CHAPTER 3

VARIOUS NOTIONS OF L-VALUED UNIFORM SPACES

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CHAPTER 3
Various Notions of $L$-Valued Uniform Spaces

3.1 Introduction


In this Chapter we introduce the notions of $L$-uniform spaces, generalised uniform spaces and $L$- Hutton (quasi) uniform spaces in a sense of [26]. We investigate the relationship between Šostak $L$-fuzzy topological spaces and the three aforementioned uniformities in a different sense of [26]. Then we show that every $[0, 1]$ - fuzzy topological space is Hutton $[0, 1]$ - quasi uniformizable. At the end of this Chapter, we will show the relation between Hutton $L$-uniformities and $L$-uniformities in the case of $(L, \leq, \wedge, ^{'})$. 
3.2 The Šostak $L$-fuzzy Topological Spaces

Definition 3.2.1[93]: A function $\mathcal{T} : L^X \rightarrow L$ is called an $SL$-fuzzy topology on $X$ if it satisfies the following conditions:

1. $\mathcal{T}(\emptyset) = \mathcal{T}(1) = 1$
2. $\mathcal{T}(\mu_1 \land \mu_2) \geq \mathcal{T}(\mu_1) \land \mathcal{T}(\mu_2)$ for $\mu_1, \mu_2 \in L^X$
3. $\mathcal{T}(\bigvee_{i \in \Gamma} \mu_i) \geq \bigwedge_{i \in \Gamma} \mathcal{T}(\mu_i)$ for any $\{\mu_i\}_{i \in \Gamma} \subset L^X$.

The pair $(X, \mathcal{T})$ is called Šostak $L$-fuzzy topological space. Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be $SL$-fuzzy topological spaces. A function $\psi : (X_1, \mathcal{T}_1) \rightarrow (X_2, \mathcal{T}_2)$ is called an $SL$-fuzzy continuous map if $\mathcal{T}_2(\mu) \leq \mathcal{T}_1(\psi^{-}(\mu))$ for all $\mu \in L^{X_2}$.

We will denote by $SL$-FTS the corresponding category. We introduce $L$-fuzzy interior operator different from Höhle and Šostak $L$-fuzzy interior operator.

Theorem 3.2.2: Let $(X, \mathcal{T})$ be an $SL$-fuzzy topological space. Define, for each $r \in L^1, \lambda \in L^X$

$$I_T(\lambda, r) = \bigvee\{\mu \in L^X | \mu \leq \lambda, \mathcal{T}(\mu) > r\}$$

Then it satisfies the following properties: for $\lambda, \mu \in L^X$ and $r, s \in L^1$

1. $I_T(1, r) = 1$
2. $I_T(\lambda, r) \leq \lambda$
3. if $\lambda \leq \mu$ and $r \leq s$ then $I_T(\lambda, s) \leq I_T(\mu, r)$
4. $I_T(\lambda \land \mu, r) \geq I_T(\lambda, r) \land I_T(\mu, r)$
5. $I_T(I_T(\lambda, r), r) \geq I_T(\lambda, r)$. 


Proof: (1), (2), (3) are trivial from the definition of $I_T$.

(4) Suppose

$$I_T(\lambda_1 \land \lambda_2, r) \not\geq I_T(\lambda_1, r) \land I_T(\lambda_2, r).$$

Since $L$ is a complete distributive lattice, by the definition of $I_T(\lambda_i, r)$, for each $i \in \{1, 2\}$, there exist $\mu_i \in L^X$ with $\mu_i \leq \lambda_i$ and $T(\mu_i) > r$ such that $I_T(\lambda_1 \land \lambda_2, r) \not\geq \mu_1 \land \mu_2$.

Since $T(\mu_1 \land \mu_2) > r$ and $\mu_1 \land \mu_2 \leq \lambda_1 \land \lambda_2$, we have $I_T(\lambda_1 \land \lambda_2, r) \geq \mu_1 \land \mu_2$.

It is contradiction.

(5) Suppose $I_T(I_T(\lambda, r), r) \not\geq I_T(\lambda, r)$. From the definitions of $I_T(\lambda, r)$, there exist $\mu \in L^X$ with $\mu \leq \lambda$ and $T(\mu) > r$ such that $I_T(I_T(\lambda, r), r) \not\geq \mu$.

Since $\mu = I_T(\mu, r) \leq I_T(\lambda, r)$, $I_T(I_T(\lambda, r), r) \geq \mu$. It is a contradiction.

Theorem 3.2.3: Let $(X, T)$ be an $SL$-fuzzy topological space. The function $T_{I_T}: L^X \to L$ is defined by $T_{I_T}(\lambda) = \vee \{ r \in L | I_T(\lambda, r) \geq \lambda \}$. Then we have the following properties:

(1) $T_{I_T}$ is an $SL$-fuzzy topology on $X$.

(2) If $L$ is an order dense chain, then $T_{I_T} = T$.

Proof: (1) We show that $T_{I_T}$ is an $SL$-fuzzy topology on $X$.

(01) It is trivial.

(02) Let $I_T(\lambda_i, r_i) \geq \lambda_i$ for each $i \in \{1, 2\}$, by Theorem 3.2.2

$$I_T(\lambda_1 \land \lambda_2, r_1 \land r_2) \geq I_T(\lambda_1, r_1) \land I_T(\lambda_2, r_2) \geq \lambda_1 \land \lambda_2.$$
Hence $\mathcal{T}_I(\lambda_1 \land \lambda_2) \geq r_1 \land r_2$. Since $L$ is a completely distributive lattice,

$$\mathcal{T}_I(\lambda_1 \land \lambda_2) \geq \mathcal{T}_I(\lambda_1) \land \mathcal{T}_I(\lambda_2).$$

(03) Let $I_T(\lambda_i, r_i) \geq \lambda_i$ for each $i \in \Gamma$, by theorem 3.2.2,

$$I_T[\lor_{i \in \Gamma} \lambda_i, \land r_i] \geq I_T(\lambda_i, r_i) \geq \lambda_i.$$ 

Hence $\mathcal{T}_I(\lor_{i \in \Gamma} \lambda_i) \geq \land_{i \in \Gamma} r_i$. Since $L$ is a complete distributive lattice,

$$\mathcal{T}_I(\lor_{i \in \Gamma} \lambda_i) \geq \land_{i \in \Gamma} \mathcal{T}_I(\lambda_i).$$

(2) We will show that $\mathcal{T}_I = \mathcal{T}$. Suppose $\mathcal{T}_I \not\preceq \mathcal{T}$. Since $L$ is an order dense chain, by the definition of $\mathcal{T}_I$, there exists $r_0 \in L^1$ with $I_T(\lambda, r_0) \geq \lambda$ such that $\mathcal{T}_I(\lambda) \geq r_0 \geq \mathcal{T}(\lambda)$.

