right) + \frac{r_0 - \beta}{\theta - \alpha} \left(s_i e^{(\theta-\alpha)s_i} - t_{i-1} e^{(\theta-\alpha)t_{i-1}} \right)$$

$$- \frac{\beta}{\theta^2} \left(s_i e^{\theta s_i} - t_{i-1} e^{\theta t_{i-1}} \right) + \frac{r_0 - \beta}{(\theta-\alpha)^2} \left(s_i e^{(\theta-\alpha)s_i} - t_{i-1} e^{(\theta-\alpha)t_{i-1}} \right) \}$$

$$+ \frac{b^2}{2} \{ \frac{z_1}{\theta} e^{\theta s_i} \left(s_i^2 - \frac{2s_i}{\theta} + \frac{2}{\theta^2} \right) - \frac{z_1}{\theta} e^{\theta t_{i-1}} \left(t_{i-1}^2 - \frac{2t_{i-1}}{\theta} + \frac{2}{\theta^2} \right)$$

$$+ \frac{z_2}{(\theta-\alpha)} e^{(\theta-\alpha)s_i} \left(s_i^2 - \frac{2s_i}{\theta-\alpha} + \frac{2}{(\theta-\alpha)^2} \right) - \frac{z_2}{(\theta-\alpha)} e^{(\theta-\alpha)t_{i-1}} \left(t_{i-1}^2 - \frac{2t_{i-1}}{\theta-\alpha} + \frac{2}{(\theta-\alpha)^2} \right)$$

$$+ \frac{z_3}{(\theta-2\alpha)} e^{(\theta-2\alpha)s_i} \left(s_i^2 - \frac{2s_i}{\theta-2\alpha} + \frac{2}{(\theta-2\alpha)^2} \right) - \frac{z_3}{(\theta-2\alpha)} e^{(\theta-2\alpha)t_{i-1}} \left(t_{i-1}^2 - \frac{2t_{i-1}}{\theta-2\alpha} + \frac{2}{(\theta-2\alpha)^2} \right) \}$$

$$-\left(t-t_{i-1}\right)+b\{\frac{\beta}{\theta}\left(te^{\theta t}-t_{i-1}e^{\theta t_{i-1}}\right)+\frac{r_0-\beta}{\theta-\alpha}\left(te^{(\theta-\alpha)t}-t_{i-1}e^{(\theta-\alpha)t_{i-1}}\right)$$
$$-\frac{\beta}{\theta^2}\left(te^{\theta t}-t_{i-1}e^{\theta t_{i-1}}\right)+\frac{r_0-\beta}{(\theta-\alpha)^2}\left(te^{(\theta-\alpha)t}-t_{i-1}e^{(\theta-\alpha)t_{i-1}}\right)\}$$
$$-\frac{b^2}{2}\{\frac{z_1}{\theta}e^{\theta t}\left(t^2-\frac{2t}{\theta}+\frac{2}{\theta^2}\right)-\frac{z_1}{\theta}e^{\theta t_{i-1}}\left(t_{i-1}^2-\frac{2t_{i-1}}{\theta}+\frac{2}{\theta^2}\right)$$
$$+\frac{z_2}{(\theta-\alpha)}e^{(\theta-\alpha)t}\left(t^2-\frac{2t}{\theta-\alpha}+\frac{2}{(\theta-\alpha)^2}\right)-\frac{z_2}{(\theta-\alpha)}e^{(\theta-\alpha)t_{i-1}}\left(t_{i-1}^2-\frac{2t_{i-1}}{\theta-\alpha}+\frac{2}{(\theta-\alpha)^2}\right)$$
$$+\frac{z_3}{(\theta-2\alpha)}e^{(\theta-2\alpha)t}\left(t^2-\frac{2t}{\theta-2\alpha}+\frac{2}{(\theta-2\alpha)^2}\right)-\frac{z_3}{(\theta-2\alpha)}e^{(\theta-2\alpha)t_{i-1}}\left(t_{i-1}^2-\frac{2t_{i-1}}{\theta-2\alpha}+\frac{2}{(\theta-2\alpha)^2}\right)\}]$$

