CHAPTER (V)

LIGHTWAVE PROPAGATION THROUGH AN OPTICAL WAVEGUIDE WITH A CORE CROSS-SECTION BOUNDED BY TWO SPIRALS AND ITS DISPERSION CHARACTERISTICS
5.1. **Introduction** :

Ever since the invention of optical transmission through dielectric waveguides [9-10,271-274], the progress made in the field of fiber optics and transmission technology has been phenomenal. The impact of fiber optics on communication technology is evident in the widespread use of the dielectric waveguide in communication technology and various other technical applications. Technologists suggested that not only the transmission properties of an optical waveguides are dictated by its refractive index profiles, but structural differences are also useful for various application in integrated optics and optical communications. Consequently various non-circular waveguide like those having elliptical, rectangular, triangular, annular, Peit-Hein, cardioid and other cross-sections [141,147-148,173-182,268-269,244] were investigated by many investigators. Currently, the study of such non-circular waveguides has assumed great importance both theoretically and experimentally. To study these non-circular waveguide one may use various analytical and numerical methods such as the perturbation method [268], the variational method [271] and the point matching method [141] etc. Among these methods, the point matching method is one of the most important and also is quite straightforward and this method has been utilized in great detail in chapters II and III. Here we have chosen the analytical method with only a few approximations which are physically and mathematically legitimate. As explained in the previous chapter I (introduction chapter), the analytical method is not applicable to most waveguides, except the planar waveguide and the fiber of circular cross-section, because of the difficulty of matching the electric or magnetic fields
at the boundary. The difficulty is not limited to solving Maxwell’s equations alone. There are other difficulties connected with the decoupling of variables and the solution of differential equations. Even the comparatively simple scalar wave equation has analytical solutions only for cross-sections that are nearly circular or for graded refractive index profiles that are unphysical [166]. Because the solution of the scalar wave equation is fundamental to weak guidance theory [166,174], the above difficulty is not significantly reduced when the refractive index difference between the core and cladding is small.

Here in the present chapter, an analytical analysis under weak guidance approximation has been made to study the modal behaviour, cutoff condition and dispersion characteristics of a new type of a waveguide having a core cross-section bounded by two spirals. Using boundary conditions relevant to the proposed waveguide under the weak guidance condition, the modal characteristic equation and modal cutoff equation have been derived. From the cutoff equation we find the number of modes propagated through the guiding region for a given size parameter. Also from the modal characteristic equation we obtain the dispersion curves for some low order modes. One reason for the study of such an unconventional waveguide is to see the effect of a distortion on the modal properties of the conventional planar waveguide. A segment of the core cross-section can be considered as a distorted planar waveguide in which a curvature and a flare have been introduced. As stated earlier, several attempts have been made to study the modal properties and cutoff conditions of several optical waveguides having unconventional non-circular cross-sections [275-281]. Recently Singh et. al. [282] studied theoretically the cutoff conditions of a double-clad helical optical fiber using
the boundary conditions relevant to a helically conducting boundary. However, as far as is known to the investigator, no theoretical analysis of an optical waveguide with a core cross-section bounded by two spirals has been tried till today. We hope that the present theoretical analysis of the proposed waveguide may have some technical use in throwing light on the effect of distortion on the conventional planar waveguide. This is the motivation of the author for the present study.

5.2. **Characteristic equation for an optical waveguide with a core cross-section bounded by two spirals and their dispersion characteristics:**

We consider here the propagation of electromagnetic waves through the waveguide with a core cross-section bounded by two spirals. The cross-section of fiber to be analyzed theoretically is shown in the Fig. (5.1); the direction of propagation is perpendicular to the plane of paper. We obtain the modal characteristic equation by the use of the boundary matching technique under the weak guidance approximation. The analytical method presented here is somewhat lengthy and tedious, but it has the merit of being straightforward. The appropriate coordinate system for the analysis of such a structure must be chosen in order to suit the geometry of fiber.

