Chapter 1

Nonlinear phenomena

1.1 Introduction

Nonlinear phenomena form the basis for all the complexities observed in nature and therefore properly defined nonlinear dynamics is required to model complex systems in nature in a satisfactory way. Such modelling, followed by careful analysis, has in recent years, improved our understanding of nature and provided us highly refined techniques that are being applied to all branches of science and engineering in a multidisciplinary manner.

The word nonlinear simply means that the output is not directly proportional to the input. A nonlinear equation involving two variables $x$ and $y$ and some control parameters does not produce a straight line graph between $x$ and $y$ on an ordinary graph paper. When viewed in this way, most of the systems in nature are nonlinear. The most universal and well established nonlinear phenomena exhibited by them are solitons, chaos and stochastic resonance (SR).

Chaos forms the third revolutionary concept and paradigm shift in the
history of the development of Physics in the last century (the other two being relativity and quantum theory) but it is the most multidisciplinary one among the three. It has the further advantage that it can explain self organisation, pattern formation and complex behaviour in many macrosystems directly. Even after 50 years of active research work, it is still a fertile field offering challenges and opportunities for new ideas and/or observations.

Stochastic resonance, in comparison, is a much more recent area of work, with better emphasis and priority in technology. It has also relevance in understanding many processes evolved by nature in biological systems in a self organised way for the existence, functioning and survival of species.

The work in this thesis centres around the study of these two phenomena in a highly organised spatiotemporal system called the coupled map lattice. We begin therefore, by introducing briefly, the basic dynamical phenomena and related concepts required for presenting our work.

1.1.1 Integrable nonlinear systems and solitons

If a system with N degrees of freedom has exactly N independent constants of motion, then the motion is reducible essentially to that of N non-interacting degrees of freedom. Such a system is integrable. The nonlinear equations used to describe the system can be completely solved and the future of the system becomes completely predictable. However, completely integrable systems occur very rarely although they can be suitably used to learn a lot about nonlinear systems.

When the variations of a particular physical property depends both on
space and time, the dynamics of such systems are governed by partial differential equations (PDE). Such systems are often considered as continuous systems with infinite degrees of freedom. These systems may be linear or nonlinear. Among nonlinear continuous systems, nonlinear dispersive systems have a wavelike solution called soliton. A soliton has finite energy and remarkable stability properties. Soliton helps in the identification of completely integrable, infinite dimensional, nonlinear dynamical system [1, 2]. A few such systems that are well studied are KdV system, nonlinear Schrodinger equations, Bowscinesq equations etc. Solutions find technological applications in loss-less transmissions through optical fibres etc.

1.1.2 Nonintegrable nonlinear systems and chaos

Nonlinear nonintegrable systems, even though mathematically difficult to solve, are much more interesting and all pervasive demanding better approaches of study. They can exhibit many interesting features like chaos, in which, even when the governing equations are known exactly, the future is unpredictable. This state that caused concern only to great masters like Maxwell and Poincaré in the 19th century, is a cause of concern and active research for a large multitude of scientists, economists, engineers, biologists, physicians, meteorologists etc. in recent times. The modern history of chaos starts from E. Lorenz, a meteorologist, who discovered evidence of longterm unpredictability in a system of equations, later called the Lorenz system.

As nonlinear equations are either difficult or impossible to solve analytically, the development of the study of chaotic systems has been slow in the beginning. Accessibility to fast and powerful computers has accelerated the developments in this field in recent times.
1.1.3 Multistable systems and Stochastic Resonance

Apart from chaos, certain types of nonlinear systems exhibit an interesting phenomenon called Stochastic Resonance (SR). Here, noise enhances the response of a nonlinear system to a weak input signal which may be periodic or aperiodic. It generally occurs in bistable nonlinear systems driven by a noise and a signal [3]. For every frequency, the information flow, in the form of a signal, through the system becomes optimum for a specific noise intensity.

Stochastic resonance has been observed in many experimental systems including electronic circuits, sensory neurons, optical systems, magnetic systems, mechanical systems, ring lasers and tunnel diode [4–8]. It has many applications for enhancing signal detection and information transmission in communication systems.

1.1.4 Spatiotemporal systems and Coupled Map Lattices

Modelling and characterisations of complex phenomena in space-time is important in the study of turbulence in general. This kind of phenomena, called spatiotemporal chaos, is an attempt to understand it from a knowledge of dynamical systems theory, especially chaos.

Coupled chaotic elements, extended in space, often display complex time evolutions whose description cannot be captured by a low-dimensional dynamical model. Spatially distributed coupled chaotic elements was proposed as a simple model for high dimensional chaos involving spatial pattern dynamics [9]. The coupled map lattice (CML) approach can be viewed as a combination of the two processes, collection of elements that locally exhibit chaotic dynamics on a lattice
and a coupling between these elements leading to a diffusive process.

For a CML with $N$-lattice elements, the dimension is $N^d$ where $d$ is the dimension of the local map used. Thus, dimensionality of CML diverges as the system size, $N$, increases. The important aspects of CML will be discussed in Chapter 2. We present below a preliminary introduction to dynamical systems, especially nonlinear ones and the special features of their dynamical states.

### 1.2 Nonlinear dynamical systems

Nonlinear dynamical systems, because of their intriguing states of behaviour like chaos and SR, need a detailed analysis and understanding starting from classification, possible dynamical states and their proper characterisation *etc.*

A dynamical system may be defined as a deterministic mathematical prescription for evolving the state of a system forward in time. The evolution over time may be discrete (*e.g.*, occurrences of earthquakes and rainstorm) or continuous (*e.g.*, air temperature and humidity, flow of water in perennial rivers). Discrete intervals may be spaced regularly or irregularly in time, continuous phenomena may be measured continuously or may be measured at discrete intervals.

#### 1.2.1 Classification of dynamical systems

Continuous-time dynamical systems are represented by differential equations.

An example of dynamical system in which time is a continuous variable is a system of $N$ first-order autonomous, ordinary differential equations, which can
be written in the vector form as

$$\frac{d \mathbf{X}(t)}{dt} = f(\mathbf{X}(t))$$  \hspace{1cm} (1.1)

where \( \mathbf{X}(t) \) is an \( N \)-dimensional vector. For any initial state \( \mathbf{X}(0) \), we can solve the equations to obtain the future state \( \mathbf{X}(t) \) for \( t > 0 \). The path followed by the system as it evolves in time is called its trajectory in the space of \( \mathbf{X}(t) \).