The expected initial stock level in the i^{th} replenishment cycle will be $E\left(I_i\left(t_{i-1}\right)\right)$

$$=-ap_0^{-b}[\left(s_i-t_{i-1}\right)-b\{\frac{\beta}{\theta}\left(s_ie^{\theta s_i}-t_{i-1}e^{\theta t_{i-1}}\right)+\frac{r_0-\beta}{\theta-\alpha}\left(s_ie^{(\theta-\alpha)s_i}-t_{i-1}e^{(\theta-\alpha)t_{i-1}}\right)$$
$$-\frac{\beta}{\theta^2}\left(s_ie^{\theta s_i}-t_{i-1}e^{\theta t_{i-1}}\right)+\frac{r_0-\beta}{(\theta-\alpha)^2}\left(s_ie^{(\theta-\alpha)s_i}-t_{i-1}e^{(\theta-\alpha)t_{i-1}}\right)\}$$
$$+\frac{b^2}{2}\{\frac{z_1}{\theta}e^{\theta s_i}\left(s_i^2-\frac{2s_i}{\theta}+\frac{2}{\theta^2}\right)-\frac{z_1}{\theta}e^{\theta t_{i-1}}\left(t_{i-1}^2-\frac{2t_{i-1}}{\theta}+\frac{2}{\theta^2}\right)$$
$$+\frac{z_2}{(\theta-\alpha)}e^{(\theta-\alpha)s_i}\left(s_i^2-\frac{2s_i}{\theta-\alpha}+\frac{2}{(\theta-\alpha)^2}\right)-\frac{z_2}{(\theta-\alpha)}e^{(\theta-\alpha)t_{i-1}}\left(t_{i-1}^2-\frac{2t_{i-1}}{\theta-\alpha}+\frac{2}{(\theta-\alpha)^2}\right)$$
$$+\frac{z_3}{(\theta-2\alpha)}e^{(\theta-2\alpha)s_i}\left(s_i^2-\frac{2s_i}{\theta-2\alpha}+\frac{2}{(\theta-2\alpha)^2}\right)-\frac{z_3}{(\theta-2\alpha)}e^{(\theta-2\alpha)t_{i-1}}\left(t_{i-1}^2-\frac{2t_{i-1}}{\theta-2\alpha}+\frac{2}{(\theta-2\alpha)^2}\right)\}]$$

And the variance of the stock level at any time *t*, is given by

$$Var\left(I_i(t)\right)=a^2p_0^{-2b}e^{-2\theta t}\int_{t_{i-1}}^{t}e^{2\theta s}Var\left(r_s\right)ds$$

$$=a^2p_0^{-2b}e^{-2\theta t}\int_{t_{i-1}}^{t}e^{2\theta s}\left(\frac{r_0\sigma^2}{\alpha}\left(e^{-\alpha s}-e^{-2\alpha s}\right)+\frac{\beta\sigma^2}{2\alpha}\left(1-e^{-\alpha s}\right)^2\right)ds$$

$$Var\left(I_i(t)\right)=a^2 p_0^{-2b}e^{-2\theta t}\int_{t_{i-1}}^{t}\left(\frac{r_0\sigma^2}{\alpha}\left(e^{(2\theta-\alpha)s}-e^{(2\theta-2\alpha)s}\right)+\frac{\beta\sigma^2}{2\alpha}\left(1+e^{(2\theta-2\alpha)s}-2e^{(2\theta-\alpha)s}\right)\right)ds$$

$$Var\left(I_i(t)\right)=a^2 p_0^{-2b}e^{-2\theta t}\left(\begin{array}{l}\frac{r_0\sigma^2}{\alpha}\left(\frac{e^{(2\theta-\alpha)t}-e^{(2\theta-\alpha)t_{i-1}}}{(2\theta-\alpha)}-\frac{e^{(2\theta-2\alpha)t}-e^{(2\theta-2\alpha)t_{i-1}}}{(2\theta-2\alpha)}\right)\\+\frac{\beta\sigma^2}{2\alpha}\left(\left(t-t_{i-1}\right)+\frac{e^{(2\theta-2\alpha)t}-e^{(2\theta-2\alpha)t_{i-1}}}{(2\theta-2\alpha)}-2\frac{e^{(2\theta-\alpha)t}-e^{(2\theta-\alpha)t_{i-1}}}{(2\theta-\alpha)}\right)\end{array}\right)$$

5.3.5 Profit Functions

The different components of the profit function and their expressions are as follows.

