CHAPTER 3

GENERALIZED INTUITIONISTIC FUZZY QUASI UNIFORM NORMAL SPACES

In this chapter, the concept of generalized intuitionistic fuzzy quasi uniform normal space is introduced based on the concepts in [18, 55]. Several interesting properties and characterizations of generalized intuitionistic fuzzy quasi uniform normal spaces are discussed. In this connection, Tietze extension theorem is established.

3.1 GENERALIZED NORMALITY IN INTUITIONISTIC FUZZY QUASI UNIFORM NORMAL SPACES

In this section, the concept of generalized intuitionistic fuzzy quasi uniform normal space is introduced. Some interesting properties are studied.

Definition 3.1.1. Let $D$ be the family of intuitionistic fuzzy mapping $f : \zeta^X \rightarrow \zeta^X$ [Remark, 1.4.2] such that

(i) $A \subseteq f(A)$, for all $A \in \zeta^X$. 

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(ii) \( f(\cup A_i) = \cup f(A_i) \), for each \( A_i \in \xi^X, i \in J \).

(iii) \( f(0_\sim) = 0_\sim \).

For \( f \in D \), the mapping \( f^{-1} \in D \) is defined by \( f^{-1}(A) = \cap \{ B \in \xi^X : f(B) \subseteq B \} \). For \( f, g \in D \), \( (f \cap g)(A) = \cap \{ f(A_1) \cup g(A_2) : A_1 \cup A_2 = A \} \) and \( (f \circ g)(A) = f(g(A)) \), for all \( A, A_1, A_2 \in \xi^X \).

**Definition 3.1.2.** Define as a subset \( U \subset D \). Then \( U \) is called an intuitionistic fuzzy quasi uniformity on \( X \), if it satisfies the following axioms:

(i) If \( f \in U, f \subseteq g \) and \( g \in D \) then \( g \in U \).

(ii) If \( f_1, f_2 \in U \) then there exists \( g \in U \) such that \( g \subseteq f_1 \cap f_2 \).

(iii) For every \( f \in U \), there exists \( g \in U \), such that \( g \circ g \subseteq f \).

Then the pair \( (X, U) \) is said to be an intuitionistic fuzzy quasi uniform space. An intuitionistic fuzzy quasi uniformity is called an intuitionistic fuzzy uniformity if \( f \in U \Rightarrow f^{-1} \in U \).

**Definition 3.1.3.** Let \( (X, U) \) be an intuitionistic fuzzy quasi uniform space. Then the operator \( Int_U : \xi^X \rightarrow \xi^X \) defined by \( Int_U(A) = \cup \{ B : f(B) \subseteq A \} \) for some \( f \in U \) and \( A, B \in \xi^X \} \) is the intuitionistic fuzzy quasi uniform interior operator.

**Definition 3.1.4.** Let \( (X, U) \) be an intuitionistic fuzzy quasi uniform space. Then the mapping \( T_U : \xi^X \rightarrow \xi^X \) is defined by, \( T_U = \{ A \in \xi^X :
\( Int_U(A) = A \). Here, \( T_U \) is a generated topology by \( U \). Then the pair \((X, T_U)\) is called an intuitionistic fuzzy quasi uniform topological space. The members of \( T_U \) are called intuitionistic fuzzy quasi uniform open sets. The complement of an intuitionistic fuzzy quasi uniform open set is called an intuitionistic fuzzy quasi uniform closed set.

**Definition 3.1.5.** Let \((X, T_U)\) be an intuitionistic fuzzy quasi uniform topological space. and let \( A \) be an intuitionistic fuzzy set. Then the intuitionistic fuzzy quasi uniform closure (\( IFQUcl \) for short) and intuitionistic fuzzy quasi uniform interior (\( IFQUint \) for short) of \( A \) are defined by,

\[
IFQUcl(A) = \{ G : G \text{ is an intuitionistic fuzzy quasi uniform closed set in } X \text{ and } A \subseteq G \},
\]

\[
IFQUint(A) = \{ G : G \text{ is an intuitionistic fuzzy quasi uniform open set in } X \text{ and } A \supseteq G \}.
\]

**Definition 3.1.6.** Let \((X, T_U)\) be an intuitionistic fuzzy quasi uniform topological space and let \( A \) be an intuitionistic fuzzy set. Then \( A \) is called generalized intuitionistic fuzzy quasi uniform closed if \( IFQUcl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is an intuitionistic fuzzy quasi uniform open set. The complement of a generalized intuitionistic fuzzy quasi uniform closed set is a generalized intuitionistic fuzzy quasi uniform open set.

