CHAPTER-1
INTRODUCTION

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Numerical analysis

The branch of mathematics concerned with finding accurate approximations to the solutions of problems whose exact solution is either impossible or infeasible to determine. In addition to the approximate solution, a realistic bound is needed for the error associated with the approximate solution\textsuperscript{[1\textendash}3. Typically, a mathematical model for a particular problem, generally consisting of mathematical equations with constraint conditions, is constructed by specialists in the area concerned with the problem. Numerical analysis is concerned with devising methods for approximating the solution to the model, and analyzing the results for stability, speed of implementation, and appropriateness to the situation.

Error considerations

Some major difficulties confront the numerical analyst when attempting to determine an effective approximation technique. The first concerns the nature of calculations in the real world. As a mathematician and scientist, one assumes that there is an infinite number of real numbers, and that each real number has an infinite number of neighbours arbitrarily close to it. Idealized computation takes into consideration all these real numbers, but practical calculation can consider only a small subset. For example, in idealized calculation one can have $(\sqrt{2})^2 = 2$. In a calculator or computer, the irrational number $\sqrt{2}$ is given an approximate representation using a fixed finite number of digits, and the squaring operation is performed on this finite-digit approximation. The result of the calculation within the computer is close to, but not exactly the same as, the real number 2. Additional operations using inexact numbers of this type can lead to significant errors in approximation\textsuperscript{[4\textendash}5]. The discipline of numerical analysis involves the design of techniques that take these and other error-producing situations into account when approximating the solution to a problem.

The difference between the true value of a real number and its finite-digit approximation within a computer is called the round-off error associated with the number. This round-off error depends not only on the real number but also on the architecture of the computer on which it is being represented and, very likely, the programming system that is used for the calculations. Approximation techniques must consider the effect of this round-off error\textsuperscript{[6]} and its propagation when the finite-digit approximations are used in subsequent calculations. In general, the total round-off error grows as a function of the number of computations, so efficient approximation techniques attempt to control the number of calculations for both computational and accuracy effectiveness.

Certain approximation procedures are more affected by round-off error difficulties than others. Methods in which small errors in representation are likely to produce large errors in the final approximation are said to be unstable. Derivative approximation techniques are generally unstable; for example, the error associated with finite-digit arithmetic can completely dominate the calculations in some derivative approximations. Techniques in which the round-off error plays little role in the accuracy of the computation are called stable\textsuperscript{[7\textendash}9. Approximating definite integrals is generally stable, since round-off is likely to have little effect on the final approximation, regardless of the number of calculations. The effect of the round-off
error on problems involving matrix operations is likely to depend on the specific matrices that are used in the computations. In some cases the round-off error will be negligible, in others it will grow dramatically and dominate the approximation. Knowing how to construct methods of approximation that will minimize the effect of round-off error on the final results is one of the basic tools of the numerical analyst.

An associated error difficulty is the representation of the problem itself. It is seldom the case that the mathematical model precisely represents the physical problem that the model's constructor wishes to study. A desirable feature for an approximation method, then, is that the solution to a realistic approximation to a problem produces a good approximation to not only the mathematical model, but the physical problem as well. Small changes in a problem are called perturbations of the problem. Perturbations might occur because of small errors in the statement of the problem or they might be designed into the approximation problem to allow a solution to be more easily determined. In a problem with many variables, such as a weather or economic prediction model, an approximate solution to a problem might only be feasible if it is assumed that certain of the variables can be ignored, and that all the other variables in the problem effect the final solution only in a linear manner. Even though this is not likely to be the true situation, the model based on these assumptions might lead to useful predictions if the model is stable with respect to the perturbations that these assumptions force on the problem.

In addition to round-off error considerations associated with the solution to a problem, there is the fact that many true solutions involve an infinite number of calculations, for example, when the true solution to a problem is known to be associated with an infinite series, such as a Taylor series. A standard procedure in approximation methods is to use only a finite number of terms of the series to represent the entire series. The error associated with the truncation of the infinite series to a finite series is called truncation error. How this truncation error effects the accuracy of the final approximation result depends on the stability of the technique to this type of error.

Approximation of data and functions

There are many situations that call for fitting a common function to a collection of data. Determining a function that agrees precisely with the data, at least to within round-off tolerances, is called interpolation. In the case of an arbitrary collection of data, polynomial interpolation is the most likely candidate for mainly pragmatic reasons. A polynomial approximation to a collection of data is easy to determine in various ways, and polynomials have easily computed derivatives and integrals that might be useful if the derivative or integral of the function underlying the data is needed. However, a polynomial of degree n-1 is generally required to satisfy a set of n conditions, and polynomials of even moderately high degree are likely to have a high degree of variation except where explicitly constrained. This means that in order to obtain stable approximating polynomials either the number of specified conditions must be kept small, which may not be a reasonable restriction, or, more likely, the conditions are only required to be satisfied in an approximate way.