On the other hand, since $I_T(\lambda, r_0) \geq \lambda$, we have $\mathcal{T}(\lambda) \geq r_0$ from the definition of $I_T(\lambda, r_0)$. It is a contradiction. Hence $\mathcal{T}_I \leq \mathcal{T}$.

Suppose $\mathcal{T}_I \not\succeq \mathcal{T}$. Since $L$ is an order dense chain, there exist $\rho \in L^X$ and $r_1 \in L^1$ such that $\mathcal{T}_I(\rho) < r_1 < \mathcal{T}(\rho)$.

On the other hand, since $\mathcal{T}(\rho) > r_1$, by the definition of $I_T(\rho, r_1) \geq \rho$. Hence $\mathcal{T}_I(\rho) \geq r_1$. It is a contradiction. Hence $\mathcal{T}_I \geq \mathcal{T}$.

Example 3.2.4: Let $L = \{0, 1, a, b\}$ be the diamond-type lattice that is, $a \lor b = 1, a \land b = 0$ and $\mu \neq \rho \in L^X - \{0, 1\}$. Let $\mathcal{T}$ be an $L$-fuzzy topology on $X$ defined as follow:

$$\mathcal{T}(\lambda) = \begin{cases} 
1 & \text{if } \lambda \in \{0, 1, \rho \lor \mu\} \\
0 & \text{if } \lambda = \rho \\
1 & \text{if } \lambda \in \{\mu, \rho \land \mu\} \\
0 & \text{otherwise}
\end{cases}$$
From Theorem 3.2.2, we have

\[ I_T(\lambda, a) = \begin{cases} 
1 & \text{if } \lambda = \frac{1}{2} \\
\mu \lor \rho & \text{if } \mu \lor \rho \leq \lambda \neq \frac{1}{2} \\
0 & \text{otherwise}
\end{cases} \]

\[ I_T(\lambda, b) = \begin{cases} 
1 & \text{if } \lambda = \frac{1}{2} \\
\mu \lor \rho & \text{if } \mu \lor \rho \leq \lambda \neq \frac{1}{2} \\
0 & \text{otherwise}
\end{cases} \]

From Theorem 3.2.3, we can obtain \( \mathcal{T}_{I_T} : L^X \to L \) as follows:

\[ \mathcal{T}_{I_T}(\lambda) = \begin{cases} 
1 & \text{if } \lambda \in \{0, \frac{1}{2}, \mu \lor \rho\} \\
0 & \text{otherwise}
\end{cases} \]

Hence, in general, \( \mathcal{T}_{I_T} \neq \mathcal{T} \).
3.3 SL-Topologies Associated to $L$-Uniform Spaces

Definition 3.3.1 [26]: (L-uniform space). Let $X$ be a nonempty set. A mapping $\xi : L^{X \times X} \rightarrow L$ is called an $L$-uniformity on $X$ if it satisfies the following axioms:

1. $\xi(1_{X \times X}) = 1, \xi(1_\phi) = 0$
2. $\xi(\mu \wedge v) \geq \xi(\mu) \wedge \xi(v)$ for all $\mu, v \in L^{X \times X}$
3. If $\mu \leq v \in L^{X \times X}$ then $\xi(\mu) \leq \xi(v)$
4. $\xi(v^{-1}) = \xi(v)$ for all $v \in L^{X \times X}(v^{-1}(x, y) = v(y, x))$
5. $\xi(v) \leq \bigvee_{\mu \mu \leq v} \xi(\mu)$ for all $v \in L^{X \times X}$
6. $\xi(v) \leq \inf_{x \in X} v(x, x)$ for all $v \in L^{X \times X}$.

The pair $(X, \xi)$ is said to be $L$-uniform space.

Example: The mapping $E : L^{X \times X} \rightarrow L$ defined for each $a \in L^{X \times X}$ by $E_0(a) = \inf a$ is a $L$-uniformity on $X$.

Definition 3.3.2: Let $(X, \xi)$ and $(Y, \xi')$ be two $L$-uniform spaces. A mapping $\phi : X \rightarrow Y$ is said to be $L$-uniformly continuous if $\xi(\phi \times \phi)^{-1}(\mu) \geq \xi'(\mu)$ for all $\mu \in L^{Y \times Y}$.

We shall denote by $L-Unif$ the corresponding category.

Theorem 3.3.3: Let $(X, \xi)$ be an $L$-uniform space for each $r \in L^1, \lambda \in L^X$, we define

$$ I_\xi(\lambda, r) = \bigvee \{ \mu \in L^X \mid \bigvee_{x \in X} \mu(x) \wedge d(x, -) \leq \lambda \text{ for some } d \text{ with } \xi(d) > r \} $$

then it satisfies the followings:

1. $I_\xi(0, r) = 0$, $I_\xi(1, r) = 1$
Proof: (1), (2) (3) are trivial from the definition of \( I_\xi(\lambda, r) \).

(4) By (2) we have \( I_\xi(I_\xi(\lambda, r), r) \leq I_\xi(\lambda, r) \). We need to show that 
\[ I_\xi(\lambda, r) \leq (I_\xi(I_\xi(\lambda, r), r) \). \]

Let \( \mu \in L^X, d \in L^{X \times X} \) s.t. \( \xi(d) > r, \forall x \in X \mu(x) \land d(x, -) < \lambda \) by \( (LU_3) \) there exist \( e \in L^{X \times X}, \xi(e) > r, eoe < d \) and \( \xi(e) > \xi(d) > r \) then it is trivial \( \forall x \in X \mu(x) \land eoe(x, -) < \lambda \) by definition of \( eoe \) we have 
\[ \forall x \in X ((\forall x \in X \mu(x) \land e(x, -))(z) \land e(z, -) \leq \lambda \) then \( \forall x \in X \mu(x) \land e(x, -) \leq (I_\xi(\lambda, r) \) then \( \mu \leq (I_\xi(I_\xi(\lambda, r), r) \) so 
\[ I_\xi(\lambda, r) = \forall \{ \mu \in L^X / \forall x \in X \mu(x) \land d(x, -) < \lambda \} < I_\xi(I_\xi(\lambda, r), r) \).

(5) We need to prove \( I_\xi(\lambda, r) \land I_\xi(\mu, r) \leq I_\xi(\lambda \land \mu, r) \) for arbitrary \( \lambda, \mu \in L^X \). In fact, since for arbitrary \( d, e \in \xi \) and arbitrary \( \lambda, \mu, \gamma, \delta \in L^X \) such that 
\[ \forall x \in \gamma \land d(x, -) < \lambda, \forall x \in \delta \land e(x, -) < \mu \] we have 
\[ \forall x \in \gamma \land \delta \land (d \land e)(x, -) < \lambda \land \mu \] so 
\[ I_\xi(\lambda, r) \land I_\xi(\mu, r) = \forall \{ \gamma \land \delta \mid \gamma, \delta \in L^X, d, e \in L^{X \times X}, \xi(d) > r, \xi(e) > r, \forall x \in \gamma \land d(x, -) < \lambda, \forall x \in \delta \land e(x, -) < \mu \} \leq \forall \{ \gamma \land \delta \mid \forall x \in \gamma \land \delta \land (d \land e)(x, -) < \lambda \land \mu \} = I_\xi(\lambda, \mu, r). \]

Theorem 3.3.4: Let \((X, \xi)\) be an \( L \)-uniform space. The function \( T_\xi : \)
\[ L^X \rightarrow L \] is defined by \( T_\xi(\lambda) = \forall \{ r \in [0, 1] | i_\xi(\lambda, r) = \lambda \} \) for \( \lambda \in L^X \) then \( T_\xi \) is an \( SL \)-fuzzy topology on \( X \).