**Definition 3.1.7.** Let \((X, T_U)\) be an intuitionistic fuzzy quasi uniform topological space and let \( A \) be an intuitionistic fuzzy set. Then the gen-
eralized intuitionistic fuzzy quasi uniform closure (\( GI F QU cl \) for short) and generalized intuitionistic fuzzy quasi uniform interior (\( GI F QU int \) for short) of \( A \) are defined by,
\[
GI F QU cl(A) = \bigcup \{ G : G \text{ is a generalized intuitionistic fuzzy quasi uniform closed set in } X \text{ and } A \subseteq G \},
\]
\[
GI F QU int(A) = \bigcap \{ G : G \text{ is a generalized intuitionistic fuzzy quasi uniform open set in } X \text{ and } A \supseteq G \}.
\]

**Definition 3.1.8.** An intuitionistic fuzzy quasi uniform topological space \((X, T_U)\) is said to be a generalized intuitionistic fuzzy quasi uniform normal space if for every generalized intuitionistic fuzzy quasi uniform closed set \( A \) and generalized intuitionistic fuzzy quasi uniform open set \( B \) in \((X, T_U)\) such that \( A \subseteq B \), there exists an intuitionistic fuzzy set \( C \), such that \( A \subseteq GI F QU int(C) \subseteq GI F QU cl(C) \subseteq B \).

**Proposition 3.1.1.** Let \((X, T_U)\) be a generalized intuitionistic fuzzy quasi uniform normal space if and only if for each generalized intuitionistic fuzzy quasi uniform closed set \( A \) and each generalized intuitionistic fuzzy quasi uniform open set \( B \) in \((X, T_U)\) such that \( A \subseteq B \), there exists a generalized intuitionistic fuzzy quasi uniform open set \( C \) in \((X, T_U)\) such that \( GI F QU cl(A) \subseteq C \subseteq GI F QU cl(C) \subseteq B \).

**Proof.** \( \Rightarrow \) Let \( A \) be any generalized intuitionistic fuzzy quasi uniform closed set and \( B \) be any generalized intuitionistic fuzzy quasi uniform open set in \((X, T_U)\), such that \( A \subseteq B \). Since \((X, T_U)\) is generalized
intuitionistic fuzzy quasi uniform normal, there exists an intuitionistic fuzzy set $D$ such that $A \subseteq GI\,FU\,QU\,int(D) \subseteq GI\,FU\,QU\,cl(D) \subseteq B$. Let $C = GI\,FU\,QU\,cl(D)$. Now, $C$ is a generalized intuitionistic fuzzy quasi uniform closed set, $B$ is a generalized intuitionistic fuzzy quasi uniform open set, such that $C \subseteq B$. Since $C$ is a generalized intuitionistic fuzzy quasi uniform closed set, $GI\,FU\,QU\,cl(C) \subseteq B$. Therefore, $GI\,FU\,QU\,cl(A) \subseteq C \subseteq GI\,FU\,QU\,cl(C) \subseteq B$.

$\Leftarrow$ Let $A$ be any generalized intuitionistic fuzzy quasi uniform closed set and $B$ be any generalized intuitionistic fuzzy quasi uniform open set in $(X, T_U)$, such that $A \subseteq B$. Since $A$ is a generalized intuitionistic fuzzy quasi uniform closed set, by assumption, there exists a generalized intuitionistic fuzzy quasi uniform open set $C$ in $(X, T_U)$ such that $GI\,FU\,QU\,cl(A) = A \subseteq C \subseteq GI\,FU\,QU\,cl(C) \subseteq B$. Also, since $C$ is a generalized intuitionistic fuzzy quasi uniform open set and $A$ is a generalized intuitionistic fuzzy quasi uniform closed set, such that $A \subseteq C$, $A \subseteq GI\,FU\,QU\,int(C) \subseteq GI\,FU\,QU\,cl(C) \subseteq B$. Therefore, $(X, T_U)$ is a generalized intuitionistic fuzzy quasi uniform normal space. \qed

**Definition 3.1.9.** Let $(X, T_U)$ and $(Y, S_U)$ be any two intuitionistic fuzzy quasi uniform topological spaces. A mapping $f : (X, T_U) \to (Y, S_U)$ is said to be generalized intuitionistic fuzzy quasi uniform continuous if for every generalized intuitionistic fuzzy quasi uniform open set $V$ in $(Y, S_U)$, $f^{-1}(V)$ is a generalized intuitionistic fuzzy quasi uniform
Definition 3.1.10. Let $(X, T_U)$ and $(Y, S_U)$ be any two intuitionistic fuzzy quasi uniform topological spaces and let $f : (X, T_U) \to (Y, S_U)$ be a mapping. Then $f$ is said to be a generalized intuitionistic fuzzy quasi uniform open if image of each generalized intuitionistic fuzzy quasi uniform open set $U$ in $(X, T_U)$, $f(U)$ is a generalized intuitionistic fuzzy quasi uniform open set in $(Y, S_U)$.

Definition 3.1.11. Let $(X, T_U)$ and $(Y, S_U)$ be any two intuitionistic fuzzy quasi uniform topological spaces. A mapping $f : (X, T_U) \to (Y, S_U)$ is said to be a generalized intuitionistic fuzzy quasi uniform homeomorphism if $f$ is bijective, generalized intuitionistic fuzzy quasi uniform continuous and generalized intuitionistic fuzzy quasi uniform open.