Spline function interpolations use piecewise polynomials of fairly low degree to overcome some of the polynomial variation weaknesses. They are
constructed so that a fixed-degree polynomial is used as an approximation between consecutive data points, with the specific polynomial of that degree changing as the pair of data points changes. For example, by fitting cubic polynomials between pairs of consecutive data points there is sufficient flexibility to ensure the approximating function has a continuous second-order derivative. By using higher-degree polynomials between the points, higher differentiability conditions can be imposed. Modifications of the spline procedure can be made to guarantee that the approximating piecewise polynomial will satisfy user-specified conditions concerning the direction that the curve will assume at the data points. This is particularly important when the resulting curve needs to satisfy aesthetic conditions needed in computer graphics routines.

Interpolation is not the only possibility for data approximation, and may not be the technique of choice, even when the approximating function is a polynomial. The method of least squares is a classic method of controlling the variability on an approximating polynomial. This technique determines the polynomial of a fixed degree that "best" approximates a given set of data. Although the least squares polynomial that is constructed may not agree precisely with any of the data, it has the property that the sum of the squares of the differences between the approximate ordinates and the ordinates associated with the data is as small as possible. Other criteria are used to construct the "best" polynomial that fits a given collection of data, but the least-squares procedure is the most popular.

The problem of approximating a known function with a simpler function follows a pattern similar to that of fitting a function to a collection of data. There are many reasons why such an approximation might be required. For example, suppose one needs to know the value of the definite integral of a given function. In only the most elementary situations will it be possible to evaluate this integral directly, so some approximation technique is required. The basic underlying technique involves fitting a polynomial to a collection of data generated by the function and using the definite integral of this polynomial to approximate the integral of the function.

It might also be the case that one needs values of a particular function to use in some approximating procedure; and evaluating the function is computationally expensive or time consuming or particularly inconvenient; for example, when the approximation is needed in a computer program and the definition of the function lies outside the program. In this case a sufficiently accurate approximation to the function may not affect the error in the final approximation and may be much easier or cheaper to determine.

When approximating a known function with a simpler function, there is flexibility in the choice of data since the data in this case is obtained by evaluating the function at specified points. In some cases one may wish the data to be equally-spaced; in others it can be chosen so that more data is available at points of special interest, perhaps when there appears to be a rapid change in the values of the function or its derivatives.

Periodic data and functions are more appropriately approximated using the basic periodic functions, the sine and cosine. These techniques are called Fourier approximation methods and are very commonly used in applications such as signal
processing. Large amounts of data are sampled, generally at equally-spaced time
intervals, with the objective of quickly determining a periodic function that
approximates the data. Because of the large amount of calculation involved, the data
are sampled in ways that permit this calculation to be done extremely efficiently,
using techniques known as fast Fourier transforms.

All of these techniques extend to functions of multiple variables, although the
amount of computation increases with the number of variables. When this increase is
a polynomial function of the number of variables, there is an inherent stability in the
procedure, but computation that increases

**Approximation of integrals and derivatives**

Integration is extremely important in applications, but very rarely is it possible
to obtain the exact value of a definite integral since this requires an anti-derivative of
the function. When one cannot find the anti-derivative, an approximation method is
needed. The definition of the definite integral involves the limit of sums involving
evaluations of the function, which provides a natural way for approximating the
integral. It is a very inefficient procedure, however, and much better approximation
methods are obtained by using an interpolating function to approximate the function
and then integrating the interpolating function to produce the approximate value for
the integral. The interpolating functions that are generally used are polynomials or
piecewise polynomials. Numerical integration procedures[27, 33] are generally stable
and can be designed to incorporate accuracy checks to modify the technique if it is
suspected that the approximation is not sufficiently accurate.

Since the definition of the derivative involves the limit of the ratio of the
change in the values of a function with respect to the change in its variable, the basic
derivative approximation techniques use a difference quotient to approximate the
derivative of a given function. However, numerical differentiation is generally
unstable with respect to round-off error, and, unless care is used, the resulting
approximations may be meaningless.

