CHAPTER 4

T- UNIFORMITY BY WEIL
ENTOURAGES

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CHAPTER 4

T- Uniformity by Weil Entourages

4.1 Introduction

Recently, a different approach to fuzzy uniformities, in terms of T-covers [30] was introduced which seems the most natural one.

In this Chapter, we investigate some properties of covering T- uniformity. The purpose of this Chapter is to present another characterization of fuzzy uniformities in the style of Weil that we call it T- Weil uniformity. We formulate and investigate a definition of entourage uniformity alternative to that one of Hutton. It is expressed in terms of coproduct of fuzzy spaces. We identify this new notion with the existing ones by proving the isomorphism between the respective categories.

4.2. Uniform Spaces

Uniformities on a set $X$ were introduced in the thirties by André Weil, in terms of subsets of $X \times X$ containing the diagonal

$$\Delta_X := \{(x, x) : x \in X\},$$
called “entourages” or “surroundings”.

Definition 4.2.1 [96]. Let $X$ be a set.

(a) A subset $E$ of $X \times X$ is called an entourage of $X$ if it contains the diagonal $\Delta_X$. 
(b) A uniformity on $X$ is a set $\mathcal{E}$ of entourages of $X$ such that

(UW1) $\mathcal{E}$ is a filter of the complete lattice $(W\ Ent(X), \subseteq)$ of all entourages of $X$;

(UW2) for each $E \in \mathcal{E}$ there exists $F \in \mathcal{E}$ such that the entourage $F \circ F := \{(x, y) \in X \times X \mid$ there is a $z \in X$ such that $(x, z), (z, y) \in F \}$ is contained in $E$;

(UW3) for every $E \in \mathcal{E}$ the set $E^{-1} := \{(x, y) \in X \times X \mid (y, x) \in E \}$ is also in $\mathcal{E}$.

The pair $(X, \mathcal{E})$ is then called a uniform space.

(c) A map $f : (X, \mathcal{E}) \to (X', \mathcal{E}')$, where $(X, \mathcal{E})$ and $(X', \mathcal{E}')$ are uniform spaces, is uniformly continuous if for every $E \in \mathcal{E}'$, $(f \times f)^{-1}(E) \in \mathcal{E}$.

A basis of a uniformity $\mathcal{E}$ is a subfamily $\mathcal{E}'$ of $\mathcal{E}$ such that $\uparrow \mathcal{E}' = \mathcal{E}$. A collection $\mathcal{E}'$ of entourages of $X$ is therefore a basis of some uniformity if and only if it is a filter basis of $(W\ Ent(X), \subseteq)$ satisfying (UW2) and the condition:

For every $E \in \mathcal{E}'$ there exists $F \in \mathcal{E}'$ such that $F^{-1} \subseteq E$.

**Example:** For $\epsilon > 0$, subsets of $\mathbb{R} \times \mathbb{R}$ of the type,

$$U_\epsilon = \{(\epsilon, \eta) : |\epsilon - \eta| < \epsilon\}$$

of $\mathbb{R} \times \mathbb{R}$
form a basis for Euclidean uniformity on \( \mathbb{R} \).

If \((X, \mathcal{E})\) is a uniform space, a topology \( \mathcal{T}_\mathcal{E} \) on \( X \) (the uniform topology) is defined as follows:

\[
A \in \mathcal{T}_\mathcal{E} \text{ if for every } x \in A \text{ there exists } E \in \mathcal{E} \text{ such that } E[x] = \{ y \in X | (x, y) \in E \} \subseteq A.
\]

For \( A, B \subseteq X \), we write \( A \triangleleft B \) if \( Eo(A \times A) \subseteq B \times B \) for some \( E \in \mathcal{E} \).

Since \( A \times A \subseteq Eo(A \times A) \), the order relation \( \triangleleft \) is stronger than the inclusion \( \subseteq \).

**Proposition 4.2.2** : [96] : Let \((X, \mathcal{E})\) be a uniform space. Then, for every \( A \in \mathcal{T}_\mathcal{E}, A = \bigcup \{ B \in \mathcal{T}_\mathcal{E} | B \triangleleft A \} \).

There is another equivalent axiomatization of the notion of uniformity, due to Tukey [96], in which the basic term is the one of “uniform cover” of \( X \):

**Definition 4.2.3** [95] : Let \( X \) be a set.

(a) A subset \( \mathcal{U} \) of \( \mathcal{P}(X) \) is a cover of \( X \) if \( \bigcup_{U \in \mathcal{U}} U = X \). For \( A \subseteq X \),

\[
st(A, \mathcal{U}) = \bigcup \{ U \in \mathcal{U} | U \cap A \neq \emptyset \}
\]

is called the star of \( A \) in \( \mathcal{U} \). A cover \( \mathcal{U} \) refines a cover \( \mathcal{V} \), and in this case one writes \( \mathcal{U} \leq \mathcal{V} \), if for each \( U \in \mathcal{U} \) there exists \( V \in \mathcal{V} \) such that \( U \subseteq V \).

(b) A covering uniformity on \( X \) is a set \( \mu \) of covers of \( X \) such that :

(U1) \( \mu \) is a filter of the preordered set \( (\text{Cov}(X), \leq) \) of all covers of \( X \);
(U2) for each $U \in \mu$ there is $V \in \mu$ such that the cover
\[
V^* := \{st(V, V) | V \in V\}
\]
refines $U$.

A basis of covering uniformity $\mu$ is a subfamily $\mu'$ of $\mu$ such that $\uparrow \mu' = \mu$.

Evidently, $\mu'$ is a basis for some covering uniformity if and only if it is a filter basis of $(Cov(X), \leq)$ such that, for every $U \in \mu'$, there exists $V \in \mu'$ satisfying $V^* \leq U$.

Example:

\[
X = \{a, b\} \quad P(X) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}
\]
\[
\mu = \{\{\emptyset, \{a\}, \{a, b\}\}, \{\emptyset, \{b\}, \{a, b\}\}, \{\emptyset, \{a\}, \{b\}, \{a, b\}\}\}.
\]

$\mu$ is a covering uniformity on $X$.

**Theorem 4.2.4**: [95]. The family $\mu$ of all uniform covers of a uniform space $(X, \mathcal{E})$, that is, the family of all covers refined by a cover of the form $\{E[x]\}$ for some $E \in \mathcal{E}$ and $x \in X$, is a filter basis for a covering uniformity on $X$.

Conversely, given any covering uniformity $\mu$ on a set $X$, the family of all sets
\[
\bigcup_{U \in \mu}(U \times U),
\]
for $U \in \mu$, is a basis for a uniformity on $X$, whose uniform covers are precisely the elements of $\mu$.

From a categorical point of view this means that there is a concrete isomorphism between the category of uniform spaces of Weil and the category
of uniform spaces of Tukey (which are concrete categories over the category of sets). Informally, this means that the description of these “structured sets”, although distinct, are essentially the same and we may substitute one structure for the other with no problem.

Thus the uniform covers also describe a uniformity and one may define a uniform space as a pair \((X, \mu)\) formed by a set \(X\) and a family \(\mu\) of covers of \(X\) satisfying axioms (U1) and (U2) of Definition 4.2.3.

With this language, the uniform topology induced by \((X, \mu)\) is the set of all \(A \subseteq X\) such that for every \(x \in A\) there exists \(U \in \mu\) satisfying \(st(x, U) \subseteq A\). Proposition 4.2.2 has now the following formulation.

**Proposition 4.2.5 [95]**: Let \((X, \mu)\) be a uniform space given in terms of covers and let \(T_\mu\) be the associated topology on \(X\). Then, for every \(A \in T_\mu\), \(A = \bigcup \{B \in T_\mu | B \mu \triangleleft A\}\), where \(B \mu \triangleleft A\) means that there is \(U \in \mu\) such that \(st(B, U) \subseteq A\).

A map \(f : (X, \mu) \rightarrow (X', \mu')\), where \((X, \mu)\) and \((X', \mu')\) are uniform spaces, is uniformly continuous if and only if, for every \(U \in \mu'\),

\[f^{-1}[U] := \{f^{-1}(U) | U \in \mathcal{U}\}\]

belongs to \(\mu\).
4.3 $\mathbb{T}$-Valued Spaces and $\mathbb{T}$-Valued Uniform Spaces

Let $\mathbb{T}$ be a complete lattice and let $\mathbb{T}_1 = \mathbb{T} \setminus \{1\}$. Recall that, for any set $X$, $\mathbb{T}^X$ with the natural order $U \leq V \equiv U(x) \leq V(x)$ for each $x \in X$ is a complete lattice (being a frame whenever $\mathbb{T}$ is itself a frame) : joins and meets are just defined pointwisely.

**Remark 4.3.1**: Recall that a (Chang-Gogun) fuzzy space we have defined in Chapter 1 is a pair $(X, L)$ consisting of a set $X$ and a family $L$ (the $\mathbb{T}$-valued topology on the set $X$) of mappings $X \to \mathbb{T}$, satisfying the three conditions (LFT1),(LFT2),(LFT3).

If $\mathbb{T}$ is a frame then the $\mathbb{T}$-valued topologies, being subframes of the frame $\mathbb{T}^X$, are frames as well.

A $\mathbb{T}$-valued continuous map $(X, L) \to (Y, M)$ is a map $f : X \to Y$ such that the correspondence $V \to V \cdot f$ maps $M$ into $L$. The resulting category will be denoted by $\mathbb{T}$-Top in this Chapter. Of course, when $\mathbb{T}$ is the frame $2$, then a $\mathbb{T}$-valued topological space is precisely a topological space and there is an isomorphism of categories between Top and $\mathbb{T}$-Top, via the characteristic functor.

If $(X, L)$ is a $\mathbb{T}$-valued topological space, the sets

$$w_t^L(U) = \{x \in X | U(x) > t\}, \quad (t \in \mathbb{T}_1, U \in L) \quad (4.1)$$
constitute a subbase for a topology \( \tau_L \) on \( X \). The space \((X, \tau_L)\) is the crisp modification of \((X, L)\) \([84,87]\). The correspondence \( (X, L) \to (X, \tau_L) \) defines a functor

\[
F_1 : \mathbb{T} - \text{Top} \to \text{Top}
\]

for each \( \mathbb{T} \)-valued continuous map \( f : (X, L) \to (Y, M) \), \( f : (X, \tau_L) \to (Y, \tau_M) \) is continuous, since \( V \cdot f \in L \) and

\[
f^{-1}(w_i^M(V)) = w_i^L(V \cdot f), \text{ for every } V \in M
\]

(4.2).