If the functions \( f \) depend explicitly on time, then the system is nonautonomous. Any nonautonomous system can be converted into an autonomous system by increasing the dimension by one. Hence most often it is sufficient to deal with autonomous ordinary differential equations only.

Discrete-time dynamical systems are represented by difference equations or iterative maps. The maps could be of any dimension depending on the number of physical variables.

Generally, for a map,

$$x_{n+1} = f(x_n, \mu),$$  \hspace{1cm} (1.2)

where \( x_n \) represents the state of the system after \( n \) iterations and \( \mu \) is the control parameter. Iteration is a mathematical way of simulating discrete time evolution. Here for an initial value \( x_0 \), we compute \( x_1, x_2 \ldots \) and so on using (1.2). The sequence \( x_0, x_1, \ldots \) defines the trajectory of the dynamical system.

Nonlinear maps can faithfully capture the salient features of the continuous systems and are easy and fast to simulate on digital computers. Most often discrete systems are mathematically obtained from their continuous versions by applying a construction called Poincaré map or section. In general, for an \( n \)-dimensional flow, the Poincaré section is an \( (n - 1) \) dimensional hypersurface, locally transverse to the flow. Subsequent penetrations of the trajectory on the
section produces a sequence of points. Thus Poincaré map is a function that is generally expressed as

\[ P_{n+1} = f(P_n) \]  

(1.3)

Defined this way, it is equivalent to the iterative scheme in 1.2

If the flow is periodic, the Poincaré section contains a fixed number of points corresponding to the period. For quasiperiodic flow, the points fill out a continuous curve on the section. If the flow is chaotic, the points show relatively more complex distribution.

Dynamical systems can be classified into conservative and dissipative systems depending on whether they conserve phase space volume or not. In equation (1.1), if \( \nabla \cdot f = 0 \), phase space volume is conserved and the system is conservative. If \( \nabla \cdot f \) is negative, the phase space volume shrinks and the system is dissipative as shown in Fig. 1.1.

![Phase space of the damped pendulum](image)

Figure 1.1: Phase space of the damped pendulum which is an example of a dissipative system
1.2.2 Attractors of dynamical systems

As a dissipative dynamical system evolves in time, its trajectory in phase space will head for some final volume in the phase space. These geometrical objects are called attractors. The properties of attractors determine the dynamical properties of the system's long term behaviour or asymptotic behaviour.

The set of initial conditions giving rise to trajectories that approach a given attractor is called the basin of attraction for that attractor. If more than one attractor exists for a system with a given set of parameter values, there will be some initial conditions that lie on the border between the basins of attraction. They form the basin boundary whose shape and nature depend on the complexity of the attractor involved.

The simplest case is when the attractor is a single point called the fixed point. If $p$ is in the domain of a function $f$ and if $f(p) = p$, then $p$ is a fixed point of $f$. A fixed point attractor is a stable attractor if it attracts nearby trajectories or small perturbations about the point eventually die out as $t \to \infty$. For example, the origin of phase space is a stable attractor for damped harmonic oscillator

\[
\frac{d^2x}{dt^2} + \nu \frac{dx}{dt} + \omega^2 x = 0
\]  

(1.4)

Next possible attractor in a two-dimensional phase space is the limit cycle. Limit cycles are possible in higher dimensional space also. They are represented by closed loop trajectories in phase space. A limit cycle is stable if neighbouring trajectories move towards it as $t \to \infty$ and unstable if neighbouring trajectories move away from it as $t \to \infty$. An example for an equation displaying a unstable limit cycle attractor is the vander Pol equation

\[
\frac{d^2x}{dt^2} + (\alpha^2 - \eta) \frac{dx}{dt} + \omega^2 x = 0
\]  

(1.5)

which models a simple vacuum tube oscillator circuit.
For a three-dimensional phase space, a possible attractor is a torus which is a doughnut shaped attractor. Here the trajectory winds round in the latitudinal as well as longitudinal direction of the torus with frequencies $f_1$ and $f_2$. If $f_1/f_2$ is rational, the motion is periodic and if it is irrational, the motion is quasiperiodic [10–13]. This can be generalised to higher dimensions with higher dimensional tori and multiple period motions.

![Diagram of different types of attractors](image)

**Figure 1.2:** Different types of attractors. (a), (b), (c) represent fixed points, (d) represents limit cycle and (e) represents torus.

All the above attractors have integer dimensions. Fixed points have dimension 0, limit cycles have dimension 1 and toroidal attractors have dimension 2 or more.
If the dynamics is chaotic, the resulting attractors have much more intricate geometrical structure. Chaotic behaviour is characterised by the divergence of nearby trajectories in phase space. At the same time, the trajectories must be bounded and unique. These requirements can be met only in three dimensional or higher dimensional phase spaces. A stretching and folding mechanism is necessary to keep the chaotic trajectories bounded despite their exponential divergence. The resulting attractor has thus a fine multilayered structure and its dimension is not an integer. Such attractors are called strange attractors. An example is the attractor obtained for the two-dimensional Henon map

\[ x_{n+1}^{(1)} = A - \left[ x_n^{(1)} \right]^2 + B x_n^{(2)}, \quad x_{n+1}^{(2)} = x_n^{(1)} \]  

(1.6)

for \( A = 1.4 \) and \( B = 0.3 \). [14]

A blow-up of the attractor reveals small-scale structure consisting of a number of parallel lines. The attractor has a self-similar structure.

![Figure 1.3: The Henon map attractor](image)
Figure 1.4: (a), (b) and (c) demonstrate the self-similarity of the Henon map attractor

1.2.3 Strange attractors and their characterisation

As mentioned in the previous section, the chaotic nature of a system asymptotically leads the system to a strange attractor. Such attractors are irregular and aperiodic. In addition, they display sensitive dependence on initial conditions ('SIC' ness). If two identical dynamical systems start at two initial conditions $x$ and $x + \epsilon$, $\epsilon$ being a very small quantity, their states will diverge very quickly from each other in phase space, i.e., the separation between their trajectories grows exponentially with time. Thus, even though the system is deterministic, long-term predictions become meaningless and impossible.