**Theory of Errors**

The branch of mathematical statistics devoted to the inference of accurate
conclusions about the numerical values of approximately measured quantities, as well
as on the errors in the measurements. Repeated measurements of one and the same
constant quantity generally give different results, since every measurement contains a
certain error. There are three basic types of error: systematic, gross and random[20].
Systematic errors always either overestimate or underestimate the results of
measurements and arise for specific reasons (incorrect set-up of measuring
equipment, the effect of environment, etc.), which systematically affect the
measurements and alter them in one direction. The estimation of systematic errors is
achieved using methods which go beyond the confines of mathematical statistics.
Gross errors (often called outliers[30,31]) arise from miscalculations[32], incorrect
reading of the measuring equipment, etc. The results of measurements which contain
gross errors differ greatly from other results of measurements[34,36] and are therefore
often easy to identify. Random errors arise from various reasons which have an
unforeseen effect on each of the measurements, both in overestimating and in underestimating results.

The theory of errors is only concerned with the study of gross and random errors. The basic problems of the theory of errors are to study the distribution laws of random errors, to seek estimates of unknown parameters using the results of measurements, to establish the errors in these estimates, and to identify gross errors. Let the values \( Y_1, Y_2, \ldots, Y_n \) be obtained as a result of \( n \) independent, equally accurate measurements of a certain unknown variable \( \mu \). The differences

\[
\delta_i = Y_i - \mu, \quad \delta_1 = Y_1 - \mu, \quad \delta_2 = Y_2 - \mu, \ldots, \quad \delta_n = Y_n - \mu
\]

are called the true errors. In terms of the probability theory of errors all \( \delta_i \) are treated as random variables; independence of measurements is understood to be mutual independence of the random variables \( \delta_1, \delta_2, \ldots, \delta_n \). The equal accuracy of the measurements is treated broadly as an identical distribution: The true errors of equally accurate measurements are identically distributed random variables. The mathematical expectation of the true errors \( b = E\delta_1 = \ldots = E\delta_n \) is then called the systematic error, while the differences \( \delta_1-b, \delta_2-b, \ldots, \delta_n-b \) are called the random errors. Thus, the absence of a systematic error means that \( b = 0 \), and, in this situation, \( \delta_1, \delta_2, \ldots, \delta_n \) are random errors. The variable \( 1/\sqrt{2} \sigma \), where \( \sigma \) is the standard deviation, is called the measure of accuracy (when a systematic error occurs, the measure of accuracy is expressed by the relation \( 1/\sqrt{2(b^2+\sigma^2)} \)). Equal accuracy of measurements is understood in a narrow sense to mean the equality of the measure of accuracy for all the results of the measurements. The incidence of gross errors signifies the disruption of equal accuracy (both in the broad and narrow sense) for certain specific measurements. As an estimator of the unknown value \( \mu \) one usually takes the arithmetic mean from the results of the measurements:

\[
\bar{Y} = \frac{1}{n} \sum_{i=1}^{n} Y_i,
\]

while the differences \( \Delta_1 = \bar{Y}_1 - \bar{Y}, \ldots, \Delta_n = \bar{Y}_n - \bar{Y} \) are called the apparent errors. The choice of \( \bar{Y} \) as an estimator for \( \mu \) is based on the fact that for any sufficiently large number \( n \) of equally accurate measurements with no systematic error, \( \bar{Y} \), with probability arbitrarily close to one, differs by an arbitrarily small amount from the unknown variable; \( \bar{Y} \) is free of systematic errors (estimators with this property are called unbiased, and its variance is

\[
DY = E(\bar{Y} - \mu)^2 = \frac{\sigma^2}{n}.
\]

Experience has shown that in practice the random errors \( \delta_i \) are very often subject to almost normal distributions (the reasons for this are revealed in the so-called limit theorems of probability theory). In this case the variable \( \bar{Y} \) has an almost normal distribution with mathematical expectation \( \mu \) and variance \( \sigma^2/n \). If the distributions of \( \delta_i \) are exactly normal, then the variance of every other unbiased
estimator for \( \mu \), for example the median, is not less than \( \bar{Y} \). If the distribution of \( \delta_i \) is not normal, then the latter property need not hold.

If the variance \( \sigma^2 \) of separate measurements is previously unknown, then the variable

\[
s^2 = \frac{1}{n-1} \sum_{i=1}^{n} \Delta_i^2
\]

is used as estimator for it (\( \text{Es}^2 = \sigma^2 \), i.e. \( s^2 \) is an unbiased estimator for \( \sigma^2 \)). If the random errors \( \delta_i \) have a normal distribution, then the relation

\[
t = \frac{(Y - \mu)Z}{s} \sqrt{n}
\]

is subject to the Student distribution with \( n-1 \) degrees of freedom. This can be used to estimate the error of the approximate equality.