(01) It is trivial \( T_\xi(0) = T_\xi(1) = 1 \)
(02) Let \( I_\xi(\lambda_i, r_i) \geq \lambda_i \) for each \( i \in \{1, 2\} \), by theorem 3.3.3,
\[
I_\xi(\lambda_1 \land \lambda_2, r_1 \land r_2) \geq I_\xi(\lambda_1, r_1) \land I_\xi(\lambda_2, r_2) \geq \lambda_1 \land \lambda_2
\]
Hence \( T_\xi(\lambda_1 \land \lambda_2) \geq r_1 \land r_2 \). Since \( L \) is completely distributive lattice
\[
T_\xi(\lambda_1 \land \lambda_2) \geq T_\xi(\lambda_1) \land T_\xi(\lambda_2).
\]
(03) Let \( I_\xi(\lambda_i, r_i) \geq \lambda_i \) for each \( i \in \{1, 2\} \), by theorem 3.3.3,
\[
I_\xi(\bigvee_{i \in \Gamma} \lambda_i \land r_i) \geq I_\xi(\lambda_i, r_i) \geq \lambda_i.
\]
Hence \( T_\xi(\bigvee_{i \in \Gamma} \lambda_i) \geq \bigwedge_{i \in \Gamma} r_i \). Since \( L \) is completely distributive lattice
\[
T_\xi(\bigvee_{i \in \Gamma} \lambda_i) \geq \bigwedge_{i \in \Gamma} T_\xi(\lambda_i).
\]

**Theorem 3.3.5**: Let \( \xi \) be an \( L \)-uniform space on \( X \) for each \( r \in L^1, \lambda \in L^X \), we define \( C_\xi(\lambda, r) = \bigwedge\{\forall y \in X \lambda(y) \land d(y, -)|\xi(d) > r\} \) then it satisfies the following:

1. \( C_\xi(\emptyset, r) = \emptyset \)
2. \( C_\xi(\lambda, r) \geq \lambda \)
3. If \( \lambda_1 \leq \lambda_2 \) then \( C_\xi(\lambda_1, r) \leq C_\xi(\lambda_2, r) \)
4. If \( r \leq s \) then \( C_\xi(\lambda, r) \leq C_\xi(\lambda, s) \)
5. \( C_\xi(\lambda_1 \lor \lambda_2, r) = C_\xi(\lambda_1, r) \lor C_\xi(\lambda_2, r) \)
6. If \( L \) is a chain \( C_\xi(C_\xi(\lambda, r), r) \leq C_\xi(\lambda, r) \).

**Proof**: (1), (2), (3) and (4) are easily proved from the definition of \( C_\xi \).

5. We need to prove \( C_\xi(\lambda_1 \lor \lambda_2, r) \leq C_\xi(\lambda_1, r) \lor C_\xi(\lambda_2, r) \) it is trivial
\[
\forall_{y \in X}(\lambda_1 \lor \lambda_2)(y) \land (d_1(y, -) \land d_2(y, -))
\]
\[ \leq \forall y \in X (\lambda_1(y) \land d_1(y, -)) \lor (\forall y \in X (\lambda_2(y) \land d_2(y, -))). \]

Suppose \( \gamma \in L^X \) such that \( \gamma \not\leq C_\xi(\lambda_1, r) \lor C_\xi(\lambda_2, r), \gamma \not\leq C_\xi(\lambda_2, r) \) then there exist \( d_1, d_2 \in L^{X \times X}_\xi(d_1) > r, \xi(d_2) > r \) such that

\[ \gamma \not\leq \forall y \in X \lambda_1(y) \land d_1(y, -), \gamma \not\leq \forall y \in X \lambda_2(y) \land d_2(y, -) \]

then

\[ \gamma \not\leq \forall y \in X (\lambda_1 \lor \lambda_2)(y) \land (d_1(y, -) \land d_2(y, -)), \gamma \not\leq C_\xi(\lambda_1 \lor \lambda_2, r) \]

so \( C_\xi(\lambda_1 \lor \lambda_2, r) \leq C_\xi(\lambda_1, r) \lor C_\xi(\lambda_2, r) \).

(6) Let \( \lambda \in L^X, d \in L^{X \times X} \) s.t. \( \xi(d) > r \). By \((IU_5)\) there exist \( e \in L^{X \times X}_\xi(e) > r \) such that \( eoe \leq d \)

\[ C_\xi(C_\xi(\lambda, r), r) \leq \forall z \in X ((\forall y \in X \lambda(y) \land e(y, -))(z) \land e(z, -)) \]

\[ \leq \forall y \in X \lambda(y) \land eoe(y, -) \leq \forall y \in X \lambda(y) \land d(y, -) \]

\[ C_\xi(C_\xi(\lambda, r), r) \leq \land \{ \forall y \in X \lambda(y) \land d(y, -) | \xi(d) < r \} = C_\xi(\lambda, r). \]

In general topology, we have \( \overline{A} = (\text{int}(A^c))^c \). In a sense, we will expand it from the following lemma.

**Lemma 3.3.6**: Let \((X, \xi)\) be an \( L \)-uniform space for each \( \lambda \in L^X, r \in L^1 \) we have \( C_\xi(\lambda, r) = (I_\xi(\lambda', r))^l \).

**Proof**:

\[ (I_\xi(\lambda', r))^l = (\forall \{ \mu \in L^X | \forall x \in X \mu(x) \land d(x, -) \leq \lambda' \} \]

for some $d$ with $\xi(d) > r$}^{'}

\begin{align*}
&= \land\{\mu' \in L^X| \forall x \in X \mu(x) \land d(x, -) \leq \lambda', \xi(d) > r\} \\
&= \land\{\mu' \in L^X| \forall x \in X \lambda(x) \land d(x, -) \leq \mu', \xi(d) > r\} \\
&= \land\{\forall x \in X \lambda(x) \land d(x, -)|\xi(d) > r\} \\
&= C_{\xi}(\lambda, r)
\end{align*}

**Theorem 3.3.7**: Let $(X, \xi)$ be an $L$-uniform space. The function $T_{\xi}: L^X \rightarrow L$ defined by, for each $\lambda \in L^X$

$$T_{\xi}(\lambda) = \lor\{r \in L|C_{\xi}(\lambda', r) \leq \lambda'\}.$$ 

Then $T_{\xi}(\lambda)$ is an $SL$-fuzzy topology on $X$.