The strange attractors can often be identified in a system using their basic features discussed above. A visual inspection of the trajectories in phase may be interesting, though not always conclusive. Their irregularities and aperiodic nature will be reflected in the power spectrum. If $x(t)$ represents the integrated output of one variable of the system, then, its Fourier transform is

$$a(\omega) = \frac{1}{2\pi} \int x(t)e^{-i\omega t}dt$$

and

$$s(\omega) = |a(\omega)|^2$$

(1.7)

(1.8)

Fourier analysis is a mathematical procedure that reveals periodicity in the
data giving a frequency-domain description of the time-domain data. The power spectrum of a periodic motion with frequency $\omega_1$ has peaks at $\omega_1$ and its harmonics. A quasiperiodic motion has peak at $\omega_1, \omega_2, \ldots, \omega_n$ and at linear combinations of these frequencies with integer coefficients. On the other hand, chaotic motion gives rise to a broadband power spectrum.

Numerically, Fourier transform calculation involves lengthy computations. Generally, for $N$ observations, $N^2$ operations are needed. Computer algorithms have been made to reduce the number of operations to $N \log_2 N$, taking advantage of certain symmetry properties in the trigonometric functions at their points of evaluation. This is called the Fast Fourier Transform or FFT [15, 16].

A more useful quantitative index to characterise the chaotic attractor is the Lyapunov Exponent that is defined to measure its ‘SIC’ness nature. This will be discussed in detail in §1.4.1.

A chaotic attractor, because of its multilayered structure, is usually a fractal. A fractal [17] is a line, surface or pattern that looks the same over a wide range of scales, i.e., it is self similar and scale invariant. Fractals have non-integer dimensions. They can be mathematically generated by repetitions of the same operation.

1.2.4 Fractals

Fractals may be deterministically-generated by iterating an equation or a mathematical construction or naturally found in nature. The Cantor set, the Koch snow flake and the Sierpinski gasket which belong to the former class are discussed in detail in this section. Rocks surface textures, coastal lines of islands, clouds etc.
belong to the latter class. Fractals can be used as ideal models for discussing complex geometry of several patterns in nature, like human circulatory system [18] and seeping of oil through porous rocks [19].

The Cantor set was introduced by the German mathematician George Cantor (1845–1918). It is constructed by starting with a line segment of length 1. For the first stage, we delete the middle one-third of the segment, leaving two segments, each of length $\frac{1}{3}$. In the next stage, the middle one-third of each of these segments is deleted, leaving four line segments, each of length $\frac{1}{9}$. In general, after the $M^{th}$ stage, we have $2^M$ segments, each of length $\frac{1}{3^M}$. This process leaves a series of points as $M \to \infty$, called Cantor dust. This is illustrated in Fig. 1.5

![Figure 1.5: An illustration of the stages in the construction of a Cantor set.](image)

The Koch curve was introduced by the Swedish mathematician Helge von Koch in 1904. To construct this fractal we start with a line segment of unit length. Then we remove the middle $\frac{1}{3}$ of the line segment and replace it with two segments, each of length $\frac{1}{3}$ to form a 'tent' as shown in Fig. 1.6. For the second stage we remove the middle $\frac{1}{3}$ of each of the smaller segments and replace those with two more segments to form tents. At the $M^{th}$ stage, we have $4^M$ segments, each of length $\frac{1}{3^M}$. As $M \to \infty$, we have a curve of infinite length. The curve is nowhere
differentiable because there are an infinite number of abrupt changes in the slope.

<table>
<thead>
<tr>
<th>Stage</th>
<th>Segments</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>16</td>
</tr>
</tbody>
</table>

Figure 1.6: An illustration of the first two stages in the construction of a Koch curve.

The Sierpinski gasket is named after the Polish mathematician Waclaw Sierpinski, who studied it half a century ago. In this case we start with an equilateral triangle with each side of unit length. Then the central triangle is deleted as shown in Fig. 1.7. This process is repeated. In each stage we have three times as many triangles as in the preceding stage with each triangle having sides of half the length of those in the preceding stage. Thus, at the $M^{th}$ stage, we have $3^M$ triangles with sides of length $\frac{1}{2^M}$.

Figure 1.7: An illustration of the first few stages in the construction of a Sierpinski gasket.
The easiest measure to characterise a fractal is the box counting dimension (also known as capacity dimension). We construct boxes of side $R$ to cover the space occupied by the geometric object under consideration. The boxes are line segments for a one-dimensional object, squares for two-dimensional objects and cubes for three-dimensional objects. We then count the number of boxes $N(R)$ needed to contain all the points of the geometric object. As $R$ decreases, $N(R)$ increases. The box counting dimension $D_b$ is defined by the equation

$$N(R) \propto R^{-D_b}$$

(1.9)

Taking log on both sides and applying the limit $R \to 0$, we get

$$D_b = \lim_{R \to 0} \frac{\log N(R)}{\log 1/R}$$

(1.10)

For a Cantor set, this definition gives

$$D_0 = -\frac{\log 2^M}{\log (\frac{1}{3})^M} = \frac{\log 2}{\log 3} = 0.63$$

(1.11)

Similarly for Koch curve, this is

$$\frac{\log 4}{\log 3} = 1.26$$

(1.12)

and for Sierpinski gasket,

$$\frac{\log 3}{\log 2} = 1.58$$

(1.13)

$D_b$ for other complicated fractals are not so direct and easy to calculate. In such cases counting has to be done using numerical algorithms in a computer. These will be introduced in detail in § 1.4.
1.3 Discrete chaotic systems

One-dimensional noninvertible maps are the simplest systems capable of chaotic motion. [20, 21] A large variety of the phenomena encountered in higher dimensional systems is already present, in some form, in one dimensional maps. So they serve as a convenient starting point in the study of chaos.

1.3.1 Logistic Map

The most popular equation is the logistic equation or the logistic map given by [22, 23]

\[ x_{n+1} = \mu x_n (1 - x_n) \]  \hspace{1cm} (1.14)

It is a one-dimensional feedback system, designed to model the long-term change in the population of a species by P. F. Verhulst in 1845. \( \mu \) is restricted to be in the range \( 0 \leq \mu \leq 4 \). It is easy to see that the, the maximum value of \( f(x) \) occurs when \( x = \frac{1}{2} \) where \( f(x) = \frac{\mu}{4} \). In order to ensure that \( 0 \leq f(x) \leq 1 \), it is necessary to restrict \( 0 \leq \frac{\mu}{4} \leq 1 \) or \( 0 \leq \mu \leq 4 \). At \( x = \frac{1}{2} \), \( f'(x) = 0 \) and \( f''(x) \) is negative. The graph of \( f(x) \) and its iterates are shown below in Fig. 1.8. We see that the

![Figure 1.8: Graphs of \( f(x) \) and its iterates for the logistic map (1.14).](image)

16
map is unimodal and non-invertible. In order to understand the important features of the dynamical states of such maps, we examine the stability of the asymptotic states of the map as $\mu$ is varied from 0 to 4.