The variable \( (n-1)s^2/\sigma^2 \) has under the same assumptions a "chi-squared" distribution with \( n-1 \) degrees of freedom. This enables one to estimate the error of the approximate equality \( \sigma = s \). It can be demonstrated that the relative error will not exceed the number \( q \) with probability

\[
\omega = F(z_2, n-1) - F(z_1, n-1),
\]

where \( F(z, n-1) \) is the \( \chi^2 \)-distribution function,

\[
z_1 = \frac{\sqrt{n-1}}{1+q}, \quad z_2 = \frac{\sqrt{n-1}}{1-q}.
\]

If certain measurements contain gross errors, then the above rules for estimating \( \mu \) and \( \sigma \) will give distorted results. It is therefore very important to be able to differentiate measurements which contain gross errors from those which are subject only to random errors \( \delta_i \). For the case where \( \delta_i \) are independent and have an identical normal distribution, a comprehensive method for identifying measurements which contain gross errors was proposed by N. V. Smirnov\(^{[37-38]} \).

Modern developments in the treatment of errors include robust estimation and outlier detection and treatment. The intuitive definition of an outlier is: an observation which deviates so much from other observations as to arouse suspicions that it was generated by a different mechanism (than the one under observation). This includes errors such as can arise for instance because data were copied incorrectly. Outliers are dangerous for many statistical procedures. One way to deal with outliers is to use outlier tests to accept or reject the hypothesis that an observation \( x^* \) belongs to a sample from a random variable or not. The observations for which this hypothesis is rejected are then removed. Other methods for dealing with outliers include censoring, the use of robust methods and Winsorization (which is robust), of which the
underlying idea is to move all excessively outlying observations in some systematic way to a position near the more central observations. One mechanism which may cause (apparent) outliers is when the data come from a heavy tailed distribution. Another one is when the data come from two distributions: a basic one which yields "good" observations and a second contaminating distribution. Which treatment of outliers is appropriate depends of course heavily on the mechanism generating the outliers.

Robust statistics\(^{[39-41]}\) in a loose sense tries to deal with the fact that many often made assumptions such as normality, linearity, independence are at best approximations to the real situation. Thus, one looks for tests, statistical procedures which are, for instance, insensitive to the assumption that the underlying distribution is normal. Let the statistical model being used be, e.g., a parametrized family of distributions \(F(\theta)\) conceived of as lying in a larger space of distributions \(S\). One main aspect of robust statistics is then the study of the effects of deformations of \(F(\theta)\) in \(S\) on the various statistical procedures being used. Similar concerns have motivated the study of deformations in other parts of mathematics, for instance deformations of dynamical systems. More generally, robust statistics is concerned with statistical concepts which describe the behaviour of statistical procedures not only under parametric models but also in the neighbourhoods of such models.

**Approximation theory**

The branch of mathematical analysis studying methods for approximating some mathematical objects by others and also studying questions related to the research and estimation of the error that arises here\(^{[7,16]}\).

The main contents of approximation theory concerns the approximation of functions. Its foundations are laid by the work of P. L. Chebyshev (1854–1859) on best uniform approximation\(^{[8]}\) of functions by polynomials and by K. Weierstrass, who in 1885 established that in principle it is possible to approximate a continuous function on a finite interval by polynomials with arbitrary pre-given error.

With the development of functional analysis, many problems in approximation theory were considered in the most general setting, e.g. as the approximation of elements of an arbitrary linear normed space \(X\)\(^{[18]}\). Three classes of problems arose here, corresponding more or less to the main chronological stages of development of research in approximation theory.

1) The approximation of a fixed element \(x\in X\) by elements of a fixed set \(\mathcal{N}\subset X\). If

\[
E(x, \mathcal{N}) = \inf_{u \in \mathcal{N}} \|x - u\|
\]

is taken as the approximation measure, i.e. the best approximation\(^{[18]}\) of \(x\) by \(\mathcal{N}\), then, along with the investigation and estimation of \(E(x, \mathcal{N})\), questions on the existence of an element of best approximation \(\mathcal{N} \subset \mathcal{N}\) (for which \(\|x - u\| = E(x, \mathcal{N})\)), its uniqueness and characteristic features arise. Any operator \(A\) mapping \(X\) into \(\mathcal{N}\) gives rise to an approximation method with error \(\|x - Ax\|\). If \(\mathcal{N}\) is a linear manifold, linear
operators are of particular importance. For sequences \( \{A_n\} \) of such operators, the question of the conditions of convergence \( A_n x \to x \) for any \( x \in X \) arises.