The transverse cross-section of a waveguide having a core refractive index \( n_1 \) and cladding refractive index \( n_2 \) (such that \( n_1 - n_2 \) is very small) is shown in Fig. (5.1). The shape of the spiral is represented by the equation

\[
 r = a \theta 
\]  

(5.1)

where \( a \) is a size parameter.
Fig (5.1)  The transverse cross-section of the proposed waveguide having a core of refractive index $n_1$ and cladding of refractive index $n_2$. 
To obtain appropriate coordinates, one uses the point of intersection of two sets of normal curves on the cross-sectional plane of the waveguide. Now the equation representing the normal curve can be written as

\[ r = ce - \frac{\theta^2}{2} \]  

(5.2)

Here we have two mutually perpendicular sets of spirals described by the parameters \( a \) and \( c \). The third orthogonal coordinate is \( z \), along the fiber axis. We omit some straightforward steps and obtain the scale factors \( h_1 \), \( h_2 \), and \( h_3 \) for the coordinates \( a \) and \( c \) as:

\[ h_1 = \frac{r}{c(\theta^2 + 1)^2} ; \quad h_2 = \frac{r\theta}{a(\theta^2 + 1)^2} ; \quad h_3 = 1 \]

But we have to express the scale factors in terms of the new coordinates \( a \) and \( c \). Solving equation (5.1) and equation (5.2) we have

\[ r = \frac{a}{2} \ln \frac{c}{a} + \frac{3a}{4} \quad \text{and} \quad \theta = \frac{1}{2} \ln \frac{c}{a} + \frac{3}{4} \]

Thus

\[ h_1 = \frac{a}{2c} \left( \frac{\ln \frac{c}{a} + \frac{3}{2}}{\left[ \frac{1}{4} \left( \frac{\ln \frac{c}{a}}{a} \right)^2 + \frac{25}{16} + \frac{3}{4} \right]^2} \right) \]
\[ h_2 = \frac{1}{4} \frac{\left( \ln \frac{c}{a} + \frac{3}{2} \right)^2}{\left[ \frac{1}{4} \left( \ln \frac{c}{a} \right)^2 + \frac{25}{16} + \frac{3}{4} \ln \frac{c}{a} \right]^2} ; \]

The scalar wave equation is given by

\[ \nabla^2 E_z + \omega^2 \mu \varepsilon E_z = 0 \quad (5.3) \]

where \( \nabla^2 = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial a} \left( h_2 h_3 \frac{\partial}{\partial a} \right) + \frac{\partial}{\partial c} \left( h_1 h_3 \frac{\partial}{\partial c} \right) + \frac{\partial}{\partial z} \left( h_1 h_2 \frac{\partial}{\partial z} \right) \right] \)

and \( E_z \) is the \( z \)-component of the electric field. To proceed further with this differential equation is going to be an extremely difficult task unless some simplifying assumption is made. If we choose \( c \to a \), we have a manageable special case. In equation (5.3) \( \omega \) is the angular frequency, \( \varepsilon_1 \) is the permittivity of the guiding region (core) and \( \mu \) is the permeability. Using this assumption in equation (5.3), the modified differential equation assumes the following explicit form.

\[ \frac{4}{3} \frac{c^2}{d^2} \left[ \frac{3}{4a^2} \frac{\partial^2 E_z}{\partial a^2} - \frac{3}{2a^3} \frac{\partial E_z}{\partial a} + \frac{4}{3c^2} \frac{\partial^2 E_z}{\partial c^2} - \frac{8}{3c^3} \frac{\partial E_z}{\partial c} + \frac{3d^2}{4c^2} \frac{\partial^2 E_z}{\partial z^2} \right] \]

\[ + \omega^2 \mu \varepsilon_1 E_z = 0 \quad (5.4) \]

where \( d = \text{constant} = \frac{3}{5} \).
The technique of separation of variables will now be applied to obtain a solution to equation (5.4). We shall assume that we can separate the variables \( a \) and \( c \), and that there is a harmonic dependence of the field on \( z \) and \( t \). We then have

\[
E_z = E_1(a) \cdot E_2(c) \exp(j(\omega t - \beta z)) \tag{5.5}
\]

where \( \beta \) is the propagation constant along \( z \)-direction.