### 1.3.2 Bifurcation sequences

The fixed point equation for the logistic map is

$$x^* = \mu x^* (1 - x^*)$$

which gives,

$$x_1^* = 0 \text{ and } x_2^* = \frac{\mu - 1}{\mu}.$$  \hfill (1.16)

The stability of these points is decided by the behaviour of the iterates when the system is slightly perturbed. If the iterates asymptotically approach $x^*$, then it is stable. The stability condition therefore is

$$|f'(x)|_{x=x^*} \leq 1.$$  \hfill (1.17)

Thus, $x_1^* = 0$ is stable for $0 \leq \mu < 1$ and $x_2^* = \frac{\mu - 1}{\mu}$ is stable for $1 < \mu < 3$. As $\mu$ increases through 1, $x_1^*$ becomes unstable and $x_2^*$ becomes stable. This sudden change in the behaviour of a system, as some parameter is varied, is commonly known as bifurcation. At the bifurcation point $\mu = 1$, $|f'(x)| = 1$.

The fixed point $x_2^*$ is stable in the range $1 < \mu < 3$. Then, at $\mu = 3$, two new fixed points are created through a period doubling bifurcation. This is shown graphically in Fig. 1.8.

The two new fixed points created when $\mu = 3$, say $x_{11}^*$ and $x_{12}^*$, are the
solutions of the equation

\[ f^2(x_n) = f(f(x_n)) = x_{n+2} = x_n. \]  \hspace{1cm} (1.18)

i.e \( f(x^*_1) = x^*_2 \) and \( f(x^*_2) = x^*_1 \).  \hspace{1cm} (1.19)

The stability of these fixed points is decided by the criterion

\[ |f'(x^*_1)f'(x^*_2)| < 1 \]  \hspace{1cm} (1.20)

As \( \mu \) increases beyond 3, the derivative of \( f^2(x) \) decreases below +1 and
the two points \( x^*_1 \) and \( x^*_2 \) gain stability. This continues until \( \mu = 1 + \sqrt{5} \), where
the derivative of \( f^2(x) \) becomes -1. Hence, for \( \mu > 1 - \sqrt{5} \), the two-cycle fixed
points become unstable. Now the system settles to a four-cycle.

This process repeats and the bifurcations continue with doubling of peri-
ods until the period becomes infinite at Feigenbaum point or accumulation point
(\( \mu_\infty \)). This bifurcation causing period doubling is also known as pitch fork bi-
furcation. The above discussion shows that the transition from regular to chaotic

![Figure 1.9: Bifurcation diagram for the logistic map.](image)

behaviour is via period-doubling. This is one of the important routes to chaos, the
other two being intermittency and quasiperiodicity. It occurs in fluid convection,
water waves, biology, chemistry, optics *etc.* [24] and is shown in the bifurcation
diagram in Fig. 1.9

The exact values of $\mu$ at which successive bifurcations occur converge in
a universal fashion. If $\mu_n$ is the value at the $n^{th}$ bifurcation, then,

$$\mu_n - \mu_{n-1} = \Delta \mu_n = C_n \delta_n^{-n} \quad (1.21)$$

and as $n \to \infty$, $\delta_n \to \delta$. Thus, $\delta$ can be defined as

$$\delta = \lim_{n \to \infty} \frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n} \quad (1.22)$$

$\delta$ converges to 4.66928 as $n \to \infty$. This means that successive bifurcation
values approach the Feigenbaum point $\mu_{\infty}$ in a geometric way. The equation can
also be used to calculate

$$\mu_{\infty} \approx \mu_n + \frac{\mu_n - \mu_{n-1}}{\delta - 1} \quad (1.23)$$

There is another index $\alpha$, is defined as the ratio of the measured width of one
fork to that of the next generation of forks. The widths are measured at the points
where the orbits are superstable ($f'(x) = 0$), to its nearest cycle element, as shown
in Fig. 1.10.

![Bifurcation diagram of the logistic map, used to calculate $\alpha$ and $\delta$.](image)

Figure 1.10: Bifurcation diagram of the logistic map, used to calculate $\alpha$ and $\delta$. 
In the limit $n \to \infty$, we get
\[ \alpha = \lim_{n \to \infty} \frac{\Delta x_n}{\Delta x_{n-1}} \] \hfill (1.24)

For the logistic map, $\alpha = 2.5029$.

This universal behaviour is called metric universality. Feigenbaum observed that the values of $\alpha$ and $\delta$ depend only on the order of maximum of the map. $\alpha$ and $\delta$ are therefore called Feigenbaum indices. They are observed in relatively complicated systems also [25, 26]. Related constants corresponding to circle map have also been observed experimentally. [27]

When the period doubling sequence continues as the parameter is tuned, period becomes infinite and the trajectory then never repeats itself. This state is called a chaotic attractor. The inhomogeneity of points on a strange attractor demands the use of a continuous spectrum of generalised dimensions and scaling indices to describe it fully. However, beyond the accumulation point of the period doubling scenario, even though the transition to chaos takes place, there are windows of periodicity that arise from other types of bifurcations.

1.3.3 Tangent bifurcation, intermittency and crisis.

In the chaotic regime beyond $\mu_\infty$, there are many periodic windows of periodicity 3, 5, 6, 7, 10... (Fig. 1.11)
Figure 1.11: Iterates of the logistic map plotted against the control parameter beyond $\mu_\infty$. The 3-cycle windows is clearly seen.

These are formed due to another mechanism called tangent bifurcation.

To understand the mechanism, we consider $\mu = \mu_c = 3.83187$, which is the critical value of $\mu$ at which the three-cycle window appears. At this point, the third iterate of the logistic map has three extrema which are tangent to the line $f(x) = x$. For $\mu > \mu_c$, the curve passes through the line producing three pairs of fixed points, three of which are stable and the other three are unstable. This behaviour is shown in Fig. 1.12.

Since the map function becomes tangent to the 45° line at $\mu = \mu_c$, the bifurcation is called tangent bifurcation.

In general, the condition for the birth of a period $-n$ cycle by tangent bifurcation is that

$$f^n(x^*) = +1$$
$$f^n(x^*) = x^*$$

(1.25)
Figure 1.12: The third iterate of the logistic map at (a) $\mu = \mu_c$ (b) $\mu > \mu_c$.