Recall also the (dual) adjunction between \( \mathbb{T} \)-\text{Top} and \( \text{Frm} \) \([84]\) : the contravariant functor

\[
O_\mathbb{T} : \mathbb{T} - \text{Top} \to \text{Frm}
\]

given by \( O_\mathbb{T}(X, L) = L \) and \( O_\mathbb{T}(f)(V) = V \cdot f \), has a right adjoint

\[
\Sigma_\mathbb{T} : \text{Frm} \to \mathbb{T} - \text{Top}
\]

defined by \( \Sigma_\mathbb{T}(L) = (\{p : L \to \mathbb{T} | p \in \text{Frm}\}, \{\hat{a} | a \in L\}) \), where \( \hat{a}(p) = p(a) \) and \( \Sigma_\mathbb{T}(h)(p) = p \cdot h \) for every \( h : L \to M \).

Here we present a different approach to fuzzy uniformities, in terms of \( \mathbb{T} \)-\text{covers} \([30]\) which seems to us the most natural one.

**Definition 4.3.2** Let \( \mathbb{1} \) denote the top element of \( \mathbb{T}^X \), that is \( \mathbb{1} : x \to 1 \). We say that

\[
U = (U_i : X \to \mathbb{T})_{i \in I}
\]
is a $T$-cover of $X$ if $\forall_i U_i = 1$. The set $T - Cov(X)$ of all $T$-covers of $X$ is preordered by $\mathcal{V} \leq \mathcal{U}$ (i.e. For each $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \leq U$).

Further let $\mathcal{U} \wedge \mathcal{V} := \{U \wedge V | U \in \mathcal{U}, V \in \mathcal{V}\}$.

**Proposition 4.3.3**[30] : Let $\mathbb{T}$ be a frame. Then, for every $T$-covers $\mathcal{U}$ and $\mathcal{V}$, $\mathcal{U} \wedge \mathcal{V}$ is again a $T$-cover and it is the meet of $\mathcal{U}$ and $\mathcal{V}$ in $(T - Cov(X), \leq)$.

**Proof** : It suffices to see that

$$\left( \bigvee_{U \in \mathcal{U}, V \in \mathcal{V}} (U \wedge V)(x) \right) = \bigvee_{U \in \mathcal{U}, V \in \mathcal{V}} (U(x) \wedge V(x)) = \left( \bigvee_{U \in \mathcal{U}} U(x) \right) \wedge \left( \bigvee_{V \in \mathcal{V}} V(x) \right) = 1.$$  

The second assertion follows immediately.

**Note** : In this Chapter $\mathbb{T}$ will always denote a frame. For each $U \in \mathcal{U}$, let

$$st(U, \mathcal{U}) := \forall \{V \in \mathcal{U} | V \wedge U \neq 0\}$$

and

$$\mathcal{U}^* := \{st(U, \mathcal{U}) | U \in \mathcal{U}\}.$$  

Clearly $\mathcal{U} \leq \mathcal{U}^*$, since $U \leq st(U, \mathcal{U})$ for every $U \in \mathcal{U}$. Therefore $\mathcal{U}^*$ is a $T$-cover whenever $\mathcal{U}$ is a $T$-cover.

We say that a pair $(X, \mu)$ consisting of a set $X$ and a non-empty family of $T$-covers of $X$ is a covering $T$-uniform space[30] whenever the following conditions are satisfied:

(CU1) $\mathcal{U} \leq \mathcal{V}, \mathcal{U} \in \mu \Rightarrow \mathcal{V} \in \mu$.

(CU2) For every $\mathcal{U}, \mathcal{V} \in \mu, \mathcal{U} \wedge \mathcal{V} \in \mu$.

(CU3) For each $\mathcal{U} \in \mu$ there exists $\mathcal{V} \in \mu$ such that $\mathcal{V}^* \leq \mathcal{U}$.  

A non-void family $\mu$ is said to be a weak covering $T$-uniform space on $L$ if there hold $CU_1$, $CU_2$ and ($CU_3'$).

($CU_3'$): for each $U \in \mu$ there is a $V \in \mu$ such that
\[
V^{(2)} = \{ U \lor U' | U, U' \in V, U \land U' \neq 0 \} \leq U.
\]

A non-void family $\mu$ of $T$-covers of $X$ is said to be a $T$-valued uniformly basis resp. (a weak $T$-valued uniformity basis) if it satisfies ($CU_3$) resp ($CU_3'$).

A map $f : (X, \mu) \rightarrow (Y, v)$ is uniform homomorphism if, for every $V := (V_i)_I \in v, f^{-1}[V] := (V_i \cdot f)_I \in \mu$. The resulting category will be denoted by $T$-Unif. Of course, for $T = 2$, this is just the category of (covering) uniform spaces of Tukey [95].

Now let $(X, \mu) \in T$ - Unif and define
\[
L_\mu := \{ U \in T^X | U(x) \neq 0 \Rightarrow \exists \mathcal{U} \in \mu : \text{st}(\hat{x}, \mathcal{U}) \leq U \},
\]
where $\hat{x} : X \rightarrow T$ is defined by $\hat{x}(y) = 1$ if $y = x$ and $\hat{x}(y) = 0$ otherwise. Note that, when $T = 2$, $L_\mu$ is just the crisp topology $T_\mu$ induced by the (classical) uniformity $\mu$ on $X$.

**Proposition 4.3.4[30]**: $L_\mu$ is a subframe of $T^X$.

**Proof**: Clearly, $0, 1$ belong to $L_\mu$. Let $U, V \in L_\mu$. Then $(U \land V)(x) \neq 0$ if and only if $U(x) \land V(x) \neq 0$ which implies $U(x) \neq 0$ and $V(x) \neq 0$. Thus there exist $\mathcal{U}, \mathcal{V} \in \mu$ such that $\text{st}(\hat{x}, \mathcal{U}) \leq U$ and $\text{st}(\hat{x}, \mathcal{V}) \leq V$. Take $W := \mathcal{U} \land \mathcal{V} \in \mu$. Then
\[
\text{st}(\hat{x}, \mathcal{U}) \leq \text{st}(\hat{x}, \mathcal{U}) \land \text{st}(\hat{x}, \mathcal{V}) \leq U \land V
\]
Finally, let \((U_i)_{i \in I} \subseteq \mu\). Then if \(\bigvee_{i} U_i(x) \neq 0\), there exists \(i \in I\) for which \(U_i(x) \neq 0\), which implies the existence of \(\mathcal{U} \in \mu\) satisfying \(st(\hat{x}, \mathcal{U}) \leq U_i \leq \bigvee_{i} U_i\). This shows that \(\bigvee_{i} U_i \in \mu\).

In conclusion, \((X, L_{\mu}) \in \mathbb{T}\text{-Top} \).

**Remark 4.3.5**[30] : Let \(f : X \to Y\) and \((V_i)_{i \in I} \subseteq \mathbb{T}^Y\). Then

\[(\bigvee_{i} V_i) \cdot f = \bigvee_{i} (V_i \cdot f)\]

Indeed,

\[(\bigvee_{i} V_i \cdot f)(x) = (\bigvee_{i} V_i)(f(x)) = \bigvee_{i} (V_i(f(x))) = \bigvee_{i} (V_i \cdot f)(x).

**Lemma 4.3.6**[30] : Let \(f : X \to Y\) and \(\mathcal{V} = (V_i)_{i \in I} \subseteq \mathbb{T}^Y\). Then, for every \(x \in X\),

\[st(\hat{x}, f^{-1}(\mathcal{V})) = st(f(\hat{x}), \mathcal{V}) \cdot f\]

**Proof** : We have

\[st(\hat{x}, f^{-1}(\mathcal{V})) = \bigvee\{V_i \cdot f | V_i \cdot f \wedge \hat{x} \neq 0\} = \bigvee\{V_i \cdot f | V_i(f(x)) \neq 0\} \cdot f.

On the other hand,

\[st(f(\hat{x}), \mathcal{V}) = \bigvee\{V_i | V_i \wedge f(\hat{x}) \neq 0\} = \bigvee\{V_i | V_i(f(x)) \neq 0\} \cdot f.

The conclusion follows immediately from the previous remark.

**Proposition 4.3.7**[30] : For any \(f : (X, \mu) \to (Y, v)\) in \(\mathbb{T}\text{-Unif}\), \(f : (X, L_{\mu}) \to (Y, L_v)\) is a morphism of \(\mathbb{T}\text{-top}\).

**Proof** : Let \(V \in L_v\). Then \(V \cdot f \in L_{\mu}\), that is, \((V \cdot f)(x) \neq 0\) implies the existence of \(\mathcal{U} \in \mu\) satisfying \(st(\hat{x}, \mathcal{U}) \leq V \cdot f\).
Indeed, since $V \in L_v, V(f(x)) \neq \emptyset$ implies the existence of $V \in v$ satisfying $st(f(x), V) \leq V$; then

$$(st(f(x), V) \cdot f \leq V \cdot f;$$

take $\mathcal{U} := f^{-1}[V] \in \mu$; by the lemma

$$st(\hat{x}, \mathcal{U}) = st(f(x), V) \cdot f \leq V \cdot f.$$

thus the correspondence

$$(X, \mu) \in \mathcal{T} - Unif \rightarrow (X, L_\mu) \in \mathcal{T} - Top$$

defines a functor $T_3 : \mathcal{T} - Unif \rightarrow \mathcal{T} - Top$ such that the diagram

$$\begin{array}{ccc}
\mathcal{T} - Unif & \xrightarrow{T_3} & \mathcal{T} - Top \\
E_3 \uparrow & & \uparrow E_1 \\
\text{Unif} & \xrightarrow{T_1} & \text{Top}
\end{array}$$

commutes (where $E_3$ denotes the uniform version of the embedding functor $E_1 : \mathcal{T} \rightarrow \mathcal{T} - Top$). This shows that the notion of $\mathcal{T}$-valued uniform space relates to a uniform space in a way similar to that in which a $\mathcal{T}$-valued space is related to a topological space.