After a few steps we obtain three equations, each of which is in one variable, instead of equation (5.4).

\[
\frac{\partial^2 E_1(a)}{\partial a^2} - \frac{2}{a} \frac{\partial}{\partial a} E_1(a) = 0 \tag{5.6}
\]

\[
\frac{\partial^2 E_2(c)}{\partial c^2} - \frac{2}{c} \frac{\partial}{\partial c} E_2(c) + \frac{9}{16} d^2 U^2 E_2(c) = 0 \tag{5.7}
\]

\[
\frac{\partial^2 E_2(c)}{\partial c^2} - \frac{2}{c} \frac{\partial}{\partial c} E_2(c) - \frac{9}{16} d^2 W^2 E_2(c) = 0 \tag{5.8}
\]

where \( U = \sqrt{k_0^2 n_1^2 - \beta^2} \); \( W = \sqrt{\beta^2 - k_0^2 n_2^2} \) and \( k_0 = \frac{2\pi}{\lambda_0} \)
Equation (5.6) is valid for both the regions and it does not contain any parameter such as $U$ which describes the modal behaviour. We therefore, concentrate only on equations (5.7) and equations (5.8). Equation (5.7) is valid for the guiding (core) region and equation (5.8) for the nonguiding (cladding) region.

Using new symbols

$$E_2(c) = y, \quad \frac{9}{16}d^2 = p^2, \quad [\text{or } p = \frac{3}{4}d = 0.45]$$

and $c = x$ in equations (5.7) and (5.8), we have

$$y'' - \frac{2}{x} y' + p^2U^2 y = 0 \quad (5.9)$$

$$y'' - \frac{2}{x} y' - p^2W^2 y = 0 \quad (5.10)$$

The above equation (5.9) can be solved by the power series solution method. Indeed, this differential equation can be reduced to Bessel's equation subject to some approximation. Thus we can get a solution to the above equation (5.9) in a closed form as given below

$$y_1 = x^{3/2} \left[ AJ_{3/2}(pUx) \right] \quad (5.11)$$
In the above expression the function \( J_{3/2}(pUx) \) is a Bessel function and \( A \) is an unknown constant.

Another solution of the above equation (5.9) which is independent of that of equation (5.11) is required, because we must have a linear combination of two independent functions which are two independent solutions of equations (5.9). We need four unknown constants, since we shall obtain four equations after matching the fields at the boundaries of the waveguide. Thus we obtain the other solution of equation (5.9) to be

\[
y_2 = x^{3/2} \left[ B Y_{3/2}(pUx) \right]
\]

(5.12)

The function \( Y_{3/2}(pUx) \) is a Bessel function of second kind and \( B \) is another arbitrary constant. Thus the final expression for the electromagnetic field in the guiding region can now be written as

\[
y_{\text{core}} = y_1 + y_2
\]

Hence

\[
y_{\text{core}} = x^{3/2} \left[ A J_{3/2}(pUx) + B Y_{3/2}(pUx) \right]
\]

(5.13)

Similarly the above equation (5.10) can easily be reduced to the general form of the modified Bessel differential equation. It has two independent solutions namely \( I_{3/2}(pWx) \) and \( K_{3/2}(pWx) \). The functions \( I_{3/2}(pWx) \) and \( K_{3/2}(pWx) \) are known as modified Bessel functions of the first and the
second kind respectively. Depending upon the nature of these two functions, we have to choose which function will be suited for which region of the waveguide. In the inner non-guiding region, one has to have the field to behave in such a manner that its value is smaller when it is far from the boundary between the inner non-guiding region and the guiding region than when it is near the same boundary. That is, when the radial distance from the axis of the waveguide is small, it should be small, and it should increase gradually as it approaches the inner nonguiding and guiding region boundary. Thus in this region, the function $I_{3/2}(pWx)$ is the function which has this property. On the other hand, the field should decay when one goes away from the boundary between the guiding region and outer nonguiding region towards the outer nonguiding region. This situation is analogous to the behaviour of the function $K_{3/2}(pWx)$. Thus we choose $I_{3/2}(pWx)$ in the inner non-guiding region and $K_{3/2}(pWx)$ in the outer non-guiding region. Thus the respective field expressions for the inner and the outer nonguiding regions are given below

$$y_1 = x^{3/2} \left[ CI_{3/2}(pWx) \right]$$  \hspace{1cm} (5.14)

$$y_2 = x^{3/2} \left[ DK_{3/2}(pWx) \right]$$  \hspace{1cm} (5.15)

Thus the field expression for the electromagnetic field in the nonguiding region can now be written as
\[ y_{clad} = y_1 + y_2 \]

\[ y_{clad} = x^{3/2} \left[ C I_{3/2}(pWx) + D K_{3/2}(pWx) \right] \]  \hspace{1cm} (5.16)

where \( C \) and \( D \) are constants to be determined from the boundary conditions imposed on the electromagnetic field and the total power in the waveguide.