(a) Inventory holding cost in the i^{th} replenishment cycle:

$$HC_i = he^{t_{i-1}E\left(r_{t_{i-1}}\right)}\int_{t_{i-1}}^{s_i}E\left(I_i(t)\right)dt$$

(b) Deteriorating cost in the i^{th} replenishment cycle:

$$DC_i = \theta ce^{t_{i-1}E\left(r_{t_{i-1}}\right)}\int_{t_{i-1}}^{s_i}E\left(I_i(t)\right)dt$$

(c) Shortage cost in the i^{th} replenishment cycle:

$$SC_i = se^{t_{i-1}E\left(r_{t_{i-1}}\right)}\int_{s_i}^{t_i}\int_{s_i}^{z}\frac{R}{1+\delta(t_i-t)}dtdz=\frac{Rs}{\delta^2}e^{t_{i-1}E\left(r_{t_{i-1}}\right)}\left(\delta(t_i-s_i)-\log\left(1+\delta(t_i-s_i)\right)\right)$$

(d) Lost sale cost in the i^{th} replenishment cycle:

$$LS_i = s_L e^{t_{i-1}E\left(r_{t_{i-1}}\right)}\int_{s_i}^{t_i}R\left(1-\frac{1}{1+\delta(t_i-t)}\right)dt$$

$$=\frac{Rs_L}{\delta}e^{t_{i-1}E\left(r_{t_{i-1}}\right)}\left(\delta(t_i-s_i)-\log\left(1+\delta(t_i-s_i)\right)\right)$$

(e) Average purchasing cost in the i^{th} replenishment cycle:

$$PC_i = ce^{t_{i-1}E\left(r_{t_{i-1}}\right)}\left(E\left(I_i(t_{i-1})\right)+\frac{R}{\delta^2}\left(\delta(t_{i-1}-s_{i-1})-\log\left(1+\delta(t_{i-1}-s_{i-1})\right)\right)\right)$$

(f) Average selling price in the i^{th} replenishment cycle:

$$SP_i = p_0 e^{t_{i-1}E\left(r_{t_{i-1}}\right)}\int_{t_{i-1}}^{s_i}ap_0^{-b}\left(1-btE(r_t)+\frac{b^2t^2E\left(r_t^2\right)}{2}\right)dt$$

$$= ap_0^2 e^{t_{i-1}E\left(r_{t_{i-1}}\right)}[(s_i - t_{i-1}) - b\{\frac{\beta}{2}\left(s_i^2 - t_{i-1}^2\right) - (r_0 - \beta)\left(\frac{s_i e^{-\alpha s_i} - t_{i-1}e^{-\alpha t_{i-1}}}{\alpha} + \frac{e^{-\alpha s_i} - e^{-\alpha t_{i-1}}}{\alpha^2}\right)\}$$

$$+\frac{b^2}{2}\{z_1 \frac{s_i^3 - t_{i-1}^3}{3} - z_2\left(e^{-\alpha s_i} - e^{-\alpha t_{i-1}}\right)\left(\frac{s_i^2 - t_{i-1}^2}{\alpha} + \frac{2(s_i - t_{i-1})}{\alpha^2} + \frac{2}{\alpha^3}\right)$$

$$- z_3\left(e^{-2\alpha s_i} - e^{-2\alpha t_{i-1}}\right)\left(\frac{s_i^2 - t_{i-1}^2}{2\alpha} + \frac{(s_i - t_{i-1})}{2\alpha^2} + \frac{1}{4\alpha^3}\right)\}]$$

(g) Ordering Cost in the i^{th} replenishment cycle: $O_i = Oe^{t_{i-1}E\left(r_{t_{i-1}}\right)}$

The total expected revenue is, therefore, given by

$$P(n,\{s_i\},\{t_i\}) = \sum_{i=1}^{n-1}\left(SP_i - \left(PC_i + HC_i + DC_i + O_i + LS_i + SC_i\right)\right) + SP_n - \left(PC_n + HC_n + DC_n + O_n\right)$$