**The Crisp Modification of a $\mathcal{T}$-Valued Uniform Space**

Let $\mathcal{T}$ be a linearly ordered complete lattice and let $(X, \mu)$ be a $\mathcal{T}$-valued uniform space. For each $\mathcal{U} \in \mu$ and $t \in \mathcal{T}_1$ let

$$w_t^X(\mathcal{U}) = \{w_t^X(U) | U \in \mathcal{U}\},$$

where $w_t^X(U) = \{x \in X | U(x) > t\}$. Further, for $t, t'$ define

$$w_t^X(\mathcal{U}) \land w_{t'}^X(\mathcal{V}) = \{w_t^X(U) \cap w_{t'}^X(V) | U \in \mathcal{U}, V \in \mathcal{V}\}.$$
The sets \( w_t^X(\mathcal{U}) \) satisfy the following properties:

**Proposition 4.3.8** [30] : For every \( \mathcal{U}, \mathcal{V} \in \mu \) and \( t, t' \in T_1 \), we have

1. \( w_t^X(\mathcal{U}) \) is a cover of \( X \).
2. \( w_t^X(\mathcal{U}) \land w_{t'}^X(\mathcal{V}) = w_{t \land t'}^X(\mathcal{U} \land \mathcal{V}) \).
3. If \( \mathcal{V}^* \leq \mathcal{U} \) then \( w_t(\mathcal{V})^* \leq w_t(\mathcal{U}) \).

**Proof :**

1. Since \( \bigvee_{U \in \mathcal{U}} U = 1 \), that is, \( \bigvee_{U \in \mathcal{U}} (U(x)) = 1 \), then, by the linearity of \( T \), for each \( t \in T_1 \) and each \( x \in X \) there exists \( U \in \mathcal{U} \) such that \( U(x) > t \) for every \( x \in X \). Consequently, \( \bigcup_{U \in \mathcal{U}} \{ x \in X | U(x) > t \} = X \), that is, \( \bigcup_{U \in \mathcal{U}} w_t(U) = X \).

2. We have

\[
\begin{align*}
w_t(U) \cap w_{t'}(V) &= \{ x \in X | U(x) > t \text{ and } V(x) > t' \} \\
 &= \{ x \in X | U(x) \land V(x) > t \land t' \} \\
 &= \{ x \in X | (U \land V)(x) > t \land t' \},
\end{align*}
\]

thus \( w_t(U) \cap w_{t'}(V) = w_{t \land t'}(U \land V) \).

3. Let

\[
st(w_t(\mathcal{V}), w_t(\mathcal{V})) \in w_t(\mathcal{V})^* = \{ st(w_t(\mathcal{V}), w_t(\mathcal{V})) | V \in \mathcal{V} \}
\]

by hypothesis, there exists \( U \in \mathcal{U} \) satisfying \( st(V, \mathcal{V}) \leq U \).

Then \( st(w_t(\mathcal{V}), w_t(\mathcal{V})) \leq w_t(U) \). Indeed:

\[
st(w_t(\mathcal{V}), w_t(\mathcal{V})) = \cup \{ w_t(V') | V' \in \mathcal{V}, w_t(V') \cap w_t(V) \neq \emptyset \}.
\]
Let $x \in w_t(V')$ for some $V' \in \mathcal{V}$ satisfying $w_t(V') \cap w_t(V) \neq \emptyset$. Then $V'(x) > t$ and there exists $y \in X$ such that $V'(y) > t$ and $V(y) > t$. This implies $(V \wedge V')(y) > t$, so $V' \wedge V \neq \emptyset$. Therefore $V' \leq st(V, V) \leq U$ and $t < V'(x) \leq U(x)$, which ensures that $x \in w_t(U)$.

Let $v_\mu = \{w_t^X(U) | U \in \mu, t \in T_1\}$. It follows immediately from Proposition 4.3.8 that

**Corollary 4.3.9** [30]: $(X, v_\mu)$ is a uniform space.

**Proposition 4.3.10** [30]: For any $f : (X, \mu) \rightarrow (Y, v)$ in $T$-Unif, $f : (X, v_\mu) \rightarrow (Y, v_\nu)$ is a morphism of Unif.

**Proof**: Straightforward.

Thus we have a functor $F_3 : T$-Unif $\rightarrow$ Unif such that $F_3 \cdot E_3 = 1_{\text{Unif}}$.

**Proposition 4.3.11** [30]: $T_1 \cdot F_3 = F_1 \cdot T_3$.

In conclusion, both squares in the following diagram commute:

$$
\begin{array}{ccc}
T \text{-Unif} & \rightarrow & T \text{-Top} \\
\downarrow E_3 & & \downarrow F_3 \\
\downarrow F_3 & & \downarrow F_1 \\
\text{Unif} & \rightarrow & \text{Top} \\
\uparrow T_1 & & \\
\end{array}
$$

**Definition 4.3.12**: For $\mu \subset T - \text{cov}(X)$ put

$$\tilde{\mu} = \{U | \exists U_1, \ldots, U_k \in \mu, U_1 \wedge U_2 \wedge \cdots \wedge U_k < U\}.$$

**Lemma 4.3.13**: We have $(U_1 \wedge \cdots \wedge U_n)^{(2)} < U_1^{(2)} \wedge \cdots \wedge U_n^{(2)}$, $(U_1 \wedge \cdots \wedge U_n)^* < U_1^* \wedge \cdots \wedge U_n^*$.

**Proof**: Obviously, it suffices to prove the statement for $n = 2$. 
(I) Let $U_1, V_1 \in U_1, U_2, V_2 \in U_2$ we want to prove that $(U_1 \wedge U_2)_{(2)} < U_1^{(2)} \wedge U_2^{(2)}, (U_1 \wedge U_2) \wedge (V_1 \wedge V_2) \neq 0$ then $U_1 \wedge V_1 \neq 0 \neq U_2 \wedge V_2$ and hence $U_i \vee V_i \in U_i^{(2)}$ we have

$$(U_1 \wedge U_2) \vee (V_1 \wedge V_2) \leq (U_1 \vee V_1) \wedge (U_2 \vee V_2).$$

(II) $(U_1 \wedge U_2)^* < U_1^* \wedge U_2^*$

$$(U_1 \wedge U_2)^* = \{st(U_1 \wedge U_2, U_1 \wedge U_2) | U_1 \wedge U_2 \in (U_1 \wedge U_2)\}$$

$st(U_1 \wedge U_2, U_1 \wedge U_2) \leq st(U_1 \wedge U_2, U_1) \wedge st(U_1 \wedge U_2, U_2) \leq st(U_1, U_2) \wedge st(U_2, U_2).$

**Theorem 4.3.14**: If $\mu$ is a $\mathbb{T}$-valued uniform basis, $\tilde{\mu}$ is a $\mathbb{T}$-valued uniformity. If $\mu$ is a weak $\mathbb{T}$-valued uniform basis, $\tilde{\mu}$ is a weak $\mathbb{T}$-valued uniformity.

**Proof**: the condition $(CU_1, CU_2)$ are obviously satisfied.

$(CU_3, CU_4)$: Let $U_1 \wedge U_2 \cdots \wedge U_k < U, U_i \in \mu$. Choose $\mathcal{V}_i \in \mu$ such that $\mathcal{V}_i^* < U_i$, resp. $\mathcal{V}_i^{(2)} < U_i$. Put $\mathcal{V} = \mathcal{V}_1 \wedge \mathcal{V}_2 \wedge \cdots \wedge \mathcal{V}_k$. By 4.2.12 we have $\mathcal{V}^* < U$ resp $\mathcal{V}^{(2)} < U$.

**Definition 4.3.15**: For a complemented $\alpha \in \mathbb{T}^X$ and $\mathcal{U} \in \mathbb{T} - \text{cov}(X)$ put $U_0 \alpha = \{U_i \wedge \alpha | U_i \in \mathcal{U}\} \vee \{U_i \wedge \overline{\alpha} | U_i \in \mathcal{U}\}$, for $\mu \subset \mathbb{T} - \text{cov}(X)$ define $\mu^0 = \{U_0 \alpha | U \in \mu, \alpha \text{ complemented}\}$.

**Proposition 4.3.16**: If $\mu$ is a $\mathbb{T}$-valued uniform basis (resp weak $\mathbb{T}$-valued uniform basis), $\mu^0$ is a $\mathbb{T}$-valued uniform basis (resp weak $\mathbb{T}$-valued uniform basis).
\textbf{Proof}: It is trivial \((U\alpha)^* < U^*o\alpha \) resp \((U\alpha)^{(2)} < U^{(2)}o\alpha \) for \( U \in \mu \), let \( U\alpha \in \mu^0 \) then choose \( V \in \mu \) such that \( V^* < U \) resp \( V^{(2)} < U \). We consider \( U\alpha \in \mu^0 \) such that \((V\alpha)^* < V^*o\alpha < U\alpha \) resp \((V\alpha)^{(2)} < V^{(2)}o\alpha < U\alpha \).
4.4 T-Weil Uniform Spaces

Definition 4.4.1 (Binary coproducts of frames)

Let $L_1$ and $L_2$ be frames. Recall ([15] or [51]) that the coproduct of frames $L_1$ and $L_2$

$$L_1 \xrightarrow{u_{L_1}} L_1 \oplus L_2 \xleftarrow{u_{L_2}} L_2$$

can be constructed as follows: take the Cartesian product $L_1 \times L_2$ with the usual order. One obtains $L_1 \oplus L_2$ as the frame $\mathcal{D}(L_1 \times L_2)/R$ where $R$ consists of all pairs of the type

$$\downarrow \{(x,0), \emptyset\},$$

$$\downarrow \{(0,y), \emptyset\},$$

$$\downarrow \{(\vee S, y) \cup \downarrow \{(s,y)\}_{s \in S}\}$$

and

$$\downarrow \{(x,\vee S) \cup \downarrow \{(x,s)\}_{s \in S}\}.$$  

Equivalently, defining a $C$-ideal of $L_1 \times L_2$ as a down-set $A \subseteq L_1 \times L_2$ satisfying

$$\{x\} \times S \subseteq A \Rightarrow (x, \vee S) \in A$$

and

$$S \times \{y\} \subseteq A \Rightarrow (\vee S, y) \in A,$$

since the intersection of $C$-ideals is again a $C$-ideal, the set of all $C$-ideals of $L_1 \times L_2$ is a frame in which

$$\bigwedge_{i \in I} A_i = \bigcap_{i \in I} A_i$$
and
\[ \bigvee_{i \in I} A_i = \cap \{ B \mid B \text{ is a } C\text{-ideal and } \bigcup_{i \in I} A_i \subseteq B \}; \]
this is the frame \( L_1 \oplus L_2 \). Observe that the case \( S = \emptyset \) implies that every \( C \)-ideal contains the \( C \)-ideal \( \downarrow (1,0) \cup \downarrow (0,1) \), which we shall denote by \( O_{L_1 \oplus L_2} \) (or just by \( O \) whenever there is no ambiguity). This is the zero of \( L_1 \oplus L_2 \).