These cycles subsequently undergo period doubling and reach chaos as $\mu$ increases.

If we decrease $\mu$ below $\mu_c$, the map becomes chaotic via intermittency. Intermittency is the behaviour exhibited by dissipative systems where regular and chaotic behaviour are seen intermittently. It was first described by Pomeau and Manneville [28, 29]. As the control parameter changes the regular interval decreases and chaotic behaviour broadens.

This is because the sequence of iterates follow a staircase path (Fig. 1.13), i.e., the trajectory spends a considerable amount of time near a previously stable fixed point, until it is repelled from this region. As $\mu$ approaches $\mu_c$, the time spent in the gap increases and at $\mu = \mu_c$, the behaviour is completely periodic.

This type of scenario has been observed in many physical experiments like single mode laser under constant driving and periodic modulation [30] and in may
Figure 1.13: Region near an extremum in Fig. 1.12, where $\mu$ is slightly less that $\mu_c$.

biological systems like pacemaker neurons under sinusoidal stimulation [31].

The type of intermittency discussed here is called Type I intermittency. This is seen in systems that show period doubling route to chaos. Type II intermittency or Hopf bifurcation intermittency is observed in a few experimental studies [32, 33]. They occur in two or higher dimensional maps. Type III intermittency is also called period doubling intermittency because the slope of the map at the transition point is $-1$, causing period doubling, but the periodic cycles are not stable. It has been observed in Rayleigh-Benard experiment [34] and in a smooth perturbation of the logistic map. [35] Other types of intermittencies including multi-intermittency, crisis-induced intermittency and on-off intermittency also exist [36–39].
Sudden changes in chaotic attractors with parameter variation are seen commonly [40, 41]. Such events in which a chaotic attractor disappears or suddenly expands in size are called crises [42, 43]. Three types of crises have been defined. When a chaotic attractor is suddenly destroyed as the parameter passes through a critical value, it is called boundary crisis. It occurs when the attractor collides with its periodic orbit on its basin boundary. The sudden increase in the size of the attractor is called interior crisis. This occurs when the periodic orbit with which the chaotic attractor collides is in the interior of its basin. The third type occurs in systems with symmetries and is called attractor merging crisis. Here two or more chaotic attractors simultaneously collide with a periodic orbit (or orbits) on the basin boundary which separates them. The chaotic attractor of the logistic maps that was present for $\mu < 4$ suddenly disappears at $\mu = 4$. This is due to boundary crisis [44].

From the discussion about intermittency and tangent bifurcation, we can see that interior crisis occurs in the logistic map near tangent bifurcation, producing stable and unstable fixed points. As parameter $\mu$ changes, the stable fixed points undergo a period doubling sequence and form chaotic bands. As $\mu$ again increases, the chaotic bands collide with the unstable fixed points and the chaotic attractor suddenly increases in size. This can be seen to the right of the period–3–period–doubling cascade in the bifurcation diagram of the logistic map. [45]

At the accumulation point, there are infinite number of bands. As the parameter value increases, the number of bands become lesser due to the collision of unstable orbits of periodicity $2^{n-1}$ with that of $2^n$. Finally, single band is created. The band merging produces a non-uniform Cantor set (Fig. 1.14).
Figure 1.14: Nonuniform Cantor set formed due to band-merging for the quadratic map \( f(x) = 1 - \mu x^2 \).

### 1.3.4 Bimodal cubic maps

We have seen that one-dimensional discrete systems modelled by nonlinear maps normally support a sequence of period doublings leading to chaos. It is also possible to take the systems back to periodicity through a sequence of period halvings by adding perturbations or modulations [46, 47]. If the system is sufficiently nonlinear, interesting phenomena like bubbling and bistability also occur. The simplest cases are maps with two control parameters, one that controls the nonlinearity and the other which is a constant additive one \( i.e. \), maps of the type

\[
x_{n+1} = f(x_n, a, b) = f_1(x_n, a) + b. \quad [48]
\]

The occurrence of bubbling or bistability is decided by the basic property of \( f(x, a, b) \), \( viz. \), the nonlinearity in \( f(x, a, b) \) must be more than quadratic. Then the derivative \( f'(x, a, b) \) is non-monotonic in \( x \) and there is at least one inflection point \( x_i \) such that

\[
f''(x, a, b) \big|_{x=x_i} = 0 \quad \text{(1.27)}
\]
Then we can have two different possibilities.

(i) if \( x_i \) corresponds to the minimum of \( f'(x, a, b) \), then

\[
f'''(x, a, b) \big|_{x=x_i} > 0
\]

For such maps, there is a value \( a_1 \) of \( a \) such that

\[
f'(x, a, b) \big|_{x=x_i} = -1
\]

By fixing \( a \) near \( a_1 \) such that \( f'(x, a, b) \big|_{x=x_i} < -1 \) and tuning \( b \), the system can be taken through a bubble structure in the bifurcation scenario.

(2) For the other set of maps, the inflection point \( x_i \) is a maximum of \( f'(x, a, b) \), such that

\[
f'''(x, a, b) \big|_{x=x_i} < 0
\]

Then, there exists a value \( a_1 \) of \( a \) such that

\[
f'(x, a, b) \big|_{x=x_i} = +1
\]

By fixing \( a \) near \( a_1 \) such that \( f'(x, a, b) \big|_{x=x_i} > +1 \) a bistability region can be observed in the system as the parameter \( b \) is varied.

The maps belonging to the above types can be studied using two specific examples.

\[
x_{n+1} = b - ax_n + x_n^3
\]

(1.32)

which belongs to the first category, and

\[
x_{n+1} = b + ax_n - x_n^3
\]

(1.33)

which belongs to the second category. These maps are plotted in Fig. 1.15 (a) and (b) respectively. Their bifurcation scenario is shown in Fig. 1.16 where (a) and (b) correspond to (1.32) and (c) and (d) correspond to (1.33).
1.4 Measures of chaos

Many important nonlinear systems display regular or chaotic behaviour depending on a number of factors like strength of the control parameter, initial conditions, nature of the external forcing etc. We need specific quantitative measures, in addition to the qualitative features, to study such systems. These quantifiers also help to distinguish between chaotic and noisy behaviour. They tell us the number of variables required to model a dynamical system. Systems belonging to different
Universality classes can be sorted out using them and their variations indicate important changes in the dynamical behaviour of the system.

The most prominent measures used as quantifiers are the Lyapunov exponent, generalised dimensions and $f^{-\alpha}$ spectrum.