2) The approximation of a fixed set \( M \subset X \) by elements of another fixed set \( N \subset X \). The best approximation here is expressed by

\[
E(M, N) = \sup_{x \in M} E(x, M),
\]

which gives the minimal possible error estimate when approximating an arbitrary \( x \in X \) by elements from \( N \). In concrete cases, the problem consists of estimating, or expressing exactly, \( E(M, M) \) by characteristic properties of the given sets \( M \) and \( N \). If the approximation is established by an operator \( A \), the supremum

\[
\sup_{x \in M} \|x - Ax\|
\]

is investigated, as well as (if \( M \) is a linear manifold)

\[
\varepsilon(M, M) = \inf_{A \in \mathcal{L}} \sup_{x \in M} \|x - Ax\|
\]

where the infimum is taken over all linear operators \( A \) mapping \( X \) into \( M \). A linear operator realizing this infimum (if it exists) gives rise to a best linear method of approximation. The case \( \varepsilon(M, M) = E(M, M) \) is of particular interest.

3) Best approximation of a fixed set \( M \subset X \) by a given class \( \{N\} \) of approximating sets in \( X \). It is assumed that in \( \{N\} \) there are, in a definite sense, "equally-expensive" classes, e.g. containing the same amount of elements or having the same dimension. The first case leads to the \( \varepsilon \)-entropy of \( M \) (relative to \( X \)), the second — to the problem of calculating the width of \( M \) (in \( X \)), in particular,

\[
d_M(M, X) = \inf_{N \in \mathcal{N}} E(M, M), \tag{1}
\]

and

\[
d_M(M, X) = \inf_{N \in \mathcal{N}} \varepsilon(M, M), \tag{2}
\]

where the infimum is taken over all subspaces \( M \) in \( X \) of fixed dimension \( N \) (or over all possible translations \( N + \alpha \) of it). Thus in (1), (2) the problem is to determine the best (respectively the best linear) approximation tool of dimension \( N \) for \( M \).

**Numerical analysis – overview**

When it comes to numerical analysis, I am an opportunist. I am motivated by problems that occur in practical engineering computations, and where possible I seize the opportunity to do some rigorous analysis to gain insight into practical calculations.
Usually, this does not involve the development of new methods of numerical analysis, but rather the application of well-established methods to new problems. The challenge is to construct an idealisation of the problem which is sufficiently simple for rigorous analysis, but yet is still relevant to the engineering application.

**Interpolation formula**

A formula for the approximate calculation of values of a function $f(x)$ by replacing it by a function

$$g(x) = g(x; \alpha_0, \ldots, \alpha_n)$$

that is simple in a certain sense and belongs to a certain class. The parameters $\alpha_i$, $i = 0, \ldots, n$, are chosen in such a way that the values of $g(x)$ coincide with the known values of $f(x)$ on a given set of $n + 1$ distinct values of the argument:

$$g(x_k) = f(x_k), \quad k = 0, \ldots, n. \quad (1)$$

This method of approximately representing a function is called interpolation and the points $x_k$ at which (1) should hold are called interpolation nodes. Instead of the simplest condition (1), the values of some quantity related to $f(x)$ may also be given, e.g. the values of a derivative of $f(x)$ at interpolation nodes.

The method of linear interpolation is the most widespread among the interpolation methods. The approximation is now looked for in the class of (generalized) polynomials

$$g(x; \alpha_0, \ldots, \alpha_n) = \sum_{i=0}^{n} \alpha_i \phi_i(x) \quad (2)$$

in some fixed system of functions $\phi_0(x), \ldots, \phi_n(x)$. In order for the interpolation polynomial (2) to exist for any function $f(x)$ defined on an interval $[a, b]$, and for any choice of $n + 1$ nodes $x_0, \ldots, x_n \subseteq [a, b]$, $x_i \neq x_j$ if $i \neq j$, it is necessary and sufficient that $\{\phi_i(x)\}$ is a Chebyshev system of functions on $[a, b]$. The interpolation polynomial will, moreover, be unique and its coefficients $\alpha_i$ can be found by directly solving (1).

For $\{\phi_i(x)\}$ one often takes: the sequences of powers of $x$,

$$1, x, x^2, \ldots,$$

the sequence of trigonometric functions,

$$1, \sin x, \cos x, \sin 2x, \cos 2x, \ldots,$$

or the sequence of exponential functions,
$1, e^{\alpha_1 x}, e^{\alpha_2 x}, \ldots,$

where $\{\alpha_i\}$ is a sequence of distinct real numbers.