**Proof**: It is trivial by Theorem 3.3.5, Lemma 3.3.6.

**Theorem 3.3.8**: Let $(X, \xi)$ and $(X', \xi')$ be $L$-uniform spaces.

Let $\phi: (X, \xi) \rightarrow (X', \xi')$ be $L$-uniformly continuous. For each $\lambda \in L^X, v \in L^{X'}$ and $r \in L^1$, we have the following properties:

1. $C_{\xi'}(\phi^{-}(\lambda), r) \geq \phi^{-}(C_{\xi}(\lambda, r))$
2. $C_{\xi'}(\phi^{-}(v), r) \leq \phi^{-}(C_{\xi'}(v, r))$

**Proof**: (1) For $\lambda \in L^X, r \in L^1$ we have

$$C_{\xi'}(\phi^{-}(\lambda), r) = \land\{\lor y \in X' \phi^{-}(\lambda)(y) \land d(y, -)|\xi'(d) > r\}$$

$$\geq \phi^{-}\left(\land\{\lor y \in X \lambda(y) \land (\phi \times \phi)^{-1}(d)(y, -)|\xi(\phi \times \phi)^{-1}(d) > \xi'(d) > r\}\right).$$

By Lemma 3.3.6 and $\phi$ is $L$-uniformly continuous

$$\geq \phi^{-}\left(\land\{\lor y \in X \lambda(y) \land d'(y, -)|\xi(d') > r\}\right) = \phi^{-}(C_{\xi}(\lambda, r)).$$
(2) For all \( r \in L^1 \) and \( v \in L^{X'} \), we have

\[
C_\xi(\phi^-(v), r) \leq \phi^-(C_\xi(\phi^-(v), r)) \\
\leq \phi^-(C_{\xi'}(\phi^-(v), r)) \leq \phi^-(C_{\xi'}(v, r)) \quad \text{by (1)).}
\]

**Theorem 3.3.9** : Let \((X, \xi), (X', \xi')\) be \( L \)-uniform spaces. If \( \phi : (X, \xi) \to (X', \xi') \) is \( L \) - uniformly continuous, then \( \phi : (X, T_\xi) \to (X', T_{\xi'}) \) is SL-fuzzy continuous.

**Proof** : Suppose that \( \phi : (X, T_\xi) \to (X', T_{\xi'}) \) is not SL-fuzzy continuous. Then there exists \( \rho \in L^{X'} \) such that \( T_{\xi'}(\rho) \nsubseteq T_\xi(\phi^-(\rho)) \). Hence there exists \( r \in L^1 \) such that

\[
r \nsubseteq T_\xi(\phi^-(\rho)), \quad C_{\xi'}(\rho', r) \leq \rho'.
\]

It implies

\[
\phi^-(\rho)' = \phi^-(\rho') \geq \phi^-(C_{\xi'}(\rho', r)) \\
\geq C_\xi(\phi^-(\rho'), r) = C_\xi(\phi^-(\rho'), r).
\]

Hence \( C_\xi(\phi^-(\rho'), r) \leq \phi^-(\rho') \). Therefore, \( T_\xi(\phi^-(\rho)) \geq r \). It is a contradiction.
3.4. SL-Topologies Associated to Generalised Uniform Spaces

**Definition 3.4.1** [12]: Let $X$ be a nonempty set. A mapping $f: 2^{X \times X} \rightarrow L$ is called generalised uniformity iff $f$ satisfies the following axioms.

(GU1) $f(X \times X) = 1, f(\emptyset) = 0$

(GU2) If $A_1 \subseteq A_2 \subseteq X \times X$ then $f(A_1) \leq f(A_2)$.

(GU3) $f(A_1 \cap A_2) \geq f(A_1) \land f(A_2)$ for all $A_1, A_2 \subseteq X \times X$.

(GU4) $f(A) \leq \vee_{B \subseteq A} f(B)$.

(GU5) $f(A^{-1}) \geq f(A)$

(GU6) $f(A) \neq 0 \Rightarrow \Delta = \{(x, x) | x \in X\} \subseteq A$.

The pair $(X, f)$ is called a generalised uniform space.

**Example:** The mapping $f: 2^{X \times X} \rightarrow L$ defined for each $A \subseteq X \times X$ by

$$f(A) = \begin{cases} 1 & A = X \times X \\ 0 & A \neq X \times X \end{cases}$$

is a generalised uniform spaces on $X$.

**Definition 3.4.2**: Let $(X, f), (Y, g)$ be two generalised uniform spaces. A mapping $\phi: X \rightarrow Y$ is said to be $L$-uniformly continuous if and only if $f(\phi \times \phi)^{-1}(B) \geq g(B)$ for all $B \subseteq Y \times Y$.

We will denote by $G$-Unif the corresponding category.

**Theorem 3.4.3**: Let $(X, f)$ be a generalised uniform space for each $r \in L^1, \lambda \in L^X$, we define

$$I_f(\lambda, r) = \vee\{\mu \in L^X | \forall x \in X \mu(x) \land 1_A(x, -) \leq \lambda \text{ for some } A \text{ with } f(A) > r\}$$
\[= \bigvee \{ \mu \in L^X \mid \forall x \in X \mu(x) \leq \lambda(y), y \in A[x] = \{ y \in X \mid (x, y) \in A \} \text{ for some } A \text{ with } f(A) > r \}\]

then it satisfies the following:

(1) \(I_f(0, r) = 0, I_f(1, r) = 1\).

(2) \(I_f(\lambda, r) \leq \lambda\)

(3) if \(\lambda \leq \mu\) and \(r < s\) then \(I_f(\lambda, s) \leq I_f(\mu, r)\)

(4) \(I_f(I_f(\lambda, r), r) = I_f(\lambda, r)\)

(5) \(I_f(\lambda \land \mu, r) = I_f(\lambda, r) \land I_f(\mu, r)\).

**Proof:** (1), (2), (3) are trivial from the definition of \(I_f(\lambda, r)\).

(4) we need to prove that

\[I_f(\lambda, r) \leq I_f(I_f(\lambda, r), r)\]

Let \(\mu \in L^X, A \subset X \times X\) s.t. \(f(A) > r, \forall x \in X \mu(x) \land 1_A(x, -) < \lambda\), by (GU4) there exist \(B \subset X \times X, f(B) > r, BoB \subset A\) then it is trivial \(\forall x \in X \mu(x) \land 1_{BoB}(x, -) < \lambda\) by definition of \(BoB\) we have

\[\forall x \in X(\forall x \in X \mu(x) \land 1_B(x, -)) \land 1_B(z, -) \leq \lambda\]

then \(\forall x \in X \mu(x) \land 1_B(x, -) \leq i(\lambda, r)\) then \(\mu \leq I_f(I_f(\lambda, r), r)\). So

\[I_f(\lambda, r) = \bigvee \{ \mu \in L^X \mid \forall x \in X \mu(x) \land 1_B(x, -) < \lambda, f(B) > r \} < I_f(I_f(\lambda, r), r)\].