### 1.4.1 Lyapunov Exponent

The Lyapunov exponent is named after A. M. Lyapunov, a Russian mathematician. It is used to define a measure of the sensitive dependence on initial conditions, which is characteristic of chaotic behaviour. This exponent can be computed easily for a one-dimensional map such as the logistic map.

If the system is allowed to evolve from two slightly differing initial states $x_0$ and $x_0 + \epsilon$, then, after $n$ iterations, the difference between the two cases will be

$$d_n = |f^n(x_0 + \epsilon) - f^n(x_0)|$$  \hspace{1cm} (1.34)

If the behaviour is chaotic, then the distance $d_n$ grows exponentially with time, so that

$$\frac{d_n}{\epsilon} = \frac{|f^n(x_0 + \epsilon) - f^n(x_0)|}{\epsilon} = e^{\lambda n}$$  \hspace{1cm} (1.35)

or

$$\lambda = \frac{1}{n} \log \frac{|f^n(x_0 + \epsilon) - f^n(x_0)|}{\epsilon}$$  \hspace{1cm} (1.36)

where $\lambda$ gives the Lyapunov exponent.

In the limit $\epsilon \to 0$, the R.H.S. of (1.36) gives the derivative of $f^n(x_0)$
with respect to \( x \). Applying chain rule, the expression (1.36) can be written as

\[
\lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log_e |f'(x_i)| \text{ where } x_i \text{ is the } i^{\text{th}} \text{ iterate of } x_0
\]  

(1.37)

Thus, \( \lambda \) gives the stretching rate per iteration averaged over a trajectory. If \( \lambda \) is negative, slightly separated trajectories converge and the dynamics is not chaotic. If \( \lambda \) is positive, nearby trajectories diverge, corresponding to a chaotic state.

For an \( n \)-dimensional system, there are \( n \) Lyapunov exponents. For a dissipative system, the sum of these exponents will be negative. If at least one of them is positive, the system is chaotic. The inverse of the magnitude of \( \lambda \) gives the time scale of possible short term predictability in a chaotic system.

This definition of Lyapunov exponent can be generalised to continuous flows or higher dimensional maps [49].

### 1.4.2 Dimensions

Another important set of quantities to measure the geometry of the complex structure of a chaotic attractor is the generalised fractal dimensions.

We have seen in \$1.2.4\$, that simple fractals, that can be constructed from simple mathematical iteration schemes, usually have fractional dimensions. However, the fractals that we come across in a chaotic attractor are much more complex fractals. To characterise such fractals completely we need generalised dimensions.
The easiest dimension is the box counting dimensions or capacity dimension \( D_0 \) (or \( D_b \)) introduced in § 1.2.4. A second dimension that is physically relevant is the information dimension \( D_1 \). If \( N_i \) is the number of points of the geometrical object in the \( i^{\text{th}} \) box and \( N \) is the total number of points, then

\[
p_i = \frac{N_i}{N} \tag{1.38}
\]

is the relative frequency of occupation of the \( i^{\text{th}} \) box. Then we can introduce a measure of missing information as [50]

\[
I(R) = -\sum_i p_i \ln p_i \tag{1.39}
\]

where \( R \) is the size of the box. Then,

\[
D_1 = \lim_{R \to \infty} \frac{I(R)}{\ln R} \tag{1.40}
\]

The number of computations required for the box-counting procedure increases exponentially with the phase space dimension. To provide a computationally simpler dimension for an attractor, Grassberger and Procaccia [51] introduced correlation dimension. Its computational advantage lies in the fact that it uses the trajectory points directly and does not require separate partitioning of the phase space. To define the correlation dimension, we first let a trajectory evolve for a long time and we collect the values of \( N \) trajectory points. Then for each point \( i \) on the trajectory, we count the number of points lying within a distance \( R \) of the point \( i \), excluding \( i \) itself. This number \( N_i(R) \) can be written in more formal terms using the Heaviside step function \( \Theta \) as

\[
N_i(R) = \sum_{j=1, j \neq i}^{N} \Theta(R - |x_i - x_j|) \tag{1.41}
\]

Then the relative number \( p_i(R) \) of points within a distance \( R \) of the \( i^{\text{th}} \) point is given by

\[
p_i(R) = \frac{N_i(R)}{N - 1} \tag{1.42}
\]
The correlation sum \( C(R) = \frac{1}{N} \sum_{i=1}^{N} p_i(R) \) (1.43)

The correlation dimension \( D_2 \) is defined to be the number that satisfies

\[
C(R) \propto R^{D_2} \quad \text{or} \quad \quad D_2 = \frac{\int_{R \to 0} \frac{\log C(R)}{\log R}}{L_t}
\] (1.44)

This dimension measures the typical number of neighbours a point has, and is generally more efficient to compute. It reflects small-scale variations of the density of points on a fractal object.

### 1.4.3 Multifractal dimensions

\( D_0, D_1, \) and \( D_2 \) are the first three important dimensions for a complex fractal set. However, these can be generalised to get \( D_q \), the generalised dimensions of order \( q \) as

\[
D_q = \int_{R \to 0} \frac{1}{q - 1} \ln \left( \frac{\sum_{i=1}^{N(R)} p_i^q}{\ln R} \right)
\] (1.46)

where \( p_i \) is defined as in (1.38)

As \( q \to \infty \), the largest probability value \( p_{\text{max}} \) will dominate the sum. Therefore,

\[
D_\infty = \int_{R \to 0} \frac{\ln p_{\text{max}}}{\ln R}
\] (1.47)

When \( q \to -\infty \), the smallest probability value, \( p_{\text{min}} \) will dominate the sum. Therefore,

\[
D_{-\infty} = \int_{R \to 0} \frac{\ln p_{\text{min}}}{\ln R}
\] (1.48)

\( D_{-\infty} \geq D_\infty \) and in general, \( D_q \geq D_{q'} \) for \( q < q' \). Fig. 1.17 shows the variation of \( D_q \) with \( q \).
For a self-similar simple fractal like the Cantor set, all cells have equal probability of \( p_i = \frac{1}{N(M)} \). \( D_q = D_0 \) for all \( q \).

### 1.4.4 \( f-\alpha \) spectrum

Different parts of a chaotic attractor may be characterised by different values of fractal dimensions. An object with a multiplicity of fractal dimensions, is a multifractal and it can be considered as a collection of overlapping fractal objects. Most of the chaotic attractors and natural fractal objects are multifractals.