Now we should match the fields in different adjacent regions at the respective boundaries. That is we should match the fields in the inner nonguiding and the guiding regions, at \( x=a \) and also we should match the fields inside the guiding region and the outer nonguiding region at \( x=b \). We represent the inner guiding region I, the guiding region by II, and the outer nonguiding region III and the respective fields are represented by \( y_I \), \( y_{II} \) and \( y_{III} \). Matching the fields and their derivatives at appropriate boundaries (since the fields along with their derivatives with respect to the radial parameter \( x \) should be continuous at the boundaries) and remembering that the boundaries are \( x=a \) and \( x=b \), we get

\[ y_{II} \bigg|_{x = a} = y_I \bigg|_{x = a} \] \hspace{1cm} (5.17a)

\[ \frac{\partial y_{II}}{\partial x} \bigg|_{x = a} = \frac{\partial y_I}{\partial x} \bigg|_{x = a} \] \hspace{1cm} (5.17b)

\[ y_{II} \bigg|_{x = b} = y_{III} \bigg|_{x = b} \] \hspace{1cm} (5.17c)
\[
\frac{\partial y_{II}}{\partial x} \bigg|_{x = b} = \frac{\partial y_{III}}{\partial x} \bigg|_{x = b}
\]  

(5.17d)

Now employing the corresponding field expressions for the electromagnetic fields in different regions in the above equations (5.17a) to (5.17d), after some simplifications, we get the following set of differential equations.

\[
A J_{3/2}(pUa) + B Y_{3/2}(pUa) - C I_{3/2}(pWa) = 0
\]  

(5.18a)

\[
A \left[ pUa J'_{3/2}(pUa) + \frac{3}{2} J_{3/2}(pUa) \right] + B \left[ pUa Y'_{3/2}(pUa) + \frac{3}{2} Y_{3/2}(pUa) \right] \\
- C \left[ pWa I'_{3/2}(pWa) + \frac{3}{2} I_{3/2}(pWa) \right] = 0
\]  

(5.18b)

\[
A J_{3/2}(pUb) + B Y_{3/2}(pUb) - D K_{3/2}(pWb) = 0
\]  

(5.18c)

\[
A \left[ pUb J'_{3/2}(pUb) + \frac{3}{2} J_{3/2}(pUb) \right] + B \left[ pUb Y'_{3/2}(pUb) + \frac{3}{2} Y_{3/2}(pUb) \right] \\
- D \left[ pWb K'_{3/2}(pWb) + \frac{3}{2} K_{3/2}(pWb) \right] = 0
\]  

(5.18d)

Equation (5.18a) to (5.18d) form a set of four equations in four unknown and these unknown can be solved easily. To obtain a non-trivial solution to the above four unknowns, one has to set the determinant formed by the coefficients of these unknowns equal to zero all the time. Thus under this
condition, we get a $4 \times 4$ determinant and set it to zero getting the following determinantal equation

\[
\Delta_1 = \begin{vmatrix}
A_{11} & A_{12} & A_{13} & A_{14} \\
A_{21} & A_{22} & A_{23} & A_{24} \\
A_{31} & A_{32} & A_{33} & A_{34} \\
A_{41} & A_{42} & A_{43} & A_{44}
\end{vmatrix} = 0
\]

(5.19)

where

\begin{align*}
A_{11} &= J_{3/2}(pUA) \\
A_{12} &= Y_{3/2}(pUA) \\
A_{13} &= -I_{3/2}(pWA) \\
A_{14} &= 0 \\
A_{21} &= J_{3/2}(pUb) \\
A_{22} &= Y_{3/2}(pUb) \\
A_{23} &= 0 \\
A_{24} &= -K_{3/2}(pWb) \\
A_{31} &= pUAJ_{3/2}(pUA) + \frac{3}{2} J_{3/2}(pUA) \\
A_{32} &= pUAy_{3/2}(pUA) + \frac{3}{2} Y_{3/2}(pUA) \\
A_{33} &= -(pWAI_{3/2}(pWA) + \frac{3}{2} I_{3/2}(pWA)) \\
A_{34} &= 0 \\
A_{41} &= pUbJ_{3/2}(pUb) + \frac{3}{2} J_{3/2}(pUb) \\
A_{42} &= pUbY_{3/2}(pUb) + \frac{3}{2} Y_{3/2}(pUb)
\end{align*}
$A_{43} = 0$