Obviously, each \( \downarrow (x,y) \cup O \) is a \( C \)-ideal. It is denoted by \( x \oplus y \). Finally, put \( u_{L_1}(x) = x \oplus 1 \) and \( u_{L_2}(y) = 1 \oplus y \). The following clear facts are useful:

- For every \( E \in L_1 \oplus L_2 \), \( E = \bigvee \{ x \oplus y \mid (x,y) \in E \} \), and so the \( C \)-ideals of the type \( x \oplus y \) generate by joins the frame \( L_1 \oplus L_2 \);
- \( x \oplus y = O \) if and only if \( x = 0 \) or \( y = 0 \);
- \( \bigvee \{ x \oplus s \mid s \in S \} = x \oplus (\bigvee S) \) and \( \bigvee \{ s \oplus y \mid s \in S \} = (\bigvee S) \oplus y \);
- \( \bigcap \{ s \oplus t \mid s \in S, t \in T \} = (\bigwedge S) \oplus (\bigwedge T) \);
- \( x \leq y \) and \( z \leq w \) imply \( x \oplus z \subseteq y \oplus w \);
- \( O \neq x \oplus y \subseteq z \oplus w \) implies \( x \leq z \) and \( y \leq w \).

Note that \( L_1 \oplus L_2 \) is isomorphic to \( L_1 \) : consider the coproduct diagram

\[ L_1 \xrightarrow{1} L_1 \xleftarrow{\sigma} 2 \]

where \( \sigma \) is the unique morphism \( 2 \to L_1 \). Under this isomorphism the element \( x \oplus 1 \) is identified with \( x \) and \( x \oplus 0 \) with \( 0 \).
For any frame homomorphism \( f_i : L_i \to L'_i \) \((i \in \{1, 2\})\), we write \( f_1 \oplus f_2 \) for the unique morphism from \( L_1 \oplus L_2 \) to \( L'_1 \oplus L'_2 \) that makes the following diagram commutative:

\[
\begin{array}{c}
L_1 \xrightarrow{u_{L_1}} L_1 \oplus L_2 \xleftarrow{u_{L_2}} L_2 \\
\downarrow f_1 \quad \quad \quad \downarrow f_2 \\
L'_1 \xrightarrow{u_{L'_1}} L'_1 \oplus L'_2 \xleftarrow{u_{L'_2}} L'_2
\end{array}
\]

Obviously,

\[
(f_1 \oplus f_2)(\bigvee_{\gamma \in \Gamma}(x_{\gamma} \oplus y_{\gamma})) = \bigvee_{\gamma \in \Gamma}(f_1(x_{\gamma}) \oplus f_2(y_{\gamma})).
\]

Consider the cartesian product \( L \times L \) with the usual order. If \( A \) and \( B \) are down-sets of \( L \times L \), we denote by \( A \cdot B \) the set

\[
\{(x, y) \in L \times L | \text{there is } z \in L \setminus \{0\} \text{ such that } (x, z) \in A \text{ and } (z, y) \in B\}
\]

and by \( \text{AoB} \) the \( C \)-ideal generated by \( A \cdot B \).

The operation \( \circ \) (which in general is not commutative) is associative and so bracketing is unnecessary for repeated compositions such as

\[
A^n = AoAo \cdots oA \quad (n \text{ factors}).
\]

Further we have:

For any \( A \in D(L \times L) \), \( A^{-1} := \{(x, y) \in L \times L | (y, x) \in A\} \) and \( A \) is symmetric if \( A = A^{-1} \). The element

\[
\bigvee\{y \in L | (y, y) \in A, y \land x \neq 0\}
\]
will be denoted by $st(x, A)$.

Let $e$ be a map from $L$ to $L$ and $\mu \in L$. $\mu$ is $e$-small if $\mu < e(\lambda)$ whenever $\mu \wedge \lambda \neq 0$.

In the following proposition we list some obvious properties of these operators.

**Proposition 4.4.2** [80]: Let $x, y \in L$ and $A, B \in D(L \times L)$. Then:

(a) $(x \oplus y)^{-1} = y \oplus x$;
(b) $(AoB)^{-1} = B^{-1}oA^{-1}$;
(c) $Ao\emptyset = \emptyset oA = \emptyset$;
(d) $st(x, A) \oplus y \subseteq Ao(x \oplus y)$ and $x \oplus st(y, A) \subseteq (x \oplus y)oA$.

The map

$$k_0 : D(L \times L) \rightarrow D(L \times L)$$

$$A \rightarrow \{(x, \vee S)|(\{x\} \times S \subseteq A) \cup \{\vee S, y)|S \times \{y\} \subseteq A\}$$

for all $A, B \in D(L \times L), A \subseteq k_0(A) \cap B \subseteq k_0(A \cap B)$ and $k_0(A) \subseteq k_0(B)$ whenever $A \subseteq B$. Consequently

$$Fix(k_0) := \{A \in D(L \times L)|k_0(A) = A\} = L \oplus L$$

is a closure system, and the associated closure operator is then given by

$$k(A) = \cap\{B \in L \oplus L|A \subseteq B\},$$

which is the $C$-ideal generated by $A$. The following technical lemma will play a crucial role in the sequel.
Lemma 4.4.3[80] : Let \( A, B \in \mathcal{D}(L \times L) \). Then

(a) \( k(A)k(B) = AoB \);

(b) \( k(A^{-1}) = k(A)^{-1} \).

Note : Since \( T^X_i, (i \in \Gamma) \) is a frame \( 0 \) and \( 1 \) denote the least and the greatest element in \( T^X_i \) we can define coproducts of \( T^X_i \) similar to coproducts of frames.

It is easy to check that the above properties for coproducts of frames satisfies for coproducts of \( T^X_i, (i \in \Gamma) \).

Definition 4.4.4 (Binary coproducts of \( T^X_i, (i \in \{1, 2\}) \)) : The coproduct of \( T^X_1 \) and \( T^X_2 \) is a \( C \)-ideal of \( T^X_1 \times T^X_2 \) as a down-set \( A \subseteq T^X_1 \times T^X_2 \) satisfying

\[
\{ f \} \times S \subseteq A \Rightarrow (f, \vee S) \in A
\]

and

\[
S \times \{ g \} \subseteq A \Rightarrow (\vee S, g) \in A.
\]

the set of all \( C \)-ideals of \( T^X_1 \times T^X_2 \) is \( T^X_1 \oplus T^X_2 \).

For any frame homomorphism \( \bar{f}_i : T^X_i \to T^X_i', (i \in \{1, 2\}) \), \( \bar{f}_1 \oplus \bar{f}_2 \) for the unique morphism from \( T^X_1 \oplus T^X_2 \) to \( T^X_1' \oplus T^X_2' \) define as follow:

\[
\bar{f}_1 \oplus \bar{f}_2 \left( \bigvee_{\gamma \in \Gamma} (f_\gamma \oplus g_\gamma) \right) = \bigvee_{\gamma \in \Gamma} (\bar{f}_1(f_\gamma) \oplus \bar{f}_1(g_\gamma)).
\]

Definition 4.4.5 : \( E \in T^X \oplus T^X \) is \( T \)-Weil entourage of \( T^X \) if and only if \( \{ \mu \in T^X | (\mu, \mu) \in E \} \) is a cover of \( T^X \). That is \( \vee \{ \mu \in T^X | (\mu, \mu) \in E \} = 1 \).

The collection \( T - W \ Ent(X) \) of all \( T \)-Weil entourage of \( T^X \) may be partially ordered by inclusion.
Definition 4.4.6: We define the composition of $T$-Weil entourage as follows:

$$EoF = \bigvee \{ f \oplus g \mid \exists h \in T^X \setminus \emptyset, (f, h) \in E, (h, g) \in F \}.$$ 

The inverse of a $T$-Weil entourage $E$ has the natural definition $E^{-1} = \{(g, f) \mid (f, g) \in E\}$.

The element $\bigvee \{ g \in T^X \mid (g, g) \in E, f \wedge g \neq 0 \}$ will be denoted by $st(f, E)$.

We also consider a new partial order in $T^X$, induced by a family $E$ of $T$-weil entourages:

$$g \overset{E}{\Delta} f \text{ (} g \text{ is } E \text{- strongly below } f \text{) if there is } E \in E \text{ such that } Eo(g \oplus g) \subseteq f \oplus f.$$ 

When $E$ is symmetric ($E \in E$ implies $E^{-1} \in E$) this is equivalent to saying there is $E \in E$ such that $(f \oplus f)oE \subseteq g \oplus g$.

The following theorem for fuzzy set theory is similar to theorem for frames proved in [80].

Proposition 4.4.7: Let $E$ be a $T$-Weil entourage. Then

(a) for any $f \in T^X$, $f \leq st(f, E)$

(b) $E^n \subseteq E^{n+1}$ for every natural $n$.

(c) For any down set $A$ of $T^X \times T^X$, $A \subseteq (EoA) \cap (AoE)$.

(d) for any $f \in T^X$, $st(st(f, F), F) \leq st(f, F^2)$.

Proof:

(a) Consider $f \in T^X$, we have

$$f = f \wedge \bigvee \{ g \in T^X \mid (g, g) \in E \} = \bigvee \{ f \wedge g \mid (g, g) \in E, f \wedge g \neq \emptyset \} \leq st(f, E).$$
(b) It suffices to prove that $E \subseteq E^2$. Consider $(f, g) \in E$. By (a) $g \leq st(g, E)$, it is trivial $f \oplus st(g, E) \subseteq (f \oplus g) o E \subseteq E^2$. Consequently, $(f, g) \in E^2$.

(c) Let $(f, g) \in A$. The case $f = 0$ or $g = 0$ are trivial. If $f, g \neq 0$, since $f = \vee\{f \land e|(e, e) \in E, f \land e \neq 0\}$ and for any $(e, e) \in E$ with $f \land e \neq 0, (e, g) \in E o A$, we have, by definition of $C$-ideal, that $(f, g) \in E o A$. Similarly $A \subseteq A o E$.

(d) we observe $st(st(\mu, F), F) \leq st(\mu, F^2)$

$$st(st(\mu, F), F) = \vee\{\lambda \in \mathbb{T}^X/((\lambda, \lambda) \in F, \lambda \land st(\mu, F) \neq \emptyset\}.$$ Consider $\lambda \in \mathbb{T}^X$ with $(\lambda, \lambda) \in F$ and $\lambda \land st(\mu, F) \neq \emptyset$. Then there is $\gamma \in \mathbb{T}^X$ such that $(\gamma, \gamma) \in F, (\gamma \land \mu) \neq \emptyset$ and $(\gamma \land \lambda) \neq \emptyset$; therefore $(\lambda, \lambda \land \gamma) \in F$ and $(\lambda \land \gamma, \gamma) \in F$, thus $(\lambda, \gamma) \in F^2$, similarly $(\gamma, \lambda) \in F^2$. Also $(\lambda, \lambda), (\gamma, \gamma) \in F^2$. But $F^2$ is a $C$-ideal so $(\lambda \lor \gamma, \lambda \lor \gamma) \in F^2$. In conclusion $(\lambda \lor \gamma, \lambda \lor \gamma) \in F^2$ and $(\lambda \lor \gamma) \land \mu \geq \gamma \land \mu \neq \emptyset$ hence $\lambda \leq st(\mu, F^2)$ and $st(st(\mu, F), F) \leq st(\mu, F^2)$.