In order to find out how many regions have a particular value or range of values of the fractal dimension, a distribution function \( f(\alpha) \) is used. It gives the distribution of scaling exponents labelled by \( \alpha \).

Using the previous definitions, the probability that a trajectory visits the \( r^{th} \) cell is \( p_i(R) = \frac{N_i}{N} \). We assume that \( p_i(R_i) \) satisfies the scaling relation

\[
p_i(R_i) = K R_i^{\alpha_i}
\]  
(1.49)
where $K$ is some proportionality constant and $R$ is the size of the $i^{th}$ cell. $\alpha_i$ is called the scaling index for cell $i$. As we make $R$ smaller, $N(R)$ increases. Then, the number of cells having a scaling index in the range between $\alpha$ and $\alpha + d\alpha$ is assumed to scale with the size of the cells with a characteristic exponent $f(\alpha)$

$$N(\alpha) = k R^{-f(\alpha)} \quad (1.50)$$

Here $f(\alpha)$ plays the role of a fractal dimension of a set of points with scaling index $\alpha$. When $f(\alpha)$ is plotted as a function of $\alpha$, we get Fig. 1.18. The degree of inhomogeneity or complexity of the geometry of the attractor can be estimated as $|\alpha_{\max} - \alpha_{\min}|$.

Both $f(\alpha)$ and $D_q$ offer analogous but parallel descriptions for the geometry of a multifractal. The change from $q$-$D_q$ description to $f$-$\alpha$ is a Legendre transformation.

$$ (q - 1)D_q = q\alpha - f(\alpha) \quad (1.51)$$

with $\alpha = \frac{d}{dq}[(q - 1)D_q] \quad (1.52)$

$$f(\alpha) = q\left[\frac{d}{dq}(q - 1)D_q\right] - (q - 1)D_q$$
When \( q = 1 \), \( f(\alpha) = \alpha \) and \( f'(\alpha) = 1 \). So the straight line \( f(\alpha) = \alpha \) is tangent to the \( f(\alpha) \) curve at \( f(\alpha) = \alpha = D_1 \). In principle, we can construct \( D_q-q \) curve from \( f(\alpha)-\alpha \) curve or vice versa. Large values of \( \alpha \) imply rarefied subsets and small \( \alpha \) implies dense subsets.

### 1.5 Stochastic Resonance

Over the last two decades, stochastic resonance (SR) has continuously attracted considerable attention. The term SR is given to a phenomenon that is manifest in nonlinear systems whereby generally feeble input information (such as a weak signal) can be amplified and optimised by the assistance of noise. If we can imagine a double well potential which is rocked by a weak periodic function, with the help of noise, the state point always has a non-zero probability to switch wells. This probability of a switching event is synchronised with the period of the signal. The switching events thus carry the information about the signal frequency and amplitude to the output of the system. Thus the inter-well transitions are synchronous with the signal and this is reflected in the power spectrum of the time-series of the output.

![Figure 1.19: The block diagram representing SR](image)
1.5.1 History

The concept of SR was originally put forward by Benzi and his collaborators [6,52,53] when they addressed the problem of the periodically recurrent ice ages in the history of global climate. The same suggestion that SR might explain the periodicity of the primary cycle of recurrent ice ages was raised independently by C. Nicolis and G. Nicolis [54–57]. In the model of Benzi et al., the global climate is represented by a double-well potential, where one minimum represents a small temperature corresponding to a largely ice-covered earth. The small modulation of the earth’s orbital eccentricity is represented by a weak periodic forcing. Short-term climate fluctuations, such as the annual fluctuations in solar radiation, are modelled by Gaussian white noise. It was found that for an optimum noise, synchronised hopping between the cold and warm climate could significantly enhance the response of the earth’s climate to the weak perturbations caused by earth’s orbital eccentricity.

A first experimental verification of SR was obtained by Fauve and Heslot [5], who studied the noise dependence of the spectral line of an a.c.-driven Schmidt trigger. The field then remained dormant until the modern age of stochastic resonance was ushered in by a key experiment in bistable ring laser [58]. Thereafter, the effect of SR has been found and studied in a variety of physical systems, namely, in magnetic systems [59], in passive optical bistable systems [60], in systems with electronic paramagnetic resonance [61], brownian particles [62], in experiments with magneto-elastic ribbons [63], in a tunnel diode [64], in superconducting quantum interference devices (SQUIDS) [65] and in ferromagnetic and ferroelectrics [66–68]. SR is recently observed also in chemical systems [69–71] and in social models [72].

SR plays a crucial role in many biological systems for extracting a weak
periodic signal embedded in a large amount of background noise [3, 73, 74]. It is also found to have a large influence in the future development of nonlinear devices and for use in communication and information transmission processes. Recently, there has been a few interesting studies on SR in spatially extended systems theoretically and experimentally. Such systems with additive and multiplicative noise, are found to show SR [75]

1.5.2 Mechanism of Stochastic Resonance.

The basic set up for SR consists of an energetic activation barrier, a weak coherent input (such as a periodic signal) and a source of noise that is inherent in the system or added to the coherent input. The underlying mechanism is simple and can be understood by considering a quartic double well potential of the form

\[ V(x) = -\frac{a}{2} x^2 + \frac{b}{4} x^4 \]  

(1.53)

The minima in this case are located at \( \pm (x_m) \) where \( x_m = \sqrt{\frac{a}{b}} \). These are separated by a potential barrier with height \( \Delta V = \frac{a^2}{4b} \). The barrier maximum is located at \( x_b = 0 \). In the presence of a periodic driving, the double-well potential \( V(x, t) = V(x) - A_0 x \cos \omega t \) is rocked back and forth, thereby raising and lowering successively the potential barriers to the left and right as shown in Fig. 1.20.

Although the periodic forcing is too weak to let the system roll from one well to the other, noise induced hopping by itself is possible, synchronised with weak periodic forcing. Obviously this synchronisation can take place with maximum probability when the average waiting time between the two noise-induced interwell transitions is comparable with half the period of the forcing. For a given period of forcing, the time-scale matching condition is fulfilled by tuning the noise level. When the noise level is too small, the mean crossing time is too large and
considerably exceeds the signal period. When the noise level is too high, there is a non-zero probability that the system switches repeatedly within one signal period and the output is irregular. By tuning the noise intensity, we can adjust the system for optimum functioning as a stochastic resonator (Fig. 1.21).