Now, $\int_{t_{i-1}}^{s_i} E(I_i(t))dt$

$$= ap_0^{-b}[(s_i - t_{i-1})\frac{e^{-\theta s_i} - e^{-\theta t_{i-1}}}{\theta} - b\{\frac{\beta}{\theta}\left(s_i e^{\theta s_i} - t_{i-1}e^{\theta t_{i-1}}\right) + \frac{r_0 - \beta}{\theta - \alpha}\left(s_i e^{(\theta-\alpha)s_i} - t_{i-1}e^{(\theta-\alpha)t_{i-1}}\right)$$

$$-\frac{\beta}{\theta^2}\left(s_i e^{\theta s_i} - t_{i-1}e^{\theta t_{i-1}}\right) + \frac{r_0 - \beta}{(\theta-\alpha)^2}\left(s_i e^{(\theta-\alpha)s_i} - t_{i-1}e^{(\theta-\alpha)t_{i-1}}\right)\}\frac{e^{-\theta s_i} - e^{-\theta t_{i-1}}}{\theta}$$

$$+\frac{b^2}{2}\{\frac{z_1}{\theta}e^{\theta s_i}\left(s_i^2 - \frac{2s_i}{\theta} + \frac{2}{\theta^2}\right) - \frac{z_1}{\theta}e^{\theta t_{i-1}}\left(t_{i-1}^2 - \frac{2t_{i-1}}{\theta} + \frac{2}{\theta^2}\right)$$

$$+\frac{z_2}{(\theta-\alpha)}e^{(\theta-\alpha)s_i}\left(s_i^2 - \frac{2s_i}{\theta-\alpha} + \frac{2}{(\theta-\alpha)^2}\right) - \frac{z_2}{(\theta-\alpha)}e^{(\theta-\alpha)t_{i-1}}\left(t_{i-1}^2 - \frac{2t_{i-1}}{\theta-\alpha} + \frac{2}{(\theta-\alpha)^2}\right)$$

$$+\frac{z_3}{(\theta-2\alpha)}e^{(\theta-2\alpha)s_i}\left(s_i^2 - \frac{2s_i}{\theta-2\alpha} + \frac{2}{(\theta-2\alpha)^2}\right) - \frac{z_3}{(\theta-2\alpha)}e^{(\theta-2\alpha)t_{i-1}}\left(t_{i-1}^2 - \frac{2t_{i-1}}{\theta-2\alpha} + \frac{2}{(\theta-2\alpha)^2}\right)\}$$

$$\times\frac{e^{-\theta s_i} - e^{-\theta t_{i-1}}}{\theta}$$

$$+\left(\frac{t_{i-1}e^{-\theta t_{i-1}} - s_i e^{-\theta s_i}}{\theta} + \frac{e^{-\theta t_{i-1}} - e^{-\theta s_i}}{\theta^2} - t_{i-1}\frac{t_{i-1}e^{-\theta t_{i-1}} - s_i e^{-\theta s_i}}{\theta}\right)$$

$$+b\{\left(\frac{\beta}{\theta} - \frac{\beta}{\theta^2}\right)\left(\frac{t_{i-1}^2 - s_i^2}{2} + t_{i-1}e^{\theta t_{i-1}}\frac{e^{-\theta s_i} - e^{-\theta t_{i-1}}}{\theta}\right)$$

$$+\left(\frac{r_0-\beta}{\theta-\alpha}+\frac{r_0-\beta}{(\theta-\alpha)^2}\right)\left(\frac{s_ie^{-\alpha s_i}-t_{i-1}e^{-\alpha t_{i-1}}}{\alpha}+\frac{e^{-\alpha s_i}-e^{-\alpha t_{i-1}}}{\alpha^2}+t_{i-1}e^{(\theta-\alpha)t_{i-1}}\frac{e^{-\theta s_i}-e^{-\theta t_{i-1}}}{\theta}\right)\}$$

$$-\frac{b^2}{2}\{-\frac{z_1}{\theta}\left(\frac{s_i^3-t_{i-1}^3}{3}-\frac{s_i^2-t_{i-1}^2}{\theta}+\frac{2(s_i-t_{i-1})}{\theta^2}\right)+\frac{z_1}{\theta}e^{\theta t_{i-1}}\left(t_{i-1}^2-\frac{2t_{i-1}}{\theta}+\frac{2}{\theta^2}\right)\frac{e^{-\theta s_i}-e^{-\theta t_{i-1}}}{\theta}$$