**Definition 4.4.8** : Let $X$ be a nonempty set and $\mathcal{E} \subset \mathbb{T} w Ent(X)$. We say $(X, \mathcal{E})$ is a $\mathbb{T}$-Weil quasi uniformity on $X$ if it satisfies the following conditions:

1. (T–WQU1) $\mathcal{E}$ is a filter of $(\mathbb{T} – W Ent(X), \subseteq)$,
2. (T–WQU2) For each $E \in \mathcal{E}$ there is $F \in \mathcal{E}$ such that $FoF \subseteq E$.

The pair $(X, \mathcal{E})$ is said to be a $\mathbb{T}$-Weil quasi-uniform space.

A $\mathbb{T}$-Weil quasi uniform space $(X, \mathcal{E})$ is called a $\mathbb{T}$-Weil uniform space if it satisfies

1. (T–WU3) for any $E \in \mathcal{E}$, $E^{-1}$ is also in $\mathcal{E}$.
It is useful to note that the symmetric $T$-Weil entourages $E$ of $\mathcal{E}$ form a basis for $\mathcal{E}$. In fact, if $E \in \mathcal{E}$ then $E^{-1} \in \mathcal{E}$ so $E \cap E^{-1}$ is a symmetric $T$-Weil entourage of $\mathcal{E}$ contained in $E$.

**Example:** A prediameter on $L^X$ is a map

$$d : L^X \to [0, +\infty]$$

satisfying

1. $d(0) = 0$
2. $d(p^a_x) \leq d(p^b_y)$ if $p^a_x \leq p^b_y$, where $p^a_x = \begin{cases} 1 & a = x \\ 0 & a \neq x \end{cases}$
3. $d(p^a_x \lor p^b_y) \leq d(p^a_x) + d(p^b_y)$ if $p^a_x \land p^b_y \neq 0$.

For any real $\epsilon > 0$ let $E_\epsilon = \cup \{p^a_x \oplus p^b_y | d(p^a_x) < \epsilon\}$. The family $\mathcal{E} = \{E_\epsilon | \epsilon > 0\}$ is a $T$-Weil uniform space.

**Definition 4.4.9:** Let $(X, \mathcal{E}), (X', \mathcal{E}')$ be two $T$-Weil uniform spaces. A mapping $f : X \to X'$ is said to be uniformly homomorphic if $f \oplus \overline{f}(E) \in \mathcal{E}'$ whenever $E \in \mathcal{E}$.

We will denote by $T$-W Unif the category whose objects are $T$-Weil uniform spaces and morphisms are uniformly homomorphism mappings.

Now we define the Fuzzy topology generated by a $T$-Weil quasi uniformity.

**Theorem 4.4.10:** Let $(X, \mathcal{E})$ be a $T$-Weil quasi uniform space. Mapping $i : T^X \to T^X$ be defined as follows:

$$\forall A \in T^X \quad i(A) = \lor \{C \in T^X | \exists E \in \mathcal{E}, \; st(C, E) \leq A\}.$$ 

Then $i$ is an interior operator on $T^X$. 
Proof (I01) Since \( st(C, E) \leq 1 \) for every \( E \in \mathcal{E} \), so \( i(1) = 1 \).

(I02) \( i(A) = \bigvee \{ C \in T^X | \exists E \in \mathcal{E}, st(C, E) \leq A \} \) since \( C \leq st(C, E) \) for all \( C \in T^X \), \( i(A) \leq A \).

(I03) We need to prove \( i(A) \wedge i(B) \leq i(A \wedge B) \), for arbitrary \( A, B \in T^X \). In fact, since for arbitrary \( E, F \in \mathcal{E} \) and arbitrary \( A, B, C, D \in T^X \) such that \( st(C, E) \leq A \) and \( st(D, F) \leq B \), we have \( st(C \wedge D, E \cap F) \leq A \wedge B \). So \( i(A) \wedge i(B) = \bigvee \{ C \wedge D \in T^X : \exists E, F \in \mathcal{E}, st(C, E) \leq A, st(D, E) \leq B \} \leq \bigvee \{ C \wedge D \in T^X : \exists E, F \in \mathcal{E}, st(C \wedge D, E \cap F) \leq A \wedge B \} = i(A \wedge B) \).

(I04) By (I02) we have \( i(i(A)) \leq i(A) \). We want to show that \( i(A) \leq i(i(A)) \). Let \( C \in T^X, E \in \mathcal{E}, st(C, E) \leq A \). By (WQE2) \( \exists F \in \mathcal{E}, FoF \subset E \). Then \( st(C, FoF) < st(C, E) \) and, by Proposition 4.4.7(d) we have \( st(st(C, F), F) < st(C, FoF) \leq A \). Then \( st(C, F) \leq i(A), C \leq i(i(A)) \) and, so \( i(A) = \bigvee \{ C \in L^X | \exists E \in \mathcal{E}, st(C, E) \leq A \} \leq i(i(A)) \).

Definition 4.4.11: Let \( \mathcal{E} \) be an \( T \)-Weil quasi uniformity on \( X \), the interior operator defined in 4.4.10 is called the interior operator on \( T^X \) generated by \( T \)-Weil quasi uniformity \( \mathcal{E} \). The \( T \)-fuzzy space generated by the \( T \)-Weil quasi-uniformity \( \mathcal{E} \), denoted by \( \mathcal{E} \), \( (X, \mathcal{E}) \) is called the \( T \)-Top corresponding to \( X, \mathcal{E} \).

Theorem 4.4.12: Let \( (X, \mathcal{E}) \) be a \( T \)-Weil quasi-uniform space, mapping \( c : T^X \to T^X \) be defined as \( \forall A \in T^X, c(A) = \bigwedge \{ st(A, E) | E \in \mathcal{E} \} \).
Then $c$ is a closure operator on $T^X$.

**Proof**: (C01) Since for every $E \in \mathcal{E}$, $st(0,E) = 0$, so $c(0) = 0$.

(C02) Since $A \leq st(A,E)$ for every $E \in \mathcal{E}$ then $A \leq c(A)$.

(C03) We need only to prove $c(A \lor B) \leq c(A) \lor c(B)$ for arbitrary $A, B \in T^X$. It is trivial,

$$st(A \lor B, E_1 \cap E_2) \leq st(A, E_1) \lor st(B, E_2).$$

Suppose $e \in T^X$ such that $e \not\leq c(A) \lor c(B)$, then $e \not\leq c(A)$, $e \not\leq c(B)$, then there exist $E, F \in \mathcal{E}$ such that $e \not\leq st(A, E), e \not\leq st(B, F)$. Then $e \not\leq st(A \lor B, E \cap F)$, $e \not\leq c(A \lor B)$, so $c(A \lor B) \leq c(A) \lor c(B)$.

(C04) For every $A \in T^X$, we have $st(st(C, F), F)) < st(C, FoF)$. For every $A \in T^X, E \in \mathcal{E}$, $c(A) = \land\{st(A, E)|E \in \mathcal{E}\}$. By (WQEs), there exist $F \in \mathcal{E}$ such that $FoF \subset E$, $c(c(A)) \leq st(st(A, F), F) \leq st(A, FoF) \leq st(A, E)$. Then $c(c(A) \leq c(A)$ and by (C02) $c(A) \leq c(c(A))$ and, so $c(c(A)) = c(c(A))$.

**Definition 4.4.13**: Let $\mathcal{E}$ be an $T$-Weil quasi-uniformity on $X$. The closure operator defined in 4.4.12 is called the closure operator on $T^X$ generated by the $T$-Weil quasi-uniformity $\mathcal{E}$.

Theorems 4.4.10, 4.4.12 shows that every $T$-Weil quasi-uniformity can generates a $T$-valued space, but the unexpected result is that its converse is also true.

**Remark**: Let $(X, \delta)$ be a $T$-valued space. For every $U \in \delta$, define a self
mapping $f_U$ on $\mathbb{T}^X$ as follows:

$$
\forall \ A \in \mathbb{T}^X, \quad f_U(A) = \begin{cases} 
1 & A \not\subseteq U \\
U & 0 \neq A \subseteq U \\
0 & A = 0 
\end{cases}
$$

It is easy to find that $f_U$ is value increasing,

$$
f_U(\lor A) = \lor_{A \in A} f_U(A), \quad f_U \circ f_U = f_U. [63]
$$

Let $D = \{ f \in \Omega(L^X) | \exists \ A \in [\delta]^{<\omega}, \ f \geq \land_{U \in A} f_U \}$ then for all $g, f \in D$ there exist $h \in D$ such that $h \leq g \land f$. \hfill (1)

Take $A \in [\delta]^{<\omega}$ such that $f \geq \land_{U \in A} f_U = \land_{U \in A}(f_U \circ f_U)$ since for every $V \in A \ f_V \circ f_V \geq (\land_{U \in A} f_U) \circ (\land_{U \in A} f_U)$ so take $g = \land_{U \in A} f_U$ we have $g \in D$ and $g \circ g \leq f$. \hfill (2)

**Theorem 4.4.14**: Let $(X, \delta)$ be a $\mathbb{T}$-valued space. Define $E_f = \cup \{\alpha \cup \alpha | \alpha \in U_f\}$ such that $U_f$ be the cover of all $f$-small elements of $\mathbb{T}^X$. Then $\mathcal{E} = \{ E_f | f \in D \}$ is a $\mathbb{T}$-Weil quasi uniformity on $X$.

**Proof**:

$\mathbb{T}$-WQU1) It is obviously satisfied by (1).

$\mathbb{T}$-WQU2) Let $E_e \in \mathcal{E}$ we can take $f \in D$ such that $f^3 \leq e$. By Lemma 4.4.3, we have $E_f \circ E_f = (\cup_{\alpha \in U_f} \alpha \cup \alpha) \circ (\cup_{\alpha \in U_f} \alpha \cup \alpha)$. Let $(a, c) \in E_f \circ E_f$ then $(a, b) \leq (\alpha, \alpha)$ and $(b, c) \leq (\beta, \beta)$ where $\alpha, \beta \in U_f$ then $a < \alpha < st(\alpha, E_f), c < B < st(\alpha, E_f)$ we prove $st(\alpha, E_f)$ is $e$-small.