Proposition 3.1.2. Let $(X, T_U)$ and $(Y, S_U)$ be any two intuitionistic fuzzy quasi uniform topological spaces. If $f : (X, T_U) \to (Y, S_U)$ is a generalized intuitionistic fuzzy quasi uniform homeomorphism and $(Y, S_U)$ is a generalized intuitionistic fuzzy quasi uniform normal space then $(X, T_U)$ is a generalized intuitionistic fuzzy quasi uniform normal space.

Proof. Let $A$ be any generalized intuitionistic fuzzy quasi uniform closed set and $B$ be any generalized intuitionistic fuzzy quasi uniform open set in $(X, T_U)$ such that $A \subseteq B$. Since $f$ is generalized intuitionistic fuzzy
quasi uniform closed, $f(A)$ is a generalized intuitionistic fuzzy quasi uniform closed set in $(Y, S_U)$ and since $f$ is generalized intuitionistic fuzzy quasi uniform open, $f(B)$ is a generalized intuitionistic fuzzy quasi uniform open set in $(Y, S_U)$. Since $(Y, S_U)$ is a generalized intuitionistic fuzzy quasi uniform normal space, there exists an intuitionistic fuzzy set $C$ in $X$, such that $f(A) \subseteq \text{GI F QU int}(C) \subseteq \text{GI F QU cl}(C) \subseteq f(B)$. Now, $f^{-1}(f(A)) \subseteq f^{-1}(\text{GI F QU int}(C)) \subseteq f^{-1}(\text{GI F QU cl}(C)) \subseteq f^{-1}(f(B))$. That is, $A \subseteq \text{GI F QU int}(f^{-1}(C)) \subseteq \text{GI F QU cl}(f^{-1}(C)) \subseteq B$. Therefore, $(X, T_U)$ is a generalized intuitionistic fuzzy quasi uniform normal space.

**Proposition 3.1.3.** Let $f : (X, T_U) \to (Y, S_U)$ be a generalized intuitionistic fuzzy quasi uniform homeomorphism from a generalized intuitionistic fuzzy quasi uniform normal space $(X, T_U)$ onto an intuitionistic fuzzy quasi uniform topological space $(Y, S_U)$. Then $(Y, S_U)$ is a generalized intuitionistic fuzzy quasi uniform normal space.

**Proof.** Let $A$ be any generalized intuitionistic fuzzy quasi uniform closed set and $B$ be any generalized intuitionistic fuzzy quasi uniform open set in $(Y, S_U)$, such that $A \subseteq B$. Since $f$ is a generalized intuitionistic fuzzy quasi uniform continuous, $f^{-1}(A)$ is a generalized intuitionistic fuzzy quasi uniform closed set and $f^{-1}(B)$ is a generalized intuitionistic fuzzy quasi uniform open set in $(X, T_U)$. Since $(X, T_U)$ is a generalized intuitionistic fuzzy quasi uniform normal space, there exists an intu-
intuitionistic fuzzy set $C$ in $X$, such that $f^{-1}(A) \subseteq GIFQUint(C) \subseteq GIFQUcl(C) \subseteq f^{-1}(B)$. Now, $f(f^{-1}(A)) \subseteq f(GIFQUint(C)) \subseteq f(GIFQUcl(C)) \subseteq f(f^{-1}(B))$. That is, $A \subseteq GIFQUint(f(C)) \subseteq GIFQUcl(f(C)) \subseteq f(f^{-1}(B))$. Therefore, $(Y, S_U)$ is a generalized intuitionistic fuzzy quasi uniform normal space. 

**Notation 3.1.1.**

(i) $\zeta^R$ is the set of all intuitionistic fuzzy sets on the real line $R$.

(ii) $\zeta^Q$ is the set of all intuitionistic fuzzy sets on the rational number $Q$.

(iii) $A(x) = (\mu_A(x), \gamma_A(x)); \quad x \in X$.

(iv) $B(x) = (\mu_B(x), \gamma_B(x)); \quad x \in X$.

(v) $1^\sim = (1, 0)$.

(vi) $0^\sim = (0, 1)$.

**Definition 3.1.12.** An intuitionistic fuzzy real line $R_I(I)$ is the set of all intuitionistic fuzzy monotonic decreasing sets $A \in \zeta^R$ for which $A = \{(t, \mu_A(t), \gamma_A(t)) : t \in R\}$ where the mappings of $\mu_A : R \rightarrow I$ and $\gamma_A : R \rightarrow I$ denote the degree of membership and degree of nonmembership as defined by, $\mu_A(t) = \alpha$, if $0 \leq t \leq 1$; and $\gamma_A(t) = 1$, if $t < 0$; $\gamma_A(t) = 0$, if $t > 1$;
\[ y_A(t) = \begin{cases} 
0, & \text{if } t < 0; \\
\beta, & \text{if } 0 \leq t \leq 1; \\
1, & \text{if } t > 1;
\end{cases} \]
respectively, and \( \alpha + \beta \leq 1 \), satisfying
\[ \cup \{A(t) : t \in \mathbb{R}\} = 1 \quad \text{and} \quad \cap \{A(t) : t \in \mathbb{R}\} = 0 \quad \text{after the identification} \]
\( A, B \in \mathcal{R}^t \) if \( A(t-) = B(t-) \) and \( A(t+) = B(t+) \) where
\[ A(t-) = \cap \{A(s) : s < t\} \quad \text{and} \quad A(t+) = \cup \{A(s) : s > t\} \quad \text{respectively.} \]