$$+\frac{z_2}{(\theta-\alpha)}\left(e^{-\alpha s_i}-e^{-\alpha t_{i-1}}\right)\left(\frac{s_i^2-t_{i-1}^2}{\alpha}+\frac{2(s_i-t_{i-1})}{\alpha^2}+\frac{2}{\alpha^3}\right)$$

$$-\frac{2z_2}{(\theta-\alpha)^2}\left(\frac{s_ie^{-\alpha s_i}-t_{i-1}e^{-\alpha t_{i-1}}}{\alpha}+\frac{e^{-\alpha s_i}-e^{-\alpha t_{i-1}}}{\alpha^2}\right)-\frac{2z_2}{(\theta-\alpha)^3}\frac{e^{-\alpha s_i}-e^{-\alpha t_{i-1}}}{\alpha}$$

$$+\frac{z_2}{(\theta-\alpha)}e^{(\theta-\alpha)t_{i-1}}\left(t_{i-1}^2-\frac{2t_{i-1}}{\theta-\alpha}+\frac{2}{(\theta-\alpha)^2}\right)\frac{e^{-\theta s_i}-e^{-\theta t_{i-1}}}{\theta}$$

$$+\frac{z_3}{(\theta-2\alpha)}\left(e^{-2\alpha s_i}-e^{-2\alpha t_{i-1}}\right)\left(\frac{s_i^2-t_{i-1}^2}{2\alpha}+\frac{(s_i-t_{i-1})}{2\alpha^2}+\frac{1}{4\alpha^3}\right)$$

$$-\frac{z_3}{(\theta-2\alpha)^2}\left(\frac{s_ie^{-2\alpha s_i}-t_{i-1}e^{-2\alpha t_{i-1}}}{\alpha}+\frac{e^{-2\alpha s_i}-e^{-2\alpha t_{i-1}}}{2\alpha^2}\right)-\frac{z_3}{(\theta-2\alpha)^3}\frac{e^{-2\alpha s_i}-e^{-2\alpha t_{i-1}}}{\alpha}$$

$$+\frac{z_3}{(\theta-2\alpha)}e^{(\theta-2\alpha)t_{i-1}}\left(t_{i-1}^2-\frac{2t_{i-1}}{\theta-2\alpha}+\frac{2}{(\theta-2\alpha)^2}\right)\frac{e^{-\theta s_i}-e^{-\theta t_{i-1}}}{\theta}\}]$$

We take the "risk" measure to be the variance of the on-hand inventory, which is given by

$$TR(n,\{s_i\},\{t_i\})$$
$$=a^2p_0^{-2b}\sum_{i=1}^{n}e^{-2\theta s_i}\left(\begin{array}{l}\frac{r_0\sigma^2}{\alpha}\left(\frac{e^{(2\theta-\alpha)s_i}-e^{(2\theta-\alpha)t_{i-1}}}{(2\theta-\alpha)}-\frac{e^{(2\theta-2\alpha)s_i}-e^{(2\theta-2\alpha)t_{i-1}}}{(2\theta-2\alpha)}\right)\\+\frac{\beta\sigma^2}{2\alpha}\left((s_i-t_{i-1})+\frac{e^{(2\theta-2\alpha)s_i}-e^{(2\theta-2\alpha)t_{i-1}}}{(2\theta-2\alpha)}-2\frac{e^{(2\theta-\alpha)s_i}-e^{(2\theta-\alpha)t_{i-1}}}{(2\theta-\alpha)}\right)\end{array}\right)$$

We aim at finding the optimum ordering policy that maximizes the expected profit subject to the total risk not exceeding a specified value v_0, say.

5.3.6 Algorithm

Here the optimization problem is

Maximize $P(n,\{s_i\},\{t_i\})$

Subject to $TR(n,\{s_i\},\{t_i\}) \le v_0$ and $0 \le t_{i-1} \le s_i \le t_i$, $i = 1,2,...,n$.