1.5.3 Measures of Stochastic Resonance.

To characterise stochastic resonance, the power spectrum of the output is plotted against frequency, by taking the FFT of the output time series [6]. Signal to noise ratio (SNR) is defined as the ratio of the intensity in the $\delta$-spike in the power
Figure 1.21: SR by tuning the noise amplitude

spectrum at the frequency of the signal, to the height of the smooth fluctuational background at the same frequency.

\[ \text{SNR} = \log_{10}(\frac{S}{N}) \]

(1.54)

where \( N \) is the average background noise around the signal \( S \).

From the above discussion, it is clear that as the noise intensity is tuned, SNR increases to a maximum value and then decreases. The presence of a peak in the output signal spectrum and a maximum zone in the signal to noise ratio thus characterises the occurrence of SR in the system. This is shown in Fig. 1.22. [76]

Figure 1.22: Variation of SNR with noise amplitude

From the signal detection point of view, SNR is a good quantifier, but
according to ordinary resonance theory, SNR should become maximum when the time scale induced by noise matches with the periodicity of the forcing term, which can be achieved either by tuning the noise or the forcing frequency. Even though SNR shows nonmonotonicity as a function of noise intensity, it is a steadily decreasing function of the forcing frequency. Therefore another quantifier, Residence time distribution function (RTDF), has been introduced later [77–79].

If \( t_i \) denotes the time at which successive switchings occur, normalised distribution \( N(t) \) of the quantities \( T(i) = t_i - t_{i-1} \) is called RTDF. The interval \( T(i) \) represents the residence time between two subsequent transitions. In symmetrical bistable systems forced by noise alone, \( T_i \) has an exponential distribution of the form

\[
N(T) = \frac{1}{T_k} e^{-\frac{T}{T_k}}
\]

where \( T_k \) is the Kramer's rate

In the presence of a periodic forcing, we get a series of peaks centered at odd multiples of half the time period of the signal. The heights of the peaks decrease exponentially with their order.

The best time for the system to switch between two potential wells is when the potential barrier becomes minimum \( \text{i.e.,} \) when well is tilted extremely to the right or to the left. If it switches at this time to the other well, then it waits in the well for half the time period until the barrier becomes minimum due to a tilt in the opposite direction. Thus, half the time period is the preferred residence interval. If the system misses this jump, then it waits for one more full period for the next switch when the potential barrier becomes minimum. Thus the next peak in RTDF is located at \( \frac{3}{2} \) of the time period. This shown in Fig. 1.23 [76]
1.5.4 Stochastic Resonance in coupled systems.

Recently, SR in spatially extended systems are being studied both theoretically and experimentally. The effect of SR can be significantly enhanced if an array of coupled bistable oscillator systems is taken instead of a single one [80]. It has been found that at optimal values of noise intensity and the coupling coefficients, the SNR in an array attains its maximum level, demonstrating an array enhanced SR effect [81,82], as shown in Fig. 1.24. [76]
A simple interpretation that can be offered for the above observation is that even when one oscillator misses the inter-state switching, the near-by oscillators may not, thus forcing the individual oscillator to switch. This co-operative behaviour should induce enhanced regularity in the switching of the oscillators and increase SNR for a given oscillator. This has been studied theoretically using two coupled bistable maps [76] also. The use of coupling to enhance output is being effectively used by biological systems in detection of signals immersed in background noise and in the development of nonlinear devices used to amplify weak signals with the help of noise, in signal processing and communications.

1.6 Conclusion

In this chapter, we have introduced the basic definitions of nonlinearity and nonlinear dynamical systems and the related concepts and measures. There are many physical situations where such nonlinear systems naturally arise. In this concluding section, we point out a few recent theoretical notions that have exciting applications in frontier technologies. They include control of chaos, synchronisation and recent applications of chaos and SR.

1.6.1 Control of chaos

Although chaos is widespread and observed in numerous physical, chemical and biological systems, in many situations, chaos is sometimes an unwanted phenomenon. Thus increasing drag in flow systems, erratic fibrillations of heart beating, erratic vibrations of mechanical and electronic systems etc. are some situations where chaos has harmful effect. Then, it is advisable to avoid chaos whenever possible, without altering the underlying system significantly. Such control
of chaos is usually done using a small feedback term that stabilises an unstable periodic orbit or creates a new periodic orbit using a weak periodic perturbation. For example, a chaotically behaving oscillator can be manipulated by a small control and the output can be made to carry information. Other examples where control of chaos has been achieved include chaotic diode resonator circuit, different types of laser systems, electronic and electrical power systems etc. It is even used to control chaotic traffic on a congested motor-way and hence to increase the speed of traffic flow [83].

1.6.2 Synchronisation

Chaotic systems have sensitive dependence on initial conditions and independently evolving chaotic systems can never synchronise with one another even if they are identical. But, by introducing a suitable coupling between the two chaotic systems, it is possible to have synchronisation. This has important technological application in spread spectrum secure communications of analog and digital signals [84]. It is also used in developing a safe and reliable cryptographic system for transmission of secret messages. Cryptography becomes most-reliable when the key used is as random as possible and also simultaneously available to both the sender and receiver. This is best achieved in synchronised chaotic systems and finds applications in systems involving electronic banking, military and diplomatic communications.

1.6.3 Applications of chaos

The study of chaos has provided new conceptual and theoretical tools enabling us to categorize and understand complex behaviour that had confounded previous
theories. Chaotic behaviour also seems to be universal as it shows up in almost all naturally occurring systems. Even more important are the universal qualitative and quantitative features shown by chaotic systems, that help us to use simple models to understand complex systems. Also, we understand the futility of long-term predictions in chaotic systems. Chaos has been applied usefully in almost every field of science and engineering. The theory of nonlinear dynamics and chaos has helped in describing, organising and quantifying complex behaviour. The notion developed are not only of theoretical and conceptual relevance but also have great application potentialities in technology.

1.6.4 Applications of Stochastic Resonance

Since its introduction, SR has become widely recognised as a paradigm for noise-induced effect in driven nonlinear dynamic systems. Most natural systems are intrinsically nonlinear and operate in noisy environments. Thus, SR finds application in physical, technological and biomedical contexts. The idea of SR is used to explain many physical and chemical processes and phenomena and for creating a number of technical devices. Applications of SR to sensory biology are most interesting. An example is the sensory process in the mechano-receptors of the crayfish which can detect an approaching predator using the periodic signal of water vibrations immersed in a background of water turbulence [73]. The neurophysiological applications have led to a new interdisciplinary field with inputs from biology, medical sciences and statistical physics and nonlinear dynamics.