When interpolating by algebraic polynomials

$$
\sum_{t=0}^{n} \alpha_t x^i
$$

the system $\{\phi_t(x)\}$ is

$$
\phi_t(x) = x^i, \quad i = 0, \ldots, n,
$$

while (1) has the form

$$
\sum_{t=0}^{n} \alpha_t x_k^i = f(x_k), \quad k = 0, \ldots, n.
$$

The system (4) is a Chebyshev system, which ensures the existence and uniqueness of the interpolation polynomial (3). A property of (4) is the possibility of obtaining an explicit representation of the interpolation polynomial (3) without immediately having to solve (5). One of the explicit forms of (3),

$$
g_\alpha(x) = L_n(x) = \sum_{t=0}^{n} f(x_t) \prod_{j \neq t} \frac{x - x_j}{x_t - x_j},
$$

is called the Lagrange interpolation polynomial. If the derivative $f^{(n+1)}(x)$ is continuous, the remainder of (6) can be written as

$$
f(x) - g_\alpha(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \omega_n(x),
$$

where $\xi \in [y_1, y_2], \quad \omega_n(x) = \prod_{t=0}^{n} (x - x_t);$

where $y_1 = \min(x_0, \ldots, x_n, x), \quad y_2 = \max(x_0, \ldots, x_n, x).$ The value of the remainder (7) depends, in particular, on the values of $\omega_n(x).$ The choice of interpolation nodes, for which $\sup_{[a,b]} |\omega_n(x)|$ is minimal, is of interest. The distribution of the nodes is optimal in this sense if the roots

$$
x_k = \frac{b + a}{2} + \frac{b - a}{2} \cos \frac{2k + 1}{2n + 2} \pi, \quad k = 0, \ldots, n,
$$
of the polynomial

\[ T_{n+1}^{[a,b]}(x) = (b-a)^{n+1} 2^{1-2n} T_{n+1}\left(\frac{2x-(b+a)}{b-a}\right), \]

which deviates least from zero on \([a,b]\) are taken as the nodes. Here \(T_{n+1}(z)\) is the Chebyshev polynomial of degree.

There is a number of other explicit representations of (3) that are more useful for solving this or another practical interpolation problem (e.g., Bessel interpolation formula; Gauss interpolation formula; Newton interpolation formula; Stirling interpolation formula; Steffensen interpolation formula; Everett interpolation formula). If it is difficult to estimate in advance the degree of the interpolation polynomial that is necessary for attaining the error desired (e.g., when interpolating a table), then one takes recourse to the Aitken scheme.\(^{[38, 54-55]}\) In this scheme interpolation polynomials of increasing degrees are constructed sequentially, thus making it possible to control the accuracy in the computational process. Another approach to the construction of interpolation formulas can be found in Fraser diagram.

The Hermite interpolation formula gives the solution to the problem of the algebraic interpolation\(^{[50]}\) of the values of a function and its derivatives at interpolation nodes.

Consider the interpolation problem of finding a polynomial \(P_N\) of degree \(\leq N\) satisfying the conditions

\[ P_N^{[i,k]}(x_i) = c_{i,k}, \quad (a1) \]

where the \(x_1, \ldots, x_m\) are \(m\) distinct knots, and there are precisely \(N+1\) equations in (a1). If for each \(i\) the orders of the derivatives occurring in (a1) form an unbroken series \(k = 0, \ldots, k_i\), one has Hermite interpolation. (In case \(k_i = 0\) for all \(i\), i.e. if no interpolation conditions involving derivatives occur in (a1), one has Lagrange interpolation.) If gaps (lacunae) occur, one speaks of lacunary interpolation or Birkhoff interpolation. The pairs \((i, k)\) which occur in (a1) are conveniently described in terms of an interpolation matrix \(E\) of size \(m \times (n+1)\), 

\[ E = (e_{i,k}), i = 1, \ldots, m, \]

\[ k = 0, \ldots, n, \]

where \(e_{i,k} = 1\) if \((i, k)\) does occur in (a1) and \(e_{i,k} = 0\) otherwise. The matrix \(E\) is called regular if (a1) is solvable for all choices of the \(x_i\) and \(c_{i,k}\) and singular otherwise.