The above problem can be written as

Minimize $-P(n,\{s_i\},\{t_i\})$

$$\text{Subject to } \begin{array}{ll} TR(n,\{s_i\},\{t_i\}) - v_0 \le 0 & \\ t_{i-1} - s_i \le 0 & i = 1,2,...,n \\ s_i - t_i \le 0 & i = 1,2,...,n \end{array}$$

The Karush-Kuhn-Tucker conditions or KKT conditions are given by

1. $0 \in -\partial P(n,\{s_i\},\{t_i\}) + \lambda \partial TR(n,\{s_i\},\{t_i\}) + \sum_{i=1}^{n} \eta_i \partial(t_{i-1} - s_i) + \sum_{i=1}^{n} \mu_i \partial(s_i - t_i)$
2. $\eta_i(t_{i-1} - s_i) = 0$ and $\mu_i(s_i - t_i) = 0$ for $i = 1,2,...,n$
3. $\eta_i, \mu_i \ge 0$ for $i = 1,2,...,n$

Using Matlab we solved the above problem we get the optimal $n^*, \{s_i^*\}, \{t_i^*\}$.

5.3.7 Numerical Example

For the CIR model, we have used the Table 5.2.1 to estimate the model parameter.

Parameter estimation is carried out on inflation rate time series with N observations $\{r_{t_i}, i = 1..N\}$

Here N=37 and $\Delta t = 1/12$ years

The log-likelihood function for inflation rate time series with N observation is

$$\ln L(\alpha,\beta,\sigma) = \sum_{i=1}^{N-1} \ln f\left(r_{t+\Delta t} \mid r_t; \alpha,\beta,\sigma,\Delta t\right)$$

$$\ln L(\alpha,\beta,\sigma) = (N-1)\ln k + \sum_{i=1}^{N-1}\left\{-u_{t_i} - v_{t_{i+1}} + \frac{q}{2}\ln\left(\frac{v_{t_{i+1}}}{u_{t_i}}\right) + \ln\left(I_q\left(2\sqrt{u_{t_i} v_{t_{i+1}}}\right)\right)\right\}$$

where $v_{t_{i+1}} = kr_{t_{i+1}}$, $u_{t_i} = kr_{t_i}e^{-\alpha\Delta t}$.

An estimate of $\alpha = 0.1575$, $\beta = 0.65648$, $\sigma = 0.1438$.

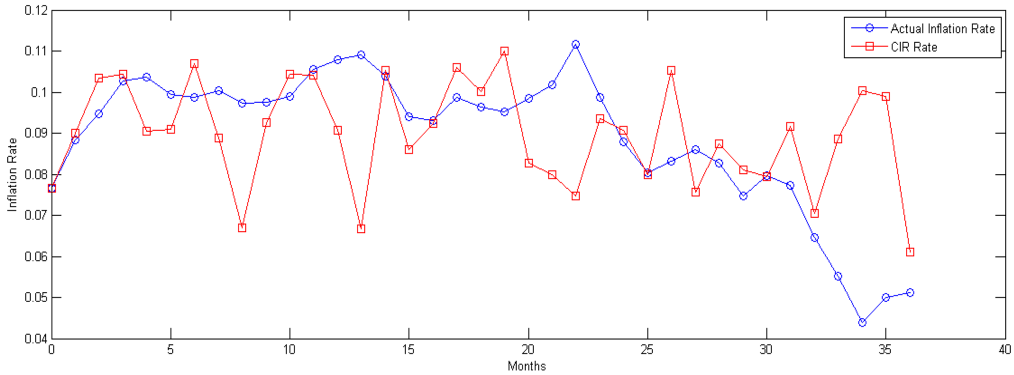

Figure 5.3.1: Movement of CIR rate and actual inflation rate with respect to time

Example 5.3.1: Suppose a=500000, b=1.5, α=0.1575, β=0.1083, r_0=0.08, σ=0.1438, θ=0.02, R=4000, c=80, p=100, S_L=2, s=3, h=5, δ=3, H=10 Years, O=2500, Permissible Variance= 3000.