The intutionistic fuzzy unit interval \( I_1(I) \) is as subset of \( R_1(I) \) such that \( A \subseteq I_1(I) \) as defined by
\[ A = \{(t, \mu_A(t), y_A(t)) : t \in \mathbb{R}\} \]
where
\[ \mu_A(t) = \begin{cases} 
1, & \text{if } t \leq 0; \\
0, & \text{if } t \leq 0;
\end{cases} \]
and
\[ y_A(t) = \begin{cases} 
\alpha, & \text{if } 0 < t < 1; \\
\beta, & \text{if } 0 < t < 1; \\
0, & \text{if } t \geq 1; \\
1, & \text{if } t \geq 1.
\end{cases} \]

The intutionistic fuzzy topology on \( R_1(I) \) is generated from the sub-basis \( \{L^1_t, R^1_t : t \in \mathbb{R}\} \) where \( L^1_t, R^1_t : R_1(I) \to I_1(I) \) are given by
\[ L^1_t(A) = A(t-) \quad \text{and} \quad R^1_t(A) = A(t+) \]
This topology is called usual intutionistic fuzzy topology for \( R_1(I) \).

**Definition 3.1.13.** Let \((X, T_U)\) be any intutionistic fuzzy quasi uniform topological space. A mapping \( f : X \to R_1(I) \) is called generalized intutionistic fuzzy quasi uniform lower (resp., upper) semi continuous if \( f^{-1}(R^1_t) \) (resp., \( f^{-1}(L^1_t) \)) is generalized intutionistic fuzzy quasi uniform open for \( t \in \mathbb{R} \). Also \( f \) is generalized intutionistic fuzzy quasi uniform continuous if it is both generalized intutionistic fuzzy quasi uniform upper (and lower) semi continuous.
Proposition 3.1.4. Let \( (X, T_U) \) be any intuitionistic fuzzy quasi uniform topological space. Let \( A \) be an intuitionistic fuzzy set of \( X \) and
\[
1^\sim, \quad \text{if } t < 0; \\
\]
let \( f : X \to R_I(I) \) be such that \( f(x)(t) = A(x), \) if \( 0 \leq t \leq 1; \) for
\[
0^\sim, \quad \text{if } t > 1; \\
\]
all \( x \in X. \) Then \( f \) is generalized intuitionistic fuzzy quasi uniform lower (resp., upper) semi continuous if and only if \( A \) is a generalized intuitionistic fuzzy quasi uniform open (resp., generalized intuitionistic fuzzy quasi uniform closed)set.

Proof. It suffices to observe that
\[
f^{-1}(R^1_I) = A, \quad \text{if } 0 \leq t < 1; \\
1^\sim, \quad \text{if } t < 0; \\
0^\sim, \quad \text{if } t \geq 1; \\
1^\sim, \quad \text{if } t \leq 0; \\
f^{-1}(L^1_I) = A, \quad \text{if } 0 < t \leq 1; \\
0^\sim, \quad \text{if } t > 1. \\
\]

Proposition 3.1.5. An intuitionistic fuzzy quasi uniform topological space \( (X, T_U) \) is generalized intuitionistic fuzzy quasi uniform normal space if and only if for every generalized intuitionistic fuzzy quasi uniform closed set \( A \) and generalized intuitionistic fuzzy quasi uniform open set \( B \) in \( (X, T_U) \) such that \( A \subseteq B, \) there exists a generalized intuitionistic fuzzy quasi uniform continuous mapping \( f : X \to I_I(I) \) such that for every \( x \in X, A(x) \subseteq f(x)(1-) \subseteq f(x)(0^+) \subseteq B(x). \)
Proof. \( \Leftarrow \) Choose \( x \in X \), since \( A(x) \subseteq f(x)(1) \subseteq f(x)(0+) \subseteq B(x) \) and for every \( t \in (0, 1) \), \( f(x)(1) \subseteq f(x)(t+) \subseteq f(x)(t-) \subseteq f(x)(0+) \), we have \( A(x) \subseteq f(x)(t+) \subseteq f(x)(t-) \subseteq B(x) \). Now, \( f^{-1}([u^1]) = f(x)(t-) \) and \( f^{-1}([r^1]) = f(x)(t+) \). Since \( f \) is generalized intuitionistic fuzzy quasi uniform continuous, \( f^{-1}([u^1]) \) is generalized intuitionistic fuzzy quasi uniform closed and \( f^{-1}([r^1]) \) is generalized intuitionistic fuzzy quasi uniform open. Hence \( A \subseteq f^{-1}([r^1]) \subseteq f^{-1}([u^1]) \subseteq B \). That is, \( (X, T_u) \) is a generalized intuitionistic fuzzy quasi uniform normal space.