Let $\lambda \land st(\alpha, E_f) \neq 0, (\gamma, \gamma) \in E_f$ with $\gamma \land \alpha \neq 0$ and $\gamma \land \lambda \neq 0$ then $\alpha, \gamma$ is $f$-smallness then $\gamma < f(\lambda), \alpha < f(\gamma)$ then $\alpha < f^2(\lambda)$. Therefore, for every $(\gamma', \gamma') \in E_f$ such that $\gamma' \land \alpha \neq 0$ we have $\gamma' \leq f(\alpha) < f^3(\lambda) \leq e(\lambda)$. 

Then \( st(\alpha, E_f) \) is \( e \)-small then \( st(\alpha, E_f) \oplus st(\alpha, E_f) \in E_e \) so \( (a, c) \in E_e \). By \( T-WQU_1 \) and \( T-WQU_2 \), \( E \) is \( T \)-Weil quasi-uniformity on \( X \).

**The Crisp Modification of a \( T \)-Weil Uniform Spaces**

We want to show that every \( T \)-(Weil) uniformity \( (\mathcal{E}) \) on \( X \) corresponds a uniformly \( (\mathcal{U}) \).

The functor

\[ \psi : T-W \text{Unif} \to \text{Unif} . \]

**Theorem 4.4.15** : Let \( \mathcal{E} \) be a \( T \)-Weil uniformity on a space \( X \), defined for each \( E \in \mathcal{E}, V_E \subset X \times X \) by \( V_E = \{(x, y) \in X \times X | \mu(x) \leq st(\mu, E)(y)\} \) and denote the set \( \{V_E | E \in \mathcal{E}\} \) by \( \mathcal{U}_E \). Then \( \mathcal{U}_E \) is Weil uniformity on a space \( X \).

**Proof** : We know for all \( x \in X \ f(x) \leq st(f, E)(x) \), then \( V_E \) contains the diagonal \( \Delta(X) \).

\( UW_1 \) Let \( V_E, V_F \in U_\mathcal{E} \). In order to prove that \( \mathcal{U}_E \) is a filter basis just take for some Weil entourage \( G \) such that \( G \subseteq E \cap F \).

\( UW_2 \) Let \( V_E \in U_\mathcal{E} \). Consider \( F \in \mathcal{E} \) such that \( F^2 \subseteq E \). We prove that \( V_F \circ V_F \subseteq V_E \).

Let \( (x, z) \in V_F \circ V_F \) There is \( y \in X \) s.t. \( (x, y) \in V_F, (y, z) \in V_F \), then \( \mu(x) < st(st(\mu, F), F)(z) \) and, by 4.4.6(d) \( st(st(\mu, F), F) \leq st(\mu, F^2) \). Then \( \mu(x) < st(\mu, F^2)(z) < st(\mu, E)(z) \) then \( (x, z) \in V_E \).

\( UW_3 \) The symmetry condition is obviously satisfied since each \( V_E \) is symmetric.
Theorem 4.4.16: Let \((X, \mathcal{E})\) and \((X', \mathcal{E}')\) be \(\mathbb{T}-\)Weil uniform spaces and let \(f : X \rightarrow X'\) be uniformly continuous. Then \(f\) is also uniformly continuous as a map between the uniform spaces \((X, \psi(\mathcal{E}))\) and \((X', \psi(\mathcal{E}'))\).

Proof: Let \(V' \in \psi(\mathcal{E}')\) and let \(E' \in \mathcal{E}'\) such that \(\psi(E') \subset V'\). Since \(f\) is \((\mathcal{E}, \mathcal{E}')\) - continuous , we have that \(E = (\overset{\rightarrow}{f} \oplus \overset{\rightarrow}{f})(E') \in \mathcal{E}\). Let \(V = \psi(E)\), we will show that \((f(x), f(y)) \in V'\) whenever \((x, y) \in V\). In fact if \((x, y) \in V\) and \(\mu \in \mathbb{T}^Y\). Then we have \(\overset{\rightarrow}{\rho} = f(\mu)\). Then \(\rho(x) \leq \text{st}(\rho, E)(y)\). Since \(\rho(x) = \overset{\rightarrow}{f}(\mu)(x) = \mu(f(x))\), We have

\[\mu(f(x)) = \rho(x) < \text{st}(\rho, E)(y) = \vee\{\alpha(y)\vert (\alpha, \alpha) \in E, \alpha \land \rho \neq 0\}\].

Since \((\alpha, \alpha) \in E\), then there is \((\alpha', \alpha') \in E'\) such that \(\alpha = \overset{\rightarrow}{f}(\alpha')\). Then we have

\[\mu(f(x)) < \vee\{\overset{\rightarrow}{f}(\alpha')(y)\vert (\alpha', \alpha') \in E', \alpha' \land \mu \neq 0\}\]

\[= \vee\{\alpha'(f(y))\vert (\alpha', \alpha') \in E', \alpha' \land \mu \neq 0\},\]

which implies that \((f(x), f(y)) \in \psi(\mathcal{E}') \subset V'\).

We want to show that every (Weil) uniformity \((\mathcal{U})\) on a set \(X\) corresponds a \(\mathbb{T}\)-Weil uniformity \((\mathcal{E})\).

The functor

\[\theta : \text{Unif} \rightarrow \mathbb{T}-\text{W Unif}.

Theorem 4.4.17: Let \(\mathcal{U}\) be a uniformity on a space \(X\), defined for each \(V \in \mathcal{U}\), \(E_V = \cup\{\alpha \oplus \alpha \mid \alpha \in U_V\}\) such that

\[U_V = \{\alpha \in \mathbb{T}^X\vert \alpha(x) < \vee\{\beta(y)\vert (x, y) \in V, \alpha \land \beta \neq 0\}\}\]
be a cover of $\mathbb{T}^X$ and denote the set $\{E_V | V \in \mathcal{U}\}$ by $\mathcal{E}_U$. $\mathcal{E}_U$ is $T$-Weil uniformity on a space $X$.

**Proof**: It is trivial that for all $V \in \mathcal{U}, E_V$ is Weil entourage.

$T - WU_1$ Let $E_V, E_U \in \mathcal{E}_U$. Consider $W \subset U \cap V$

$$E_W \subset E_{U \cap V} = \bigcup \{\alpha \oplus \alpha | \alpha \in U \cap V\}.$$  

It is trivial

$$\bigcup \{\alpha \oplus \alpha | \alpha \in U \cap U\} \subset \bigcup \{\alpha \oplus \alpha | \alpha \in U\} \cap \bigcup \{\alpha \oplus \alpha | \alpha \in U\} \subset E_U \cap E_V.$$  

$T - WU_2$ Let $E_V \in \mathcal{E}_U$. Take $U \in \mathcal{U}$ such that $U^3 \subset V$. By Lemma 4.4.3, we have

$$E_V \circ E_U = (\bigcup_{\alpha \in U_U} (\alpha \oplus \alpha)) \circ (\bigcup_{\alpha \in U_V} (\alpha \oplus \alpha)).$$  

Let $(a, c) < \alpha \oplus \alpha, (c, b) < \beta \oplus \beta$

$$a < \alpha < st(\alpha, E_U) \quad b < \beta < st(\alpha, E_U).$$  

We prove that $st(\alpha, E_U) \in U_V$. Let $\lambda \land st(\alpha, E_U) \neq 0$. Then there is $\gamma \in \mathbb{T}^X, (\gamma, \gamma) \in E_U$. Then $\lambda \land \gamma \neq 0$ and $\alpha \land \gamma \neq 0$. Then $\alpha, \lambda \in U_U$.

$$\alpha(x) < \bigvee \{\gamma(y) | (x, y) \in U, \alpha \land \gamma \neq 0\}$$  

$$< \bigvee \{\lambda(z) | (y, z) \in U, (x, y) \in U, \alpha \land \gamma \neq 0, \gamma \land \lambda \neq 0\}$$  

$$< \bigvee \{\lambda(z) | (x, z) \in U \circ U, \alpha \land \lambda \neq 0\},$$  

then $\alpha \in U_U^2$.

Therefore, for every $(\gamma', \gamma') \in E_U$ such that $\gamma' \land \alpha \neq 0$, we have

$$\gamma'(x) < \bigvee \{((\alpha(y)) | (x, y) \in U, \gamma' \land \alpha \neq 0\}$$
\[
< \bigvee \lambda \{ (x, z) \mid (x, z) \in U^3 (\lambda \land \alpha) \neq 0 \} \\
< \bigvee \lambda \{ (x, z) \mid (x, z) \in V (\lambda \land \alpha) \neq 0 \}.
\]

Then \( st(\alpha, E_U) \in U_V \), then \( st(\alpha, E_U) \oplus st(\alpha, E_U) \in E_V \), then \( (a, b) \in E_V \).

\( T - WU_3 \) It is trivial each \( E_V \) is symmetric.

**Theorem 4.4.18** : Let \((X, \mathcal{U})\) and \((X', \mathcal{U}')\) be uniform spaces and let \( f \) be a function from \( X \to Y \). Then \( f \) is uniformly continuous iff \( f \) is uniformly continuous as a map between the \( T \)-Weil uniform spaces \((X, \theta(\mathcal{U})) \to (X', \theta'(\mathcal{U}'))\).

**Proof** : Suppose \( f \) is \((\mathcal{U}, \mathcal{U}')\)-continuous and Let \( E' \in \theta(\mathcal{U}') \), there exist \( V' \in \mathcal{U}' \) s.t.

\[
E' = E_V = \bigcup \{ \alpha \oplus \alpha \mid \alpha \in U_{V'} \}.
\]

Let \( V \in \mathcal{U} \) such that \( (f(x), f(y)) \in V' \) whenever \( (x, y) \in V \). If \( \theta(V) = F \), we have

\[
(f \oplus f)(E_{V'}) = (f \oplus f)(\bigcup_{\alpha \in U_{V'}, \alpha \oplus \alpha} = \bigcup_{\alpha \in U_{V'}, f(\alpha) \oplus f(\alpha)).
\]

We prove that \( U_{V'} \subset f(U_{V'}) \). It is trivial \( f(U_{V'}) \) is a cover of \( L^X \). Let

\[
\alpha \in U_{V'}, \alpha(x) < \bigvee \{ (\beta(y))(x, y) \in V, \alpha \land \beta \neq 0 \}
\]

,then

\[
f(\alpha)(f(x)) < \bigvee \{ f(\beta)(f(y))(f(x), f(y)) \in f(V), f(\alpha) \land f(\beta) \neq 0 \}.
\]

Therefore \( f(\alpha) \in U_{V'} \), then \( \alpha \in f(U_{V'}) \). We conclude that \( F \subset (f \oplus f)(E) \) then \( (f \oplus f)(E) \in \theta(\mathcal{U}) \).
Conversely, let $f$ be a $(\theta(\mathcal{U}), \theta(\mathcal{U}'))$ continuous and let $V' \in \mathcal{U}'$ we have $\theta(V') \in \theta(\mathcal{U}')$, then $(\overline{f} \oplus \overline{f})(\theta(V')) \in \theta(\mathcal{U})$. There exist $V \in \mathcal{U}$ such that $\theta(V) \leq \overline{f} \oplus \overline{f}(\theta(V'))$, then we have $\alpha \in U_V$ then $f(\alpha) \in U_{V'}$. It is trivial $(f(x), f(y)) \in V'$ whenever $(x, y) \in V$.