(5) We need to prove that

\[I_f(\lambda, r) \land I_f(\mu, r) \leq I_f(\lambda \land \mu, r)\] for arbitrary \(\lambda, \mu \in L^X\).

For arbitrary \(B, A \subset X \times X\) and arbitrary \(\lambda, \mu, \gamma, \delta \in L^X\) such that

\[\forall x \in X \gamma(x) \leq \lambda(y), y \in B[x], \forall x \in X \delta(x) \leq \mu(y), y \in A[x]\]
then we have $\forall_{x \in X} (\gamma \land \delta)(x) \leq (\lambda \land \mu)(y), y \in (A \cap B)[x]$.

So $I_f(\lambda, r) \land I_f(\mu, r)
= \vee\{\gamma \land \delta|, \gamma, \delta \in L^X, A, B \subset X \times X f(A) > r, f(B) > r, \forall_{x \in X} \gamma(x) \leq \lambda(y)
y \in B[x], \forall_{x \in X} \delta(x) \leq \mu(y) y \in A[x]\}
\leq \forall\{\gamma \land \delta| \forall_{x \in X} (\gamma \land \delta)(x) \leq (\lambda \land \mu)(y), y \in (A \cap B)[x], f(A \cap B) > r\}
= I_f(\lambda \land \mu, r).

**Theorem 3.4.4**: Let $(X, f)$ be a generalised uniform space. The function $T_{I_f}: L^X \to L$ is defined by $T_{I_f}(\lambda) = \vee\{r \in L|I_f(\lambda, r) \geq \lambda\}$ for each $\lambda \in L^X$ then $T_{I_f}$ is an SL-fuzzy topology on $X$.

**Proof**: The proof technique is exactly the same as in Theorem 3.3.4.

**Theorem 3.4.5**: Let $f$ be a generalised uniform space on $X$ for each $r \in L^1, \lambda \in L^X$, we define $C_f(\lambda, r) = \land\{\forall_{y \in X} \lambda(y) \land 1_A(y, -)|f(A) > r\}$ then it satisfies the following:

1. $C_f(0, r) = 0$
2. $C_f(\lambda, r) \geq \lambda$
3. If $\lambda_1 \geq \lambda_2$ then $C_f(\lambda_1, r) \leq C_f(\lambda_2, r)$
4. $C_f(\lambda_1, \lor \lambda_2, r) = C_f(\lambda_1, r) \lor C_f(\lambda_2, r)$
5. If $L$ is a chain $C_f(C_f(\lambda, r), r) \leq C_f(\lambda, r)$.

**Proof**: It is easily proved from the definition of $C_f(\lambda, r)$.

**Lemma 3.4.6**: Let $(X, f)$ be a generalised uniform space for each $\lambda \in L^X, r \in L^1$ we have $C_f(\lambda, r) = (I_f(\lambda', r))'$.

**Proof**: 


(I_{f}(\lambda', r))' = (\vee \{ \mu \in L^X | \bigwedge_{x \in X} \mu(x) \wedge 1_A(x, -) \leq \lambda' \})'

for some A with f(A) > r}

= \land \{ \mu' \in L^X | \bigvee_{x \in X} \lambda(x) \wedge 1_A(x, -) \leq \mu', f(A) > r \}

= \land \{ \forall x \in X \lambda(x) \wedge 1_A(x, -)[f(A) > r] \}

= C_f(\lambda, r)

**Theorem 3.4.7**: Let \((X, f)\) be a generalised uniform space. The function \(T_f: L^X \to L\) defined by, for each \(\lambda \in L^X\)

\(T_f(\lambda) = \vee \{ r \in L \mid C_f(\lambda', r) \leq \lambda' \} \).

Then \(T_f(\lambda)\) is an \(SL\)-fuzzy topology on \(X\).

**Proof**: It is trivial by Theorem 3.4.5, Lemma 3.4.6.

**Theorem 3.4.8**: Let \((X, f)\) and \((X', f')\) be \(L\)-uniform spaces.

Let \(\phi: (X, f) \to (X', f')\) be \(L\)-uniformly continuous. For each \(\lambda \in L^X, v \in L^{X'}\) and \(r \in L^1\), we have the following properties:

1. \(C_{f'}(\phi^-(\lambda), r) \geq \phi^-(C_f(\lambda, r))\)
2. \(C_f(\phi^-(v), r) \leq \phi^-(C_{f'}(v, r))\)

**Proof**: (1) For \(\lambda \in L^X, r \in L^1\) we have

\[ C_{f'}(\phi^-(\lambda), r) = \land \{ \forall y \in X | \phi^-(\lambda)(y) \wedge 1_A(y, -)[f'(A) > r] \} \]

\[ \geq \phi^-(\land \{ \forall y \in X | (\phi \times \phi)^{-1}(1_A)(y, -)[f(\phi \times \phi)^{-1}(A) > f'(A) > r] \}) \].
By Lemma 3.4.6 and \( \phi \) is generalised uniformly continuous

\[
\geq \phi^{-}(\lambda \{ \forall y \in X \lambda(y) \wedge 1_{A'}(y, -)|f(A') > r \}) = \phi^{-}(C_{f}(\lambda, r)).
\]

(2) For all \( r \in L^{1} \) and \( v \in L^{X'} \), we have

\[
C_{f}(\phi^{-}(v), r) \leq \phi^{-}(C_{f}(\phi^{-}(v), r)) \\
\leq \phi^{-}(C_{f'}(\phi^{-}(v), r)) \leq \phi^{-}(C_{f'}(v, r)) \text{ by (1)}.
\]

**Theorem 3.4.9:** Let \( (X, f), (X', f') \) be generalised uniform spaces. If \( \phi : (X, f) \to (X', f') \) is generalised uniform continuous then \( \phi : (X, T_{f}) \to (X', T_{f'}) \) is SL-fuzzy continuous.

**Proof:** Suppose that \( \phi : (X, T_{f}) \to (X', T_{f'}) \) is not SL-fuzzy continuous. Then there exists \( \rho \in L^{X'} \) such that \( T_{f'}(\rho) \nleq T_{f}(\phi^{-}(\rho)) \). Hence there exists \( r \in L^{1} \) such that

\[
r \nleq T_{f}(\phi^{-}(\rho)), \quad C_{f'}(\rho', r) \leq \rho'.
\]

It implies

\[
\phi^{-}(\rho)' = \phi^{-}(\rho') \geq \phi^{-}(C_{f'}(\rho', r)) \\
\geq C_{f}(\phi^{-}(\rho'), r) = C_{f}(\phi^{-}(\rho)', r).
\]

Hence \( C_{f}(\phi^{-}(\rho'), r) \leq \phi^{-}(\rho') \). Therefore, \( T_{f}(\phi^{-}(\rho)) \geq r \). It is a contradiction.
3.5. SL-Topologies Associated to Hutton L-(quasi) Uniform Spaces

Definition 3.5.1[90,26] : A function \( D : \Omega(L^X) \rightarrow L \) is said to be a Hutton L-quasi-uniformity on \( X \) if it satisfies the following conditions :

(HLQU1) If \( f \leq g \), then \( D(f) \leq D(g) \).