\( \Rightarrow \) Construct \( \{A_r/r \in (0, 1)\} \), so that for each \( r \in (0, 1), \ A \subseteq A_r \subseteq B \) and \( r < s \) implies \( GI F QUcl(A_s) \subseteq GI F QUint(A_r) \). Define \( f(x)(t) = A_t(x) \) for all \( x \in X \) and \( t \in R \). Clearly \( A(x) \subseteq f(x)(1) \subseteq f(x)(0+) \subseteq B(x) \). Now, \( f^{-1}([r^1]) = \bigcup_{r>t} A_r = \bigcup_{r>t} GI F QUint(A_r) \) is a generalized intuitionistic fuzzy quasi uniform open set and \( f^{-1}([u^1]) = \bigcap_{r<t} A_r = \bigcap_{r<t} GI F QUcl(A_r) \) is a generalized intuitionistic fuzzy quasi uniform closed set in \( (X, T_u) \). Hence \( f \) is generalized intuitionistic fuzzy quasi uniform continuous. 

\[ \square \]

**Proposition 3.1.6.** Let \( (X, T_u) \) be a generalized intuitionistic fuzzy quasi uniform normal topological space. Let \( \{A_i\}_{i=1}^{\infty} \) and \( \{B_j\}_{j=1}^{\infty} \) be countable families of intuitionistic fuzzy sets of \( X \). If there exist intuitionistic fuzzy sets \( A \) and \( B \), such that \( GI F QUcl(A_i) \subseteq GI F QUcl(A) \subseteq GI F QUint(B_j) \) and \( GI F QUcl(A_i) \subseteq GI F QUint(A_i) \subseteq GI F QUint(B) \subseteq GI F QUint(B_j) \) for all \( i, j = 1, 2, \ldots \) then there exists an intuitionistic fuzzy set \( C \) such
that $\text{GI F QU} cl(A_i) \subseteq \text{GI F QU} int(C) \subseteq \text{GI F QU} cl(C) \subseteq \text{GI F QU} int(B_j)$ for all $i, j = 1, 2, \ldots$.

**Proof.** First, we shall show by induction that for all $n \geq 2$ there exists a collection of intuitionistic fuzzy sets $\{C_i, D_j | i \leq n\}$, such that the conditions:

\[
\begin{align*}
\text{GI F QU} cl(A_i) & \subseteq \text{GI F QU} int(C_i), \\
\text{GI F QU} cl(D_j) & \subseteq \text{GI F QU} int(B_j), \\
\text{GI F QU} cl(A) & \subseteq \text{GI F QU} int(D_j), \\
\text{GI F QU} cl(C_i) & \subseteq \text{GI F QU} int(B), \\
\text{GI F QU} cl(C_i) & \subseteq \text{GI F QU} int(D_j),
\end{align*}
\]

hold for all $i, j = 1, 2, \ldots, n - 1$.

Clearly, $P_2$ follows at once from the generalized intuitionistic fuzzy quasi uniform normality of $(X, T_\emptyset)$. Now, suppose that for $n \geq 2$. We have defined intuitionistic fuzzy sets $C_i, D_j(i < n)$, such that $P_n$. Since $\text{GI F QU} cl(A_n) \subseteq \text{GI F QU} cl(A) \subseteq \text{GI F QU} int(D_j) \ (j < n)$ and $\text{GI F QU} cl(A_n) \subseteq \text{GI F QU} int(B)$, by generalized intuitionistic fuzzy quasi uniform normality of $(X, T_\emptyset)$, there exists an intuitionistic fuzzy set $C_n$ such that $\text{GI F QU} cl(A_n) \subseteq \text{GI F QU} int(C_n) \subseteq \text{GI F QU} cl(C_n) \subseteq \text{GI F QU} int(\bigcup_{j=n}^{T} D_j \cap B)$.