$A_{44} = -(pWbK'_{3/2}(pWb) + \frac{3}{2} K_{3/2}(pWb))$

The equation (5.19) is described as the characteristic equation for the proposed waveguide. The left hand side of equation (5.19) is solved numerically for obtaining allowed values of the propagation constant and the result of analysis is presented in section (5.3) of this chapter.

5.2.1 Cutoff condition:

A mode is said to be cut off when the particular mode ceases to remain bounded in the guiding region of the guided structure. The cutoff condition of the waveguide bounded by two spirals can be obtained by solving the above equation (5.19) under the limiting condition $W \to 0$ or $k_z \to \beta$ at cutoff. Expanding the characteristic equation given by equation (5.19), and taking limits at $W \to 0$, we get the efficient cutoff equation which can be explicitly written as

$$
\Delta_2 = \begin{vmatrix}
A_{11} & A_{12} & A_{13} & A_{14} \\
A_{21} & A_{22} & A_{23} & A_{24} \\
A_{31} & A_{32} & A_{33} & A_{34} \\
A_{41} & A_{42} & A_{43} & A_{44}
\end{vmatrix} = 0
$$

(5.20)

where

$A_{13} = -1$

$A_{24} = -1$

$A_{33} = -3$

$A_{44} = -3$

and all other $A_{ij}$'s remain the same as in equation (5.19).
Equation (5.20) represents the cutoff equation which is an eigenvalue equation and the solutions of this equation give different values of the propagation constant at cutoff. A numerical estimate is made in the following section of this chapter.

5.3. Modal dispersion curves and some numerical estimates of cutoff V-values by using the characteristic equation and the cutoff equation:

In the above sections, we have studied the modal characteristic equation and the modal cutoff equation of an waveguide bounded by two spirals. Here we have two distinct interfaces which define the boundaries of the guiding structure. The parameter \(a\) and \(b\) (\(a>b\)) define these two boundaries. We assume that the refractive index is \(n_1\) at \(x=a\) and it is \(n_2\) at \(x=b\). The refractive index of the material in the waveguide region is also \(n_2\).

Figure (5.2) shows the dispersions curves for the first four low order modes sustained by the waveguide described above. The standard method of plotting these graphs has been adopted. This method is as follows. The L.H.S. of the characteristic equation (5.19) is first plotted against \(\beta\) by taking equally spaced valued of \(\beta\) in the admissible range \(n_1k_0 > \beta > n_2k_0\) and choosing suitable values for the different parameters. Here we have chosen \(n_1=1.50, n_2=1.48, \lambda_0=1.54\mu m \ a=2.0\mu m\) and the width \((b-a)\) is allowed to change by changing the parameter \(b\). The zero crossings of this graph represent the \(\beta\) values for the successive modes for these parameters. This process is repeated by choosing different values of the parameter \((b-a)\). The normalized propagation constant \(b'\) and the V-parameter are defined as follows:
Fig (5.2) Normalised dispersion curves ($b'$ versus $V$-parameter) for the first four low order modes sustained by the proposed waveguide for $a=2.0\mu m$. 
Fig (5.3) Variation of the L.H.S of the cutoff equation (5.20) with $U(b-a)$ for $a=2.0\mu m$. 
\[
\frac{\beta^2}{k_0^2} - n_2^2 = \frac{b'}{n_1^2 - n_2^2}, \quad \text{and} \quad V = \frac{2\pi}{\lambda_0} (b - a)(n_1^2 - n_2^2) \frac{1}{2}. \]

From these values of \( b' \) and \( V \) the characteristic curves can be obtained.