More generally, let \(G = \{g_0, \ldots, g_N\}\) be a system of linearly independent \(n\)-times continuously-differentiable real-valued functions on an interval \([a, b]\) or on the circle. Instead of polynomials now consider linear combinations \(P = \sum_{j=0}^N a_j g_j\). A matrix of zeros and ones \(E = (e_{i,k}), i = 1, \ldots, m, k = 0, \ldots, n\) is an interpolation matrix if there are precisely \(N+1\) ones in \(E\) (and, usually, if there are no rows of zeros in \(E\); this means that all knots do occur at least once in an interpolation
condition). Let \( K = \{ x_1, \ldots, x_m \} \) be a set of knots, i.e. \( m \) distinct points of the interval or circle. Finally, for each \((i, k)\) such that \( \varepsilon_{i,k} = 1 \) let there be given a number \( c_{i,k} \). These data \( (G, E, K, c_{i,k}) \) define a Birkhoff interpolation problem:

\[
P^{(k)}(x_i) = c_{i,k}
\]

for all \((i, k)\) such that \( \varepsilon_{i,k} = 1 \).

The pair \((E, K)\) is called regular if (a2) is solvable for all choices of the \( c_{i,k} \).

For each \((i, k)\) such that \( \varepsilon_{i,k} = 1 \), consider the row vector of length \( N + 1 \),

\[
\mathfrak{g}^{(k)}(x_i), \ldots, \mathfrak{g}^{(k)}(x_N).
\]

For varying \((i, k)\) such that \( \varepsilon_{i,k} = 1 \) one thus finds \( N + 1 \) row vectors which together make up an \((N + 1) \times (N + 1)\) matrix. The pair \((E, K)\) is regular if and only if this matrix is invertible. Its determinant, where the pairs \((i, k)\) with \( \varepsilon_{i,k} = 1 \) are ordered lexicographically, is denoted \( D(E, K) \).

Suppose that the \( c_{i,k} \) are the values \( f^{(k)}(x_i) \) of the derivatives of some function \( f \) at the knots. Then a simple formula for the solution of the interpolation problem (a2) follows from Cramer's rule. Indeed, if \( D^{(j)}(E, K) \) denotes the determinant obtained by replacing \( \mathfrak{g}^j \) with \( f \) in the formula for \( D(E, K) \), then

\[
P(t) = \sum_{j=0}^{N} \frac{D^{(j)}(E, K)}{D(E, K)} \mathfrak{g}^j(t)
\]

Interpolation in numerical mathematics

A method for approximating or precisely finding some quantity by known individual values of it or of other quantities related to it. On the basis of interpolation a whole series of approximate methods for solving mathematical problems has been developed.

Most significant in numerical mathematics is the problem of constructing means for the interpolation of functions. The interpolation of functionals and operators is also widely used in constructing numerical methods.

The approximate representation and calculation of functions.

Interpolation of functions is considered as one of the ways of approximating them. Interpolating a function \( f(x) \) on a segment \([a, b]\) by its values at the nodes \( x_k \) of a grid \( \Lambda_n = \{ a = x_0 < \ldots < x_n = b \} \) means constructing another function \( L_n(x) \equiv L_n(f; x) \) such that \( L_n(x_k) = f(x_k), k = 0, \ldots, n. \) In a more general
setting, the problem of interpolating a function \( f(x) \) consists of constructing \( L_n(x) \) not only by prescribing values on a grid \( \Delta_n \), but also derivatives at individual nodes, up to a certain order, or by describing some other relation connecting \( f(x) \) and \( L_n(x) \).

Usually \( L_n(x) \) is constructed in the form

\[
L_n(x) = \sum_{i=0}^{n} a_i \phi_i(x),
\]

where \( \{ \phi_i(x) \} \) is a certain system of linearly independent functions, chosen in advance. Such an interpolation is called linear with respect to \( \{ \phi_i(x) \} \), while \( L_n(x) \) is called an interpolation polynomial in the system \( \{ \phi_i(x) \} \) or an interpolation function.

The choice of \( \{ \phi_i(x) \} \) is determined by the properties of the function class for which the interpolation formula is constructed. E.g., for the approximation of \( 2\pi \)-periodic functions on \([0, 2\pi]\) one naturally takes the trigonometric system for \( \{ \phi_i(x) \} \), for the approximation of bounded or increasing functions on \([0, \infty)\) one takes the system of rational or exponential functions, taking into account the behaviour of the functions to be approximated at infinity, etc.

Most often one uses algebraic interpolation: \( \phi_i(x) = x^i \); its simplest variant (linear interpolation with two nodes \( x_k \) and \( x_k + 1 \)) is defined by the formula

\[
L_1(x) = \frac{x - x_k}{x_k + 1 - x_k} [f(x_k + 1) - f(x_k)] + f(x_k),
\]

\( x_k \leq x \leq x_k + 1 \).