In conclusion, we have following diagram:

$$
\begin{array}{c}
\text{T-W(Q)Unif} \quad \leftrightarrow \quad \text{T-Top} \\
\theta \downarrow \psi \quad \quad \quad \quad E_1 \downarrow F_1 \\
\text{Unif} \quad \rightarrow \quad \text{Top} \\
\quad \quad \quad \quad \quad T_1
\end{array}
$$
4.5 The Isomorphism Between the Categories $T$-Unif, $T - W$ Unif and Hutt-Unif.

The functor $\psi : T-Unif \to T-W Unif$.

Let $\mu$ be a family of $T-Cov(X)$. For each $\mathcal{U} \in \mu$ consider the $T$-Weil entourage $E_{\mathcal{U}} := \bigvee_{U \in \mathcal{U}} (U \oplus U)$, and denote the set $\{E_{\mathcal{U}}|\mathcal{U} \in \mu\}$ by $\mathcal{E}_\mu$.

**Proposition 4.5.1**: Let $\mu$ be a $T$-valued uniformity on a space $X$. Then $\mathcal{E}_\mu$ is a $T$-Weil uniformity on $X$.

**Proof**: 

$T$-WU1, $T$-WU2) : Let $E_{\mathcal{U}}, E_{\mathcal{V}} \in \mathcal{E}_\mu$. Take $w \in \mu$ such that $w \leq \mathcal{U} \wedge \mathcal{V}$. Clearly $E_w \subseteq E_{\mathcal{U}} \cap E_{\mathcal{V}}$.

$T$-WU3) Let $E_{\mathcal{U}} \in \mathcal{E}_\mu$, and take $\mathcal{V} \in \mu$ such that $\mathcal{V}^* \leq \mathcal{U}$. Then $E_{\mathcal{V} \circ E_{\mathcal{V}}} \subseteq E_{\mathcal{U}}$ : By Lemma 4.4.3(a) it follows that

$$E_{\mathcal{V} \circ E_{\mathcal{V}}} = (\bigcup_{U \in \mathcal{V}} (U \oplus U)) \circ (\bigcup_{U \in \mathcal{V}} (U \oplus U)).$$

Let $(f, g) \in E_{\mathcal{V} \circ E_{\mathcal{V}}}$ then $(f, h) \leq (U, U)$, and $(h, g) \leq (V, V)$ where $U, V \in \mathcal{V}$ and $h \neq 0$. Then $\mathcal{V} \wedge U \neq 0$, $f \leq U \leq st(U, \mathcal{V})$ and $g \leq V \leq st(U, \mathcal{V})$. As $st(U, \mathcal{V}) \in \mathcal{V}^* \leq \mathcal{U}$, this says that there is $W \in \mathcal{U}$ such that $f \leq W, g \leq W$ and consequently, that $(f, g) \in E_{\mathcal{U}}$.

$T$-WU4) the symmetry condition is obviously satisfied since each $E_{\mathcal{U}}$ is symmetric.

In the sequel, for every uniformity $\mu$, $\psi(\mu)$ denotes the $T$-Weil uniformity $\mathcal{E}_\mu$. The correspondence $(X, \mu) \to (X, \psi(\mu))$ is functorial. Indeed, it is the
function on objects of a functor $\psi : T - \text{Unif} \rightarrow T - W \text{Unif}$ whose function on morphisms is described in the following proposition:

**Proposition 4.5.2**: Let $(X, \mu)$ and $(X', \mu')$ be covering $T$-uniform spaces and let $f : (X, \mu) \rightarrow (X', \mu')$ be a uniform homomorphism. Then $f : (X, \psi(\mu)) \rightarrow (X', \psi(\mu'))$ is a $T$-Weil uniform homomorphism.

**Proof**: It is obvious since, for every $U \in \mu$,

$$ (f^{-} \oplus f^{-})(E_U) = \bigvee_{v \in U} (f^{-}(U) \oplus f^{-}(U)) = E_{f^{-}|U}. $$

The functor $\Phi : T - W \text{Unif} \rightarrow \text{Hutt-Unif}.

Let $E \subseteq T^X \oplus T^X$. For each $E \in E$ define $e_E : T^X \rightarrow T^X$ by $e_E(U) = st(U, E)$ and denote the set $\{e_E|E \in E\}$ by $D_E$. It is obvious that each $e_E$ preserves arbitrary joins and that it is an entourage of $T^X$ whenever $E$ is a $T$-Weil entourage.

**Proposition 4.5.3**: Let $E$ be a $T$-Weil uniformity on a space $X$. Then $D_E$ is a Hutton uniformity on $X$.

**Proof**: $D_E$ satisfies the axiom (1) - (3).

(1) for all $E \in E$, $e_E(\emptyset) = st(\emptyset, E) = \emptyset$

(2) for all $E \in E$, $e_E(U) = st(U, E) \geq U$

(3) for all $E \in E$ $e_E(\bigvee_i U_i) = st(\bigvee_i U_i, E) = \bigvee_i st(U_i, E)$. Then $D_E \subset \Omega(T^X)$.

HU1) $D_E \neq \emptyset$ since every $C$-ideal contains the $C$-ideal $\emptyset$ then there exist $\emptyset \neq E \in E$ such that $e_E \in D_E$.

HU2, HU3) $e_E, e_F \in D_E$ then $E, F \in E$ in order to prove that $D_E$ is a
filter basis just take for some $T$-Weil entourage $G$ such that $G \subseteq E \cap F$.

HU4) For $e_E \in D_E$ consider $F \in \mathcal{E}$ such that $F^2 \subset E$ by 4.4.7(d). We observed that $st(st(U,F),F) \leq st(U,F^2)$. Hence $e_F \leq e_{F^2} \leq e_E$.

HU5) $e_E \in D_E$ we have that

$$V \land e_E(U) = V \land st(U,E) = 0 \iff \{V \land W | (W,W) \in E, W \land U \neq 0\} = \emptyset$$

$$e_E(V) \land U = st(V,E) \land U = 0 \iff \{W \land U | (W,W) \in E, W \land V \neq 0\} = \emptyset.$$

These two formula are equivalent, then we have,

$$e_E^{-1}(U) = \land \{V | e_E(V') \leq U'\} = \land \{V | e_E(V') \land U \leq 0\} = \land \{V \land U \leq 0\} = \land \{V | e_E(U) \leq V\} = e_E(U).$$

In what follows, if $\mathcal{E}$ is a $T$-Weil uniformity on $X$, then $\phi(\mathcal{E})$ denotes the Hutton uniformity. The correspondence $(X,\mathcal{E}) \rightarrow (X,\phi(\mathcal{E}))$ is functorial.

**Proposition 4.5.4**: Let $(X,\mathcal{E})$ and $(X',\mathcal{E}')$ be $T$-Weil uniform spaces and let $f : (X,\mathcal{E}) \rightarrow (X',\mathcal{E}')$ be a $T$-Weil uniform homomorphism. Then $f : (X,\phi(\mathcal{E})) \rightarrow (X',\phi(\mathcal{E}'))$ is a Hutton uniform homomorphism.

**Proof**: Let $e_E \in D_E$ where $E \in \mathcal{E}$. Take a symmetric $F \in \mathcal{E}$ such that $F^2 \subseteq E$. Since $f$ is a Weil uniform homomorphism, $(f^- \oplus f^-)(F) \in \mathcal{E}'$. In order to show that $f : (X,\phi(\mathcal{E})) \rightarrow (X',\phi(\mathcal{E}'))$ is uniform it suffices to show that

$$e_{(f^- \oplus f^-)(F)} \cdot f^- \leq f^- \cdot e_E.$$
so, fix $\alpha \in \mathbb{T}^X$ and take $\beta \in \mathbb{T}^{X'}$ such that $(\beta, \beta) \in (f^- \oplus f^-)(F)$ and $(\beta \land f^- (\alpha)) \neq 0, \beta < e_{(f^- \oplus f^-)(F)}$. Then $(\beta, \beta \land f^- (\alpha)) \in (f^- \oplus f^-)(F)$ and $(\beta \land f^- (\alpha), f^- (\alpha)) \in f^- (\alpha) \oplus f^- (\alpha)$ and consequently $(\beta, f^- (\alpha)) \in ((f^- \oplus f^-)(F) o (f^- (\alpha) \oplus f^- (\alpha)))$. Further, since $F$ is of the form $\forall \gamma (a_{\gamma} \oplus b_{\gamma})$ for some subset $\{(a_{\gamma}, b_{\gamma})| \gamma \in \Gamma\} \times \mathbb{T}^X$.

$$(f^- \oplus f^-)(F) o (f^- (\alpha) \oplus f^- (\alpha)) = (f^- \oplus f^-)((\forall a_{\gamma} \oplus b_{\gamma}))(f^- (\alpha) \oplus f^- (\alpha))$$

$= k(\bigcup_{\gamma \in \Gamma} f^- (a_{\gamma}) \oplus f^- (b_{\gamma})) o k(\downarrow (f^- (\alpha), f^- (\alpha))).$

By Lemma 4.4.3

$$= \cup_{\gamma \in \Gamma} f^- (a_{\gamma}) \oplus f^- (b_{\gamma}) o (\downarrow (f^- (\alpha), f^- (\alpha))) \subset (f^- \cdot e_{E})(\alpha) \oplus f^- (\alpha).$$

For any $(a, b) \in (\cup_{\gamma \in \Gamma} f^- (a_{\gamma}) \oplus f^- (b_{\gamma})) o (\downarrow (f^- (\alpha), f^- (\alpha))) \setminus \{0\} \quad \exists c \in L^X \setminus \emptyset$

and $\gamma \in \Gamma \quad (a, c) \leq (f^- (a_{\gamma}), f^- (b_{\gamma}))$ and $(c, b) \leq (f^- (\alpha), f^- (\alpha))$ it follows that $a \leq f^- (a_{\gamma} \lor b_{\gamma})$ and therefore $a \leq (f^- \cdot e_{E})(\alpha)$ indeed $(a_{\gamma} \lor b_{\gamma}) \land \alpha \neq \emptyset$ because $f^- (b_{\gamma} \lor a_{\gamma}) \geq c \neq \emptyset$ and by the symmetry of $F \quad (a_{\gamma} \lor b_{\gamma}, a_{\gamma} \lor b_{\gamma}) \in F^2 \subseteq E$ inclusion $a \leq f^- (a_{\gamma} \lor b_{\gamma}) < f^- (e_{E}(\alpha))$ then $(a, b) \in (f \cdot e_{E})(\alpha) \oplus f(\alpha)$. We have that $(\beta, f(\alpha)) \in (f^- \oplus f^-)(F) o (f^- (\alpha) \oplus f^- (\alpha)) \subseteq (f^- \cdot e_{E})(\alpha) \oplus f^- (\alpha)$. Hence $\beta \leq (f^- \cdot e_{E})(\alpha)$ which implies that $e_{(f^- \oplus f^-)(F)} f^- (\alpha) \leq f^- (e_{E})(\alpha)$ then $e_{(f^- \oplus f^-)(F)} f^- \leq f^- \cdot e_{E}$.