(HLQU2) For each \( f, g \in \Omega(L^X) \), \( D(f \land g) \geq D(f) \land D(g) \).

(HLQU3) For each \( f \in \Omega(L^X) \), \( \vee \{ D(g) | g \leq f \} \geq D(f) \).

(HLQU4) There exists \( f \in \Omega(L^X) \) such that \( D(f) = 1 \).

The pair \((X, D)\) is said to be a Hutton L-quasi-uniform space. A Hutton L-quasi-uniform space \((X, D)\) is called a Hutton L-uniform space if it satisfies

(HLU) For each \( f \in \Omega(L^X) \), \( \vee \{ D(g) | g \leq f^{-1} \} \geq D(f) \).

Let \( D_1 \) and \( D_2 \) be Hutton L-(quasi) uniformities on \( X \). \( D_1 \) is finer than \( D_2 \) (or \( D_2 \) is coarser than \( D_1 \)), denoted by \( D_2 \leq D_1 \) iff for any \( f \in \Omega(L^X) \), \( D_2(f) \leq D_1(f) \).

Remark 3.5.2 :

(1) We define for \( f \in \Omega(L^X) \), \( D^{-1}(f) = D(f^{-1}) \cdot D^{-1} \) is a Hutton L-quasi-uniformity on \( X \) from 2.4.1.

(2) From (HLQU1), (HLQU2) and 2.4.1, we have \( D(f \land g) = D(f) \land D(g) \).

(3) From 2.4.1 and (HLQU4), since \( f \leq f_1 \) for all \( f \in \Omega(L^X) \), we have \( U(f_1) = 1 \).

(4) If \((X, D)\) is a Hutton L-uniform space, then by (HLU), (HLQU1) and
from 2.4.1 we have $D(f) = D(f^{-1})$.

**Definition 3.5.3**: Let $(X, D), (X', D')$ be Hutton $L$-(quasi) - uniform spaces. A function $\phi : (X, D) \to (X', D')$ is Hutton $L$-(quasi) - uniformly continuous if $D'(f) \leq D(\phi^{-1}(f))$, for every $f \in \Omega(L^X)$.

We will denote by Hutt $L - (Q)$-Unif the category whose objects are Hutton $L$-(quasi)-uniform spaces and morphisms $L$-uniformly continuous mappings.

**Theorem 3.5.4**: Let $D$ be a Hutton $L$-quasi-uniformity on $X$. For each $r \in L^1, \lambda \in L^X$, we define $C_D(\lambda, r) = \land \{f^{-1}(\lambda) | D(f) > r\}$. Then it satisfies the following:

1. $C_D(0, r) = 0$
2. $C_D(\lambda, r) \geq \lambda$
3. If $\lambda_1 \leq \lambda_2$, then $C_D(\lambda_1, r) \leq C_D(\lambda_2, r)$
4. If $r \leq s$, then $C_D(\lambda, r) \leq C_D(\lambda, s)$
5. $C_D(\lambda_1 \lor \lambda_2, r) \leq C_D(\lambda_1, r) \lor C_D(\lambda_2, r)$
6. If $L$ is a chain $C_D(C_D(\lambda, r), r) \leq C_D(\lambda, r)$.

**Proof**: (1), (2), (3) and (4) are easily proved from the definition of $C_D$.

(5) Suppose $C_D(\lambda_1 \lor \lambda_2, r)(x) \not\leq C_D(\lambda_1, r)(x) \lor C_D(\lambda_2, r)(x)$.

Since $L$ is a completely distributive lattice, by the definitions of $C_D(\lambda_i, r)$, for each $i \in \{1, 2\}$, there exist $f_i \in \Omega(L^X)$ with $D(f_i) > r$ such that

$$C_D(\lambda_1 \lor \lambda_2, r)(x) \not\leq f_i^{-1}(\lambda_1)(x) \lor f_i^{-1}(\lambda_2)(x).$$
Since $\mathcal{D}(f_1 \wedge f_2) > r$, we have

$$C_D(\lambda_1 \lor \lambda_2, r)(x) \leq (f_1 \wedge f_2)^{-1}(\lambda_1 \lor \lambda_2)(x)$$

$$= (f_1^{-1} \wedge f_2^{-1})(\lambda_1 \lor \lambda_2)(x)$$

$$\leq f_1^{-1}(\lambda_1)(x) \lor f_2^{-1}(\lambda_2)(x).$$

It is a contradiction.

(6) Let $\lambda \in L_X, f \in \Omega(L_X)$ s.t. $\mathcal{D}(f) > r$. By (HLQU3) there exist $e \in \Omega(L_X), \mathcal{D}(e) > r$ such that $eoe \leq f$

$$C_D(C_D(\lambda, r), r) \leq e^{-1}(e^{-1}(\lambda)) \leq (eoe)^{-1}(\lambda) \leq f^{-1}(\lambda)$$

$$C_D(C_D(\lambda, r), r) \leq \land\{f^{-1}(\lambda) | \mathcal{D}(f) > r\} = C_D(\lambda, r).$$

**Theorem 3.5.5**: Let $(X, \mathcal{D})$ be a Hutton $L$-quasi-uniform space. Define, for each $r \in L^1, \lambda \in L_X,$

$$I_D(\lambda, r) = \lor\{\mu \in L_X | f(\mu) \leq \lambda, \mathcal{D}(f) > r\}$$

Then we have $(I_D(\lambda, r))' = C_D(\lambda', r)$.

**Proof**: It is proved from

$$(I_D(\lambda, r))' = (\lor\{\mu \in L_X | f(\mu) \leq \lambda, \mathcal{D}(f) > r\})'$$

$$= \land\{\mu' \in L_X | f(\mu) \leq \lambda, \mathcal{D}(f) > r\}$$

$$= \land\{\mu' \in L_X | f^{-1}(\lambda') \leq \mu', \mathcal{D}(f) > r\}$$

$$= \land\{f^{-1}(\lambda') | \mathcal{D}(f) > r\}$$

$$= C_D(\lambda', r).$$
**Theorem 3.5.6**: Let $(X, D)$ be a Hutton $L$-quasi-uniform space. The function $T_D : L^X \to L$ is defined by,

for each $\lambda \in L^X$

$$T_D(\lambda) = \lor \{ r \in L | I_D(\lambda, r) \geq \lambda \} = \lor \{ r \in L | C_D(\lambda', r) \leq \lambda' \}.$$  

Then $T_D$ is an SL-fuzzy topology on $X$.