If the first (n-1) replenishment cycles, which allow shortages, are assumed to be equal and $s_i - t_{i-1} = T_1$, for $i = 1(1)n$, the optimum policy is to place 23 orders at time 0 and thereafter at time points each of which is at a distance of T = 0.43544 units from the previous ordering point, and the order quantity in any reorder interval is just sufficient to meet the backorders in the previous. The last replenishment cycle is of length 0.42037 units of time and the stock height reduces to zero at the end of the cycle. The total expected profit for the policy is Rs. 7966604.

On the other hand, when the cycle lengths are allowed to vary, the optimum number of cycles is 27, and the ordering policy is as given in Table 5.3.1. The total expected profit comes out to be Rs. 8696570.8.

Table 5.3.1: Optimum replenishment policy under CIR model

Cycle No.	Start Point	End Point	Cycle Length(T_i=t_{i+1}-t_i)	Shortage period(=t_i-s_i)
1	0	0.44581	0.44581	0.11857
2	0.44581	0.88529	0.43948	0.11755
3	0.88529	1.31858	0.43329	0.11653
4	1.31858	1.74583	0.42725	0.11551
5	1.74583	2.16715	0.42132	0.11449
6	2.16715	2.58261	0.41546	0.11347
7	2.58261	2.99226	0.40965	0.11245
8	2.99226	3.39615	0.40389	0.11143
9	3.39615	3.79432	0.39817	0.11041
10	3.79432	4.1868	0.39248	0.10939
11	4.1868	4.57363	0.38683	0.10837
12	4.57363	4.95485	0.38122	0.10735
13	4.95485	5.33049	0.37564	0.10633
14	5.33049	5.70057	0.37008	0.10531
15	5.70057	6.06511	0.36454	0.10429
16	6.06511	6.42413	0.35902	0.10327
17	6.42413	6.77764	0.35351	0.10225
18	6.77764	7.12564	0.348	0.10123
19	7.12564	7.46815	0.34251	0.10021
20	7.46815	7.80517	0.33702	0.09919
21	7.80517	8.1367	0.33153	0.09817
22	8.1367	8.46276	0.32606	0.09715
23	8.46276	8.78335	0.32059	0.09613
24	8.78335	9.09849	0.31514	0.09511
25	9.09849	9.4082	0.30971	0.09409
26	9.4082	9.71249	0.30429	0.09307
27	9.71249	10.0114	0.29891	0

5.3.8 Sensitivity Analysis

In the following tables we examine how sensitive the optimum policy is to a change in the parameter values in Example 5.3.1.

Table 5.3.2: Changes in the values of the decision variables with change in *c*, and the corresponding % change in the expected profit from that when *c* = 80

c	*n*	Profit	% Change
65	34	16390024.6	88.4654
70	32	13421073.6	54.326
75	31	10286412.1	18.2812
80	27	8696570.8	0
85	27	6143645.4	-29.356
90	26	4798710.5	-44.821
95	24	2795163.2	-67.859

Table 5.3.3: Changes in the values of the decision variables with change in *p*, and the corresponding % change in the expected profit from that when *p* = 100

p	*n*	Profit	% Change
85	24	2297468.5	-73.582
90	26	4963178.1	-42.929
95	27	6832458.2	-21.435
100	27	8696570.8	0
105	29	10647887.9	22.4378
110	30	12978911.3	49.2417
115	33	15634789.4	79.7811

Table 5.3.4: Changes in the values of the decision variables with change in *a*, and the corresponding % change in the expected profit from that when *a* = 500000

a	*n*	Profit	% Change
350000	27	5187346.4	-40.352
400000	27	6798417.7	-21.826
450000	27	7591230.3	-12.71
500000	27	8696570.8	0
550000	27	9870054.5	13.4936
600000	27	10934181.8	25.7298
650000	27	12556036.2	44.3792

Table 5.3.5: Changes in the values of the decision variables with change in *b*, and the corresponding % change in the expected profit from that when *b* = 1.5

b	*n*	Profit	% Change
1.2	28	9436871.1	8.51255
1.3	28	9266478.1	6.55324
1.4	27	8976412.3	3.21784
1.5	27	8696570.8	0
1.6	26	7741687.4	-10.98
1.7	24	5047947.7	-41.955
1.8	23	3863478.2	-55.575