Algebraic interpolation of a very high order is seldom used in practice in the problem of approximating functions on an entire segment \([a, b]\). One usually restricts oneself to linear interpolation by (1) or to quadratic interpolation with three nodes on particular segments of the grid, by the formula

\[
L_2(x) = \frac{(x - x_k)(x - x_k + 1)}{(x_k - 1 - x_k)(x_k - 1 - x_k - 1)} f(x_k - 1) +
\]

\[
\frac{(x - x_k - 1)(x - x_k + 1)}{(x_k - x_k - 1)(x_k - x_k + 1)} f(x_k) + \frac{(x - x_k - 1)(x - x_k)}{(x_k + 1 - x_k - 1)(x_k + 1 - x_k)} f(x_k + 1),
\]

\( x_k - 1 \leq x \leq x_k + 1 \).
There are several ways of writing the algebraic interpolation polynomials. Interpolation by splines gains increasing application.

Parabolic or cubic splines are most often used in practice. An interpolation spline of defect 1 for a function \( f(x) \) with respect to a given grid \( \Delta x \) is a function \( S_2(x) = S_2(f; x) \) that is a polynomial of degree three on each segment \([x_{k-1}, x_k]\), belongs to the class of twice continuously-differentiable functions, and satisfies the conditions

\[
S_2(x_k) = f(x_k), \quad k = 0, \ldots, n, \quad n \geq 2.
\]

There are still two free parameters in this definition; these are determined by additional boundary conditions: \( S_2^{(i)}(a) = S_2^{(i)}(b) \), \( i = 1, 2 \), \( S_2^{(i)}(a) = a_i \), \( S_2^{(i)}(b) = b_i \), or other conditions.

As well as immediately in the problem of approximating functions, splines are also used in solving other problems; the splines are required then not only to coincide on a grid \( \Delta x \) with the values of a function \( f(x) \), but also with those of the derivatives of this function, up to a certain order.

When processing empirical data \( \{y_k\} \) one often determines the coefficients of in \( L_n(x) \) by requiring

\[
S = \sum_{k=1}^{m} \left[ y_k - L_n(x_k) \right]^2, \quad m \geq n,
\]

to be minimal. The function \( L_n(x) \) thus constructed is called the interpolation function in the sense of least squares.

The interpolation of functions in several variables meets with a number of principal and numerical difficulties. E.g., in the case of algebraic interpolation the Lagrange interpolation polynomial of fixed degree need not, generally speaking, exist for an arbitrary system of different nodes. In particular, for a function \( f(x, y) \) in two variables such a polynomials \( L_n(x, y) \) of total degree at most \( n \) can be constructed for nodes \((x_k, y_k)\) only if these nodes do not lie on an algebraic curve of order \( n \).

Another approach to the interpolation of functions \( f(x_1, \ldots, x_m) \) in several variables is that one first interpolates the function with respect to \( x_k \) for fixed \( x_{k_2}, \ldots, x_{k_m} \), then with respect to the next variable \( x_{k_2} \) for fixed remaining nodes, etc. Now the interpolation polynomial \( L_{n_1 \ldots n_m}(x_{1}, \ldots, x_m) \) for \( f(x_{1}, \ldots, x_m) \) with nodes

\[
(x_{1}^{n_1}, \ldots, x_{m}^{n_m}),
\]

is constructed.
\[ x_j^{(r)} \neq x_j^{(\ell)} \quad \forall r \neq \ell; \quad k_j = 0, \ldots, n_j; \quad j = 1, \ldots, m, \]

has the form:

\[
L_{n_1 \ldots n_m}(x_1, \ldots, x_m) =
\sum_{k_1^{(1)}, \ldots, k_m^{(m)} = 0}^{n_1 \ldots n_m} \frac{\omega_{n_1}(x_1) \ldots \omega_{n_m}(x_m) \ f(x_1^{k_1^{(1)}}, \ldots, x_m^{k_m^{(m)}})}{f(x_1^{k_1^{(1)}}, \ldots, x_m^{k_m^{(m)}})}
\]

where

\[
\omega_{n_j}(x_j) = \prod_{k_j^{(j)} = 0}^{n_j} (x_j - x_j^{k_j^{(j)})}, \quad j = 1, \ldots, m.
\]

Interpolation splines \[^{[58]}\] for functions of several variables are defined on a multi-dimensional grid \[^{[59]}\], with corresponding changes, in analogy with the one-dimensional case. Interpolation of functions is used for replacing complicate functions by simpler ones that can be calculated quicker; for the approximate recovery of functions on their entire domain of values by their values at individual points; and for obtaining smoother functions described by a running process. This kind of problems is of both independent interest and arises in an auxiliary fashion in many branches of science and technology in solving complex problems. Interpolation of functions is also used in approximately finding limit values of functions, in problems of accelerating the convergence of series or sequences, etc.