Similarly, since $\text{GI F QU} cl(A) \subseteq \text{GI F QU} int(B_n)$ and $\text{GI F QU} cl(C_i) \subseteq \text{GI F QU} int(B_n)(i \leq n)$, there exists an intuitionistic fuzzy set $D_n$ of
such that \( \bigcup_{i \in n} G I F Q U c l(C_i) \subseteq G I F Q U c l(A) \subseteq G I F Q U i n t(D_n) \subseteq G I F Q U c l(D_n) \subseteq G I F Q U i n t(B_n) \). Thus, \((P_{n+1})\) holds. Let \( C = \bigcup_{i=1}^\infty C_i \). Then \( G I F Q U c l(A_i) \subseteq G I F Q U i n t(C_i) \subseteq G I F Q U i n t(C) \) for all \( i = 1, 2, \ldots \). Thus, \( G I F Q U c l(A_i) \subseteq G I F Q U i n t(C) \) for all \( i = 1, 2, \ldots \). Since \( G I F Q U c l(C_i) \subseteq G I F Q U i n t(D_j)(i, j = 1, 2, \ldots) \), we have \( C_i \subseteq D_j \) for all \( i = 1, 2, \ldots \), so that \( G I F Q U c l(C) \subseteq G I F Q U c l(D_j) \subseteq G I F Q U i n t(B_j) \) \((j = 1, 2, \ldots)\). Thus, \( G I F Q U c l(C) \subseteq G I F Q U i n t(B_j) \) for all \( j = 1, 2, \ldots \). The proof is complete. 

**Proposition 3.1.7.** Let \((X, T_U)\) be a generalized intuitionistic fuzzy quasi uniform normal space. If \( \{A_r\}_{r \in Q} \) and \( \{B_r\}_{r \in Q} \) are monotone increasing collection of respectively, generalized intuitionistic fuzzy quasi uniform closed and generalized intuitionistic fuzzy quasi uniform open sets of \((X, T_U)\) \((Q \text{ is the set of all rational numbers})\) such that \( A_r \subseteq B_s \), whenever \( r < s \), then there exist a collection of intuitionistic fuzzy sets \( \{C_r\}_{r \in Q} \) of \( X \), such that \( A_r \subseteq G I F Q U i n t(C_s) \), \( G I F Q U c l(C_r) \subseteq G I F Q U i n t(C_s) \) and \( G I F Q U c l(C_i) \subseteq B_s \) whenever \( r < s \).

**Proof.** Let us arrange into sequence \( \{r_n\} \) of all rational numbers (without repetitions). For every \( n \geq 2 \) we shall define inductively a collection of intuitionistic fuzzy sets \( \{C_{r_i} \}_{1 \leq i < n} \) of \( X \), such that
\[ A_r \subseteq \text{GI F QU int}(C_{r_i}) \quad \text{if } r < r_i, \]
\[ \text{GI F QU cl}(C_{r_i}) \subseteq B_r \quad \text{if } r_i < r, \quad (P_n) \]
\[ \text{GI F QU cl}(C_{r_j}) \subseteq \text{GI F QU int}(C_{r_j}) \quad \text{if } r_i < r_j, \]

for all \( 1 \leq i, j < n \).

It is clear that the countable collections \( \{A_r/r < r_i\} \) and \( \{B_r/r > r_i\} \) together with \( A_{r_i} \) and \( B_{r_i} \) satisfy all hypothesis of Proposition 3.1.6., so that there exists an intuitionistic fuzzy set \( C_1 \), such that \( A_r \subseteq \text{GI F QU int}(C_{r_i}) \) for all \( r < r_i \) and \( \text{GI F QU cl}(C_{r_i}) \subseteq B_r \) for all \( r_i < r \).

Letting \( C_{r_i} = D_1 \), we get \((P_2)\).

Assume that the intuitionistic fuzzy sets \( C_{r_i} \) are already defined for \( i < n \) and satisfy \((P_n)\) holds. Define

\[ A = \cup \{C_{r_i}/i < n, r_i < r_n\} \cup A_r \]

and

\[ B = \cap \{C_{r_j}/j < n, r_j > r_n\} \cap B_r. \]

Then \( \text{GI F QU cl}(C_{r_j}) \subseteq \text{GI F QU cl}(A) \subseteq \text{GI F QU int}(C_{r_j}) \) and \( \text{GI F QU cl}(C_{r_j}) \subseteq \text{GI F QU int}(B) \subseteq \text{GI F QU int}(C_{r_j}) \)

whenever \( r_i < r_n < r_j \quad (i, j < n) \), as well as \( A_r \subseteq \text{GI F QU cl}(A) \subseteq B_s \)

and \( A_r \subseteq \text{GI F QU int}(B) \subseteq B_s \) whenever \( r < r_n < s \).

This shows that countable collections \( \{C_{r_i}/i < n, r_i < r_n\} \cup \{A_r/r < r_n\} \)

and \( \{C_{r_j}/j < n, r_j > r_n\} \cup \{B_r/r > r_n\} \) together with \( A \) and \( B \) satisfy
all hypothesis of Proposition 3.1.6. Hence, there exists an intuitionistic fuzzy set $D_n$ such that

$$A_r \subseteq GI F QU int(D_n) \quad \text{if } r < r_n,$$

$$GI F QU cl(C_r) \subseteq GI F QU int(D_n) \quad \text{if } r_i < r_n,$$

$$GI F QU cl(D_n) \subseteq B_r \quad \text{if } r_n < r,$$

$$GI F QU cl(D_n) \subseteq GI F QU int(C_{r_j}) \quad \text{if } r_n < r_j,$$

where $1 \leq i, j \leq n - 1$. Now, letting $C_{r_n} = D_n$, we obtain intuitionistic fuzzy sets $C_{r_1}, C_{r_2}, C_{r_3}, \ldots, C_{r_n}$ that satisfy the result $(P_{n+1})$. Therefore, the collection of intuitionistic fuzzy sets $\{C_{r_i}/i = 1, 2, \ldots\}$ has the required properties. The proof is complete. $\square$

**Proposition 3.1.8.** Let $(X, T_U)$ be any intuitionistic fuzzy quasi uniform topological space. Then the following statements are equivalent:

(i) $(X, T_U)$ is a generalized intuitionistic fuzzy quasi uniform normal space.