Table 5.3.6: Changes in the values of the decision variables with change in *h*, and the corresponding % change in the expected profit from that when *h* = 5

h	*n*	Profit	% Change
2	26	9063741.8	4.22202
3	26	8817549.1	1.3911
4	27	8764844.5	0.78506
5	27	8696570.8	0
6	27	8571447.5	-1.4388
7	28	8367521.4	-3.7837
8	28	8073600.2	-7.1634

Table 5.3.7: Changes in the values of the decision variables with change in *s*, and the corresponding % change in the expected profit from that when *s* = 3

s	*n*	Profit	% Change
0.5	29	9843546.1	13.1888
1	28	9244145.2	6.29644
2	27	8851464.5	1.78109
3	27	8696570.8	0
4	26	8458801.3	-2.7341
5	25	8167415.7	-6.0846
6	24	7865163.6	-9.5602

Table 5.3.8: Changes in the values of the decision variables with change in s_L, and the corresponding % change in the expected profit from that when $s_L = 2$

s_L	*n*	Profit	% Change
0.25	27	8796715.9	1.15155
0.5	27	8712337.1	0.18129
1	27	8700664.5	0.04707
2	27	8696570.8	0
4	26	8591166.2	-1.212
6	25	8468141.6	-2.6267
8	24	8365140.5	-3.811

Table 5.3.9: Changes in the values of the decision variables with change in r_0, and the corresponding % change in the expected profit from that when $r_0 = 0.08$

r_0	*n*	Profit	% Change
0.05	21	8064703.8	-7.2657
0.06	23	8331796.4	-4.1945
0.07	26	8501738.7	-2.2403
0.08	27	8696570.8	0
0.09	29	9037687.6	3.92243
0.1	33	9679810.2	11.3061
0.11	37	9954641.1	14.4663

Table 5.3.10: Changes in the values of the decision variables with change in θ, and the corresponding % change in the expected profit from that when θ= 0.02

θ	*n*	Profit	% Change
0.0025	19	11836710.7	36.1078
0.005	22	10737268.3	23.4655
0.01	26	9375971	7.81228
0.02	27	8696570.8	0
0.04	31	8067347.5	-7.2353
0.06	35	7237964.2	-16.772
0.1	41	5131465.9	-40.994

Table 5.3.11: Changes in the values of the decision variables with change in O, and the corresponding % change in the expected profit from that when O= 2500.

O	*n*	Profit	% Change
500	27	8907056.9	2.42033
1000	27	8811753.4	1.32446
2000	27	8729347.5	0.37689
2500	27	8696570.8	0
5000	27	8634671	-0.7118
10000	27	8594761.2	-1.1707
15000	27	8514799.7	-2.0901

Table 5.3.12: Changes in the values of the decision variables with change in δ, and the corresponding % change in the expected profit from that when δ= 3.

δ	*n*	Profit	% Change
0.5	26	8816473.2	1.37873
1	26	8795339.5	1.13572
2	27	8763710.7	0.77203
3	27	8696570.8	0
5	28	8467988.1	-2.6284
10	30	8276984.4	-4.8247
15	34	7837128.5	-9.8825

On the basis of the results of Table 5.3.2-12, the following observations can be made.

i) The percentage change in the optimal profit is approximately the same for equally positive and negative changes of all the parameters except *b*.

ii) The optimal profit increases with the increase in the values of the parameters *p*, *a* and r_0.

iii) The optimal profit decreases with the increase in the values of the parameters δ, O, θ, s_L, s, h, b and c.

iv) It is observed that the model is highly sensitive to changes in the parameters θ, *a*, *b*, *c* and *p*.

v) The model is moderately sensitive to changes in δ, r_0, *s* and *h*, while it remains more or less robust to changes in O and s_L.

5.3.9 Discussion

In this subsection, an inventory model is developed for deteriorating items, permitting shortage and with time inversely proportional backlogging rate. The demand rate is assumed to be iso-elastic and is dependent on the inflation rate, which has been modeled as a CIR model. Different costs associated with the system change in every replenishment cycle due to inflation. The main objective here is to find the optimal ordering policy so as to maximize the total expected profit over the planning horizon. The reorder intervals have been optimally determined, and are of unequal lengths. The backlogging rate, which has been considered to be a decreasing function of the waiting time, is a feature that is often observed in real life, and adds credibility to the model.