**Proof**: The operator $I_D$ satisfies the condition (1-4) of Theorem 3.2.2. Thus, we can obtain an SL-fuzzy topology $T_D$ as a similar method as in Theorem 3.2.3.

**Theorem 3.5.7**: Let $(X, D), (X', D')$ be Hutton $L$-quasi uniform spaces. Let $\phi : (X, D) \to (X', D')$ be Hutton $L$-quasi-uniformly continuous. For each $\lambda \in L^X, v \in L^{X'}$ and $r \in L^1$, we have the following properties.

(1) $C_{D'}(\phi^-(\lambda), r) \geq \phi^-(C_D(\lambda, r))$

$$C_{D'}^{-1}(\phi^-(\lambda), r) \geq \phi^-(C_D^{-1}(\lambda, r))$$

(2) $C_D(\phi^-(v), r) \leq \phi^-(C_{D'}(v, r)).$

**Proof**: (1) For $\lambda \in L^X, r \in L^1$, we have

$$C_{D'}(\phi^-(\lambda), r) = \land \{ f^{-1}(\phi^-(\lambda)) | D'(f) > r \}$$

$$\geq \land \{ \phi^{-1}(\phi^{-1}(f)^{-1}(\lambda)) | D(\phi^{-1}(f)) \geq D'(f) > r \}$$

(by Lemma 2.4.3 and $\phi$ is Hutton $L$-quasi-uniformly continuous)

$$\geq \phi^-(\land \{ \phi^{-1}(f)^{-1}(\lambda) | U(\phi^{-1}(f)) > r \})$$
\[
\geq \phi^{-}(\land \{f^{-1}(\lambda)|D(f) > r\}) \\
= \phi^{-}(C_D(\lambda, r)).
\]

Similarly, we have \( C_{D^{-1}}(\phi^{-}(\lambda), r) \geq \phi^{-}(C_{D^{-1}}(\lambda, r)) \).

(2) For all \( r \in L^1 \) and \( v \in L^{X'} \), we have

\[
C_D(\phi^{-}(v), r) \leq \phi^{-}(\phi^{-}(C_D(\phi^{-}(v), r))) \\
\leq \phi^{-}(C'_D(\phi^{-}(\phi^{-}(v)), r)) \quad \text{(by (1))} \\
\leq \phi^{-}(C_D(v, r)).
\]

**Theorem 3.5.8** : Let \((X, D), (X', D')\) be Hutton \(L\)-(quasi)-uniform spaces. If \( \phi : (X, D) \to (X', D') \) is Hutton \(L\)-quasi-uniformly continuous, then :

\( \phi : (X, T_D) \to (X', T_{D'}) \) is SL-fuzzy continuous.

**Proof** : Suppose that \( \phi : (X, T_D) \to (X', T_{D'}) \) is not SL-fuzzy continuous. Then there exists \( \rho \in L^{X'} \) such that \( T_{D'}(\rho) \not\leq T_D(\phi^{-}(\rho)) \). Hence there exists \( r \in L^1 \) such that

\[
r \not\leq T_D(\phi^{-}(\rho)), \quad C_{D'}(\rho', r) \leq \rho'.
\]

It implies

\[
\phi^{-}(\rho')' = \phi^{-}(\rho') \\
\geq \phi^{-}(C_{D'}(\rho', r)) \\
= C_D(\phi^{-}(\rho'), r).
\]

Hence \( C_D(\phi^{-}(\rho'), r) \leq \phi^{-}(\rho')' \). Therefore, \( T_D(\phi^{-}(\rho)) \geq r \) from Theorem 3.5.6. It is a contradiction.
3.6. Hutton $L$-quasi Uniformizable Spaces

Lemma 3.6.1[57] : Let $(X, T)$ be an $L$-fuzzy topological space and $T_0 = \{0 \neq \rho \in L^X | T(\rho) \neq 0 \}$ for every $\rho \in T_0$, we define $f_\rho : L^X \to L^X$ as follows :

$$f_\rho(\lambda) = \begin{cases} 0 & \text{if } \lambda = 0 \\ \rho & \text{if } 0 \neq \lambda \leq \rho \\ 1 & \text{otherwise} \end{cases}$$

Then we have the following properties :

(1) $f_\rho \in \Omega(L^X)$ and $f_\rho^{-1} = f'_\rho$

(2) $f_\rho o f_\rho = f_\rho$ and $f \leq f_1$ for all $f \in \Omega(L^X)$.

(3) If $f_\rho$, for $i = 1, \cdots, n, \Gamma = \{J \subset \{1, \cdots, n\} \lambda \leq \vee_{j \in J} \rho_j \}$ and $\rho_J = \vee_{j \in J} \rho_j$, then

$$\vee_{i=1}^n f_\rho_i(\lambda) = \begin{cases} 0 & \text{if } \lambda = 0 \\ \vee_{j \in \Gamma} \rho_j & \text{if } \Gamma \neq \emptyset \\ 1 & \text{if } \Gamma = \emptyset \end{cases}$$

Theorem 3.6.2[57] : Let $(X, T)$ be an SL-fuzzy topological space. Define a function $D_T : \Omega(L^X) \to L$ by

$$D_T(f) = \vee \{\vee_{i=1}^n T(\rho_i) | \vee_{i=1}^n f_\rho_i \leq f \}$$

where the first $\vee$ is taken over every finite family $\{f_\rho_i | i = 1, \cdots, n\}$. Then $D_T$ is an Hutton L-quasi-uniformity on $X$.

Proof :

(HLQU1) is trivial from definition.

(HLQU2) Suppose there exist $f, g \in \Omega(L^X)$ such that

$$D_T(f \wedge g) \not\leq D_T(f) \wedge D_T(g).$$
There exist two finite families \( \{ \rho_i \in T_0 \mid \wedge_{i=1}^m f \rho_i \leq f \} \) and \( \{ v_i \in T_0 \mid \wedge_{i=1}^m f v_i \leq g \} \) such that
\[
D_T(f \wedge g) \not\geq (\wedge_{i=1}^m T(\rho_i)) \wedge (\wedge_{j=1}^n T(v_j)).
\]
On the other hand, since \( f \wedge g \geq (\wedge_{i=1}^m f \rho_i) \wedge (\wedge_{j=1}^n f v_j) \),
we have \( D_T(f \wedge g) \geq (\wedge_{i=1}^m T(\rho_i)) \wedge (\wedge_{j=1}^n T(v_i)) \). It is contradiction.