These characteristic curves exhibit the standard behavior showing the cutoff values of successive modes at \( V = 4.4, V = 12.0, V = 19.2 \) and \( V = 26.2 \). We find that these values are regularly spaced. For \( V \) less than 14 there is only a single sustained mode. It can thus be said that a single mode performance can be obtained for a relatively large value of the \( V \)-parameter. The \( b' \) value of the first mode attains a saturation value at \( V \approx 20 \). The cutoff values can also be determined directly by plotting the L.H.S. of the cutoff equation (5.20) against \( U(b-a) \) where \( U \) is the core parameter given by \( U = \sqrt{k_0^2 n_1^2 - \beta^2} \). This graph is shown in figure (5.3). Remembering that at cutoff, \( V = U(b-a) \), the zero crossings of this curve, give us the successive cutoff values namely \( U(b-a) = 4.8, U(b-a) = 12.4, U(b-a) = 19.2 \) and \( U(b-a) = 25.9 \). We find that the cutoff values determined from the characteristic equations above agree well with the cutoff value determined directly. It is known that a planar waveguide can sustain several modes for moderately large values of \( V \). The present analysis shows that if the cross-section of the planar guide is bent along a spiral such that one side is wider than other, this results in a reduction of the number of sustained modes.

The above discussion pertains to the case \( a = 2.0 \mu m \). The width \( b-a \) is here allowed to change by changing the parameter \( b \). Hence in this case the average curvature will be less than the curvature corresponding to \( r = 2\theta \). If we want to study how the dispersion characteristics change with curvature,
Fig (5.4) Normalised dispersion curves (b' versus V-parameter) for the first four low order modes sustained by the proposed waveguide for a=4.0\mu m.
Fig (5.5) Normalised dispersion curves ($b'$ versus $V$-parameter) for the first four low order modes sustained by the proposed waveguide for $a=6.0\mu m$. 
we have to consider a few more fixed values of \( a \). We choose \( a=4.0\mu m \) and \( a=6.0\mu m \) and then plot the dispersion curves by changing the width (\( b-a \)). Then the resulting dispersion curves are shown in Fig. [5.4] and [5.5] respectively.

As we go from \( a=2.0\mu m \) to \( a=4.0\mu m \) and then to \( a=6.0\mu m \) we are gradually decreasing the average curvature of the guide. By comparing Fig. [5.5] and [5.4] with Fig. [5.2] we find that the cutoff values of the lowest mode changes from \( V=0.4 \) to \( V=2.2 \) and finally to \( V=4.4 \). There is, thus, a shift of the curve from lower values of \( V \) to higher values of \( V \), as the curvature is increased from \( a=6.0\mu m \) to \( a=4.0\mu m \) to \( a=2.0\mu m \). This is true for the other higher modes also. Thus we find that for \( V=23 \) Fig. [5.5] which corresponds to \( a=6.0\mu m \) shows four sustained modes whereas Fig. [5.4] which corresponds to \( a=4.0\mu m \) shows three sustained modes. Again, for \( V=11 \) Fig.[5.4] shows two sustained modes and Fig.[5.2] only one sustained mode. We can thus say that as the curvature of guide increases, the number of sustained modes for a given value of \( V \) shows a tendency to decrease.

It is also necessary that the cutoff values determined from the characteristic equations must agree with the cutoff value determined directly. All these dispersion curves, like those for \( a=2.0\mu m \), show the usual expected behaviour exhibiting the cutoff values which must correspond to the values of \( V \) for \( b'=0 \). In the case of dispersion curves, the cutoff values for some low-order modes are given in Table (I). Table (II) shows the cutoff values obtained directly from the graphs shown in Fig(5.6) for \( a=4.0\mu m \) and in Fig(5.7) for \( a=6.0\mu m \).
Fig (5.6) Variation of the L.H.S of the cutoff equation (5.20) with $U(b-a)$ for $a=4.0\mu m$. 
Fig (5.7) Variation of the L.H.S of the cutoff equation (5.20) with $U(b-a)$ for $a=6.0\mu m$. 
### Table I

<table>
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<th>Mode Number</th>
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<th>$a = 6.0\mu m$</th>
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### Table II

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<th>Mode Number</th>
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<th>$a = 6.0\mu m$</th>
</tr>
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We find that the cutoff values obtained from the characteristic equations above agree well with the cutoff values determined directly. This shows that the cutoff equation is consistent with the characteristic equation.