(ii) If $g, h : X \to R_1(I)$, $g$ is generalized intuitionistic fuzzy quasi uniform upper semi continuous, $h$ is generalized intuitionistic fuzzy quasi uniform lower semi continuous and $g \subseteq h$ then there exists a generalized intuitionistic fuzzy quasi uniform continuous mapping $f : X \to R_1(I)$ such that $g \subseteq f \subseteq h$

**Proof.** (ii) $\Rightarrow$ (i) Let $\bar{A}$ and $B$ are generalized intuitionistic fuzzy quasi uniform open sets in $(X, T_U)$ such that $A \subseteq B$. 

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Define \( g, h : X \to R_I(\mathcal{I}) \) by
\[
\begin{align*}
g(x)(t) &= 1^- , \quad \text{if } t < 0; \\
h(x)(t) &= 1^- , \quad \text{if } t < 0; \\
g(x)(t) &= A(x), \quad \text{if } 0 \leq t \leq 1; \\
h(x)(t) &= B(x), \quad \text{if } 0 \leq t \leq 1; \\
g(x)(t) &= 0^-, \quad \text{if } t > 1; \\
h(x)(t) &= 0^-, \quad \text{if } t > 1.
\end{align*}
\]
By Proposition 3.1.4., \( g \) is generalized intuitionistic fuzzy quasi uniform upper semi continuous and \( h \) is generalized intuitionistic fuzzy quasi uniform lower semi continuous. Clearly \( g \subseteq h \) holds. So that there exists a generalized intuitionistic fuzzy quasi uniform continuous mapping \( f : X \to R_I(\mathcal{I}) \) such that \( g \subseteq f \subseteq h \). Suppose \( t \in (0, 1) \). Then \( A = g^{-1}(\overline{R_1}) \subseteq f^{-1}(\overline{R_1}) \subseteq h^{-1}(\overline{R_1}) = B \). Thus, \((X, T_U)\) is a generalized intuitionistic fuzzy quasi uniform normal space.

(i) \( \Rightarrow \) (ii) Define \( A, B \in \xi^Q \) by \( \{A_r\}_{r \in Q} = \{h^{-1}(\overline{R_1})\} \) and \( \{B_r\}_{r \in Q} = \{g^{-1}(\overline{L_1})\} \) (the set of all rational ). Clearly \( A_r \) and \( B_r \) are intuitionistic fuzzy monotonic increasing families of generalized intuitionistic fuzzy quasi uniform closed and generalized intuitionistic fuzzy quasi uniform open sets of \((X, T_U)\). Moreover \( A_r \subseteq B_{r_1} \), if \( r < r_1 \). Now, by Proposition 3.1.7., there exists a collection of intuitionistic fuzzy sets \( \{C_r\}_{r \in Q} \subseteq \xi^Q \) such that
\[
\begin{align*}
A_r &\subseteq GI F QU \text{ int}(C_{r_1}), \\
GI F QU \text{ cl}(C_r) &\subseteq GI F QU \text{ int}(C_{r_1}), \\
GI F QU \text{ cl}(C_r) &\subseteq B_{r_1},
\end{align*}
\]
whenever \( r < r_1 (r, r_1 \in Q) \). Letting \( D_t = \cap_{r < t} C_r \) for each \( t \in R \), we
define an intuitionistic fuzzy monotone decreasing family \( \{D_t/t \in R\} \) of \( X \). Moreover \( GIFT \subseteq GIFT(D_t) \) whenever \( s < t \). Indeed, for \( s < r < r_1 < t(s, t \in R \text{ and } r, r_1 \in Q) \) we have \( D_s \subseteq GIFT(C_r) \subseteq GIFT(C_{r_1}) \subseteq D_t \), hence \( GIFT(D_t) \subseteq GIFT(D_s) \). we have also

\[
\bigcup_{t \in R} D_t = \bigcup_{t \in R} \left( \bigcap_{r < t} B_r \supseteq \bigcap_{r < t} \overline{g^{-1}(L^I_t)} \right)
\]

Similarly \( \bigcup_{t \in R} D_t = 0_\sim \). Now, define a mapping \( f : X \to R_I(I) \) satisfying the required properties.