We shall denote the functor defined above by $\phi$.

The Functor

$\theta : \text{Hutt-Unif} \rightarrow \mathbb{T} - \text{Unif}$:

Finally, we want to study that the functors $\theta$ between the categories Hutton uniform space and $\mathbb{T}$-valued uniform space. For each entourage $e$ of $\mathbb{T}^X$
let \( U_e \) be the cover of all \( e \)-small elements of \( \mathbb{T}^X \) i.e. \( U_e = \{ U \in \mathbb{T}^X | U \leq e(V), U \land V \neq 0 \} \).

**Proposition 4.5.5**: Let \( D \) be Hutton uniformity on \( X \), then \( \mu_D = \{ U_e | e \in D \} \) is a \( \mathbb{T} \)-valued uniformity on \( X \).

**Proof**:

\( CU_1, CU_2 \) Consider \( U_e, U_f \in \mu_D \) and let \( g \in D \) such that \( g \leq e \land f \) then it is obvious that \( U_g \leq U_e \land U_f \).

\( CU_3 \) Let \( U_e \in \mu_D \) and take \( f \in D \) such that \( f^3 \leq e \) we claim \( U_f^* \leq U_e \) consider \( st(U, U_f) \in U_f^* \). It suffices to show that \( st(U, U_f) \) is \( e \)-small. So, consider \( V \in \mathbb{T}^X \) such that \( V \land st(U, U_f) \neq 0 \) then there is \( W \in U_f \) with \( W \land U \neq 0 \) and \( V \land W \neq 0 \) then \( f \)-smallness of \( U, W \) then \( U \leq f(W) \leq f^2(V) \).

Therefore for every \( W' \in U_f \) such that \( W' \land U \neq 0 \) we have \( W' \leq f(U) \leq f^3(V) \leq e(V) \). Then \( st(U, U_f) \) is \( e \)-small.

In the sequel, if \( D \) is a Hutton uniformity, \( \theta(D) \) denotes the uniformity generated by \( U_D \).

**Proposition 4.5.6**: Let \( (X, D) \to (X', D') \) be a Hutton uniform homomorphism then \( f : (X, \theta(D)) \to (X', \theta(D')) \) is a covering \( \mathbb{T} \)-uniform homomorphism.

**Proof**: Let \( U \in \theta(D) \) and \( e \in D \) such that \( U_e \leq U \) there exists \( g \in D' \) with \( g \cdot f \leq f \cdot e \). We show that \( U_g \leq f[ U_e ] \). Let \( U \) be a non-zero \( g \)-small element of \( \mathbb{T}^X \). Since \( f[ U_e ] \) is a cover of \( \mathbb{T}^X \), there exists \( V \in U_e \) satisfying \( U \land f(V) \neq 0 \). Consequently \( U \leq g \cdot f(V) \leq f \cdot e(V) \). But as can be easily
proved, the fact \( V \) is \( e \)-small implies that \( e(V) \) is \( e^3 \)-small. In conclusion \( \mathcal{U}_g \leq f [\mathcal{U}_s] \leq f (\mathcal{U}) \) and \( f (\mathcal{U}) \in \theta(D') \).

**The Isomorphism**

Finally, let us show that the functors \( \psi, \Phi \) and \( \Theta \) define an isomorphism between the categories \( \mathcal{T} \)-Unif, \( \mathcal{T} \)-W Unif and Hutt-Unif.

**Lemma 4.5.7** : For any \( \mathcal{T} \)-cover \( \mathcal{U} \) of \( X \) we have that

(a) \( \mathcal{U} \leq \mathcal{U}_{e_{E_d}} \)

(b) \( \mathcal{U}_{e_{E_d}} \leq \mathcal{U}^* \).

**Proof** : (a) Let \( U \in \mathcal{U} \). For any \( V \in \mathcal{T}^X \) satisfying \( U \land V \neq 0 \), since \( (U, U) \in E_{d_d}, U \leq s_t(V, E_d) = e_{E_d}(V) \), that is, \( U \) is \( e_{E_d} \)-small and therefore \( U \in \mathcal{U}_{e_{E_d}} \).

(b) for any non-zero \( e_{E_d} \)-small member \( U \) of \( \mathcal{T}^X \), there exists \( V \in \mathcal{U} \) such that \( U \land V \neq 0 \).

Then \( U \leq s_t(V, E_d) \). But, for every \( (W, W) \in E_d, W \) is \( \mathcal{U} \)-small so \( W \leq s_t(V, \mathcal{U}) \) in case \( W \land V \neq 0 \). This means that \( s_t(V, E_d) \leq s_t(V, \mathcal{U}) \). Hence \( U \leq s_t(V, \mathcal{U}) \in \mathcal{U}^* \).

**Lemma 4.5.8** : For any symmetric \( \mathcal{T} \)-Weil entourage \( E \) of \( X \) we have

(a) \( E \subseteq E_{d_{E_d}} \)

(b) \( E_{d_{E_d}} \subseteq E^2 \).

**Proof** : (a) Consider \( (U, V) \in E \setminus \mathcal{O} \) because \( (V, U) \in E \setminus \mathcal{O} \) then \( (U, U), (V, V), (U \lor V, U \lor V) \) belong to \( E^2 \). Since every member of \( E^2 \) with equal coordinates is \( e_{E^2} \)-small, \( (U \lor V, U \lor V) \in E_{d_{E_d}} \), and consequently, \( (U, V) \in E_{d_{E_d}} \).
(b) Let us verify that \( \cup\{U \oplus U|U \text{ is } e_E\text{-small } \} \subseteq E^2 \) or, which is the same, that \((U, U) \in E^2 \) whenever \( U \) is \( e_E\text{-small} \) consider \( U \neq 0, e_E\text{-small} \). We have that \( U \leq \lor\{W \in T^X|(W, W) \in E, U \land W \neq 0\} \). Since \( V \) is \( e_E\text{-small} \), we get \( U \leq e(W) = \lor\{V \in T^X|(V, V) \in E, V \land W \neq 0\} \), for any \( W \) such that \((W, W) \in E \) and \( U \land W \neq 0\). For each \( V \) in this set we have that \((W, W \land V), (W \land V, V) \in E \), which implies that \((W, V) \in E^2 \). Therefore \((W, U) \in E^2 \) and, consequently \((U, U) \in E^2 \).

Lemma 4.5.9 : Let \( e \) be a symmetric entourage of \( T^X \). Then

(a) \( e \leq e_{E_{U, 3}} \)

(b) \( e_{E_{U, 3}} \leq e \)

Proof : For any \( U \in T^X, e_{E_{U, 3}} (U) = st(U, E_{U, 3}) \). On the other hand, \( e(U) = e(\lor\{U \land V|V \text{ is } e\text{-small } \}) = \lor\{e(U \land V)|V \text{ is } e\text{-small and } U \land V \neq 0\} \).

Consider any \( e\text{-small} \) element \( V \) such that \( U \land V \neq 0 \). We have \( e(U \land V) \) is \( e^3\text{-small} \) so \( (e(U \land V), e(U \land V)) \in E_{U, 3} \). Since \( e(U \land V) \land (U \land V) = U \land V \neq 0 \).

It follows that

\[
e(U \land V) \leq e_{E_{U, 3}}(U \land V) \leq e_{E_{U, 3}}(U).
\]

Hence \( e(U) \leq e_{E_{U, 3}}(U) \).

(b) \( V \) is \( U_e \text{- small} \) whenever \((V, V) \in E_{U_e}\). Therefore, \( e_{E_{U_e}} = \lor\{V \in T^X|(V, V) \in E_{U_e}, V \land U \neq 0\} \leq st(U, U_e) \leq e(U) \).

Proposition 4.5.10 : Let \( \mu, \mathcal{E} \) and \( D \) denote, respectively, a covering \( T\)-uniformity and \( T\)-Weil uniformity and Hutton - uniformity on \( X \). Then \( \Theta\phi\psi(\mu) = \mu, \psi\Theta\phi(\mathcal{E}) = \mathcal{E} \) and \( \phi\psi\Theta(D) = D \).
Proof: The uniformity $\theta \phi \psi(\mu)$ has $\{U_e | e \in \phi \psi(\mu)\}$ as a basis. It suffices to prove this is a basis for $\mu$, by Lemma 4.5.7 (a) $\{U_e | e \in \phi \psi(U)\} \subseteq \mu$, and, by (b), for any $U \in \mu$ there is some $V \in \mu$ such that $U_{e_{V,U}} \subseteq U$.

By Lemma 4.5.8 implies that the basis $\{E_U | U \in \theta \phi(E)\}$ of $\psi \theta \phi(E)$ is also a basis for $E$, which proves the second equality, and Lemma 4.5.9 implies that the basis $\{e_E | E \in \psi \theta(D)\}$ of $\phi \psi \theta(D)$ is also a basis for $D$, which proves the equality $\phi \psi \theta(D) = D$.

Theorem 4.5.11: The categories $T$-Unif, $T$-W Unif and Hutt -Unif are isomorphic.

Proof: Follows by proposition 4.5.10.