Let \( f(x)(t) = D_t(x) \) for all \( x \in X \) and \( t \in R \). Then above discussion shows that \( f \) is well defined, that is \( f(x) \in R_I(I) \) for every \( x \in X \). To prove \( f \) is generalized intuitionistic fuzzy quasi uniform continuous, observe that

\[
\bigcup_{s > t} D_s = \bigcup_{s > t} \text{GIFT}(D_s)
\]

and

\[
\bigcap_{s < t} D_s = \bigcap_{s < t} \text{GIFT}(D_s).
\]
Then $f^{-1}(R^t_I) = S_{s>t} D_s = S_{s>t} GI FQUint(D_s)$ is a generalized intuitionistic fuzzy quasi uniform open set. Now, $f^{-1}(L^t_I) = T_{s<t} GI FQUcl(D_s)$ is a generalized intuitionistic fuzzy quasi uniform closed set. So that $f$ is generalized intuitionistic fuzzy quasi uniform continuous. To show that $g \subseteq f \subseteq h$, that is, $g^{-1}(L^t_I) \subseteq f^{-1}(L^t_I) \subseteq h^{-1}(L^t_I)$ and $g^{-1}(R^t_I) \subseteq f^{-1}(R^t_I) \subseteq h^{-1}(R^t_I)$ for each $t \in R$. We have

$$g^{-1}(L^t_I) = \bigcap_{s<r<s} g^{-1}(L^t_r) = \bigcap_{s<r<s} D_s = f^{-1}(L^t_I)$$

and

$$f^{-1}(L^t_I) = \bigcap_{s<r<s} D_s = \bigcap_{s<r<s} h^{-1}(R^t_r) = \bigcap_{s<r<s} h^{-1}(L^t_s) = h^{-1}(L^t_I).$$

Similarly, we obtain

$$g^{-1}(R^t_I) = \bigcup_{s>t} g^{-1}(R^t_s) = \bigcup_{s>t} g^{-1}(L^t_r) = \bigcup_{s>t} B_r \subseteq \bigcup_{s>r<s} C_r = \bigcup_{s>r<s} D_s = f^{-1}(R^t_I).$$
and

\[
\begin{align*}
\mathcal{D}_s & = \bigcap_{s \succ r \prec s} \mathcal{C}_r \subseteq \bigcap_{s \succ r \succ s} \mathcal{A}_r \\
\mathcal{C}_r & = \bigcap_{s \succ r \prec s} h^{-1}(R^t_i) = \bigcap_{s \succ r \succ s} h^{-1}(R^t_i) \\
\mathcal{A}_r & = h^{-1}(R^t_i).
\end{align*}
\]

The proof is complete. \qed

## 3.2 Tietze Extension Theorem

In this section, Tietze extension theorem is established.

**Notation 3.2.1.** Let \((X, T_U)\) be an intuitionistic fuzzy quasi uniform topological space. Let \(A \subset X\). Then an intuitionistic fuzzy set \(X_A\) is of the form \(X_A(x) =\)

\[
\begin{cases}
1^\sim, & x \in A; \\
0^\sim, & x \notin A.
\end{cases}
\]

**Proposition 3.2.1.** Let \((X, T_U)\) be a generalized intuitionistic fuzzy quasi uniform normal normal topological space and let \(A \subset X\) be such that \(X_A\) is a generalized intuitionistic fuzzy quasi uniform closed set in \(X\). Let \(f : (A, T/A) \rightarrow \mathcal{I}_1(\mathcal{I})\) be a generalized intuitionistic fuzzy quasi uniform continuous mapping. Then \(f\) has a generalized intuitionistic fuzzy quasi uniform continuous extension over \((X, T_U)\).

**Proof.** Let \(g, h : X \rightarrow \mathcal{I}_1(\mathcal{I})\) be such that \(g = f = h\) on \(A\) and \(g(x) = 0^\sim, h(x) = 1^\sim\) if \(x \notin A\). For every \(t \in R\), we have
$g^{-1}(L^t) = U_t \cup X_A$ if $t > 0$; 
where $U_t$ is a generalized intuitionistic fuzzy quasi uniform open set, such that $U_t/A = f^{-1}(L^t)$ and 
$h^{-1}(R^t) = V_t \cup X_A$ if $t < 1$; 
where $V_t$ is a generalized intuitionistic fuzzy quasi uniform open set, such that $V_t/A = f^{-1}(R^t)$. Thus, 
g is generalized intuitionistic fuzzy quasi uniform upper semi continuous, 
h is generalized intuitionistic fuzzy quasi uniform lower semi continuous and $g \subseteq h$. By Proposition 3.1.8., there exists a generalized intuitionistic fuzzy quasi uniform continuous mapping $F : X \to I_I(\mathcal{I})$ 
such that $g(x) \subseteq f(x) \subseteq h(x)$ for all $x \in X$. That is, for all $x \in A$, 
$f(x) \subseteq F(x) \subseteq f(x)$. So that $F$ is the required extension of $f$ over $(X, T_U)$.