(HLQU3) Suppose there exists \( f \in \Omega(L^X) \) such that
\[
\forall \{ D_T(g) \mid g \leq f \} \not\geq D_T(f).
\]
There exists a finite family \( \{ \rho_i \in T_0 \mid \wedge_{i=1}^m f \rho_i \leq f \} \) such that \( \forall \{ D_T(g) \mid g \leq f \} \not\geq \wedge_{i=1}^m T(\rho_i) \).
On the other hand, since \( f \rho_i \circ f \rho_i = f \rho_i \) for each \( i \in \{1, \ldots, m\} \), we have
\[
(\wedge_{i=1}^m f \rho_i) \circ (\wedge_{i=1}^m f \rho_i) \leq f \rho_i \circ f \rho_i = f \rho_i.
\]
From 2.4.1, it implies
\[
(\wedge_{i=1}^m f \rho_i) \circ (\wedge_{i=1}^m f \rho_i) \leq (\wedge_{i=1}^m f \rho_i) \leq f.
\]
Put \( g = \wedge_{i=1}^m f \rho_i \). Then \( g \leq f \) and \( \forall \{ D_T(g) \mid g \leq f \} \geq \wedge_{i=1}^m T(\rho_i) \).
It is contradiction.

(HLQU4) Since \( T(1) = 1 \), there exists \( f_1 \in \Omega(X) \) such that \( D_T(f_1) = 1 \).
Hence \( D_T \) is a Hutton \( L \)-quasi-uniformity on \( X \).

**Example 3.6.3** [57]: Let \( T \) be an \([0, 1]\) - fuzzy topology on \( X \) defined as follows:
\[
T(\lambda) = \begin{cases} 
1 & \text{if } \lambda = 1 \text{ Or } 0 \\
\frac{1}{2} & \text{if } \lambda = \rho \\
0 & \text{otherwise}
\end{cases}
\]
From theorem, we have

\[ D_T(f) = \begin{cases} 
  1 & \text{if } f = f_1 \\
  \frac{1}{2} & \text{if } f_\rho \leq f \leq f_1 \\
  0 & \text{otherwise}
\end{cases} \]

**Definition 3.6.4**: An L-fuzzy topological space \((X, T)\) is said to be L-fuzzy quasi-uniformizable if there exists a Hutton L-quasi-uniformity \(D\) on \(X\) such that \(T = T_D\).

**Theorem 3.6.5**: Let \(L\) be an order dense chain. Let \((X, T)\) be an L-fuzzy topological space and \(D_T\) denoted by the Hutton L-quasi uniformity induced by \(T\). Then we have

1. \(T_{D_T} = T\).
2. \(D_T\) is the coarsest number of all Hutton L-quasi uniformities which are compatible with an L-fuzzy topology \(T\) i.e. \((T = T_D)\).

**Proof**: Since \(L\) is an order dense chain, by Theorem 3.2.3 \(T_{I_T} = T\). Thus, we only show that each \(\lambda \in L^X\) and \(r \in L^1\), \(I_{D_T}(\lambda, r) = I_T(\lambda, r)\) defined as follows:

\[ I_{D_T}(\lambda, r) = \vee\{\mu \in L^X | f(\mu) \leq \lambda, D_T(f) > r\} \]
\[ I_T(\lambda, r) = \vee\{\rho \in L^X | \rho \leq \lambda, T(\rho) > r\} \]

If \(\lambda = 0\) or \(1\), it is trivial. If \(T(\rho) > r\) with \(\rho \leq \lambda\), then there exists \(f_\rho \in \Omega(L^X)\) such that \(\rho = f_\rho(\rho) \leq \lambda, D_T(f_\rho) \geq T(\rho) > r\). Hence \(I_{D_T}(\lambda, r) \leq I_T(\lambda, r)\).

Suppose that there exist \(\lambda \in L^X\) and \(r \in L^1\) such that \(I_T(\lambda, r) \not< I_{D_T}(\lambda, r)\).
From the definition of $I_D(\lambda, r)$, there exists $f \in \Omega(L^X)$ with $D_T(f) > r$ such that $f(\mu) \leq \lambda, I_T(\lambda, r) \not\geq \mu$. On the other hand, since $D_T(f) > r$ and $L$ is an order chain, there exists $r_1 \in L^1$ and a family $\{ f_{\rho_1}, \cdots, f_{\rho_n} | \land_{i=1}^n f_{\rho_i} \leq f \}$ such that

$$\land_{i=1}^n T(\rho_i) \geq r_1 > r, \land_{i=1}^n f_{\rho_i}(\mu) \leq f(\mu).$$

Since $\land_{i=1}^n f_{\rho_i}(\mu) \leq \lambda$ and $\lambda \neq 1$, by Lemma 3.6.2 there exists

$$\Gamma = \{ J \subset \{1, \cdots, n\} | \mu \leq \rho_J = \lor_{j \in J} \rho_j \}$$

such that

$$\land_{i=1}^n f_{\rho_i}(\mu) = \land_{j \in J} \rho_j \leq \lambda$$

and

$$T(\rho_J) = T(\lor_{j \in J} \rho_j) \geq \land_{j \in J} T(\rho_j) \geq r_1 > r.$$ 

Since $T(\lor_{j \in J} \rho_J) \geq \land_{J \in \Gamma} T(\rho_J) \geq r_1 > r$ we have $I_T(\lambda, r) \geq \lor_{J \in \Gamma} \rho_J \geq \mu$. It is a contradiction. Hence, $I_T(\lambda, r) \geq I_D(\lambda, r)$.

(2) By (1), we have that $D_T$ is compatible with $T$. Let $D$ be Hutton $L$-quasi uniformity such that $T = T_D$. We will show that $D_T(f) < D(f)$, for all $f \in \Omega(L^X)$.

Suppose that there exists $f \in \Omega(L^X)$ such that $D_T(f) \not\leq D(f)$. Then, by the definition of $D_T$, there exists a family $\{ f_{\rho_i} | \land_{i=1}^n f_{\rho_i} \leq f \}$ such that

$$\land_{i=1}^n T(\rho_i) \not\leq D(f).$$

Put $D(f) = r$. Since $T_D = T$ and $L$ is a chain, $T_D(\rho_i) > r$ for all $i = 1, \cdots, n$, that is, $I_D(\rho_i, r) = \rho_i$. From the definition of $I_D(\rho_i, r)$, for
each $i = 1, \ldots, n$ there exist $g_i \in \Omega(L^X)$ with $D(g_i) > r$ and $\mu_i \in L^X$
with $g_i(\mu_i) \leq \rho_i$ such that $\mu_i \leq I_D(\rho_i, r) = \rho_i$. Thus, by the definition
of $f_{\rho_i}, g_i \leq f_{\rho_i}$ for all $i = 1, \ldots, n$. Put $g = \wedge_{i=1}^n g_i$. We have $D(g) > r$.
Hence $f \geq g$ and $D(f) \geq D(g) > r$. Thus, $D(f) > r$. It is a contradiction.
Therefore, $D_T(f) \leq D(f)$, for all $f \in \Omega(L^X)$.

**Corollary 3.6.6**: Since $L = [0, 1]$ is a completely distributive and order
dense chain every $[0, 1]$ - fuzzy topological space is Hutton $[0, 1]$ quasi-
uniformizable.
3.7. Relation Between Hutton Uniformities and Hutton $L$-uniformities and $L$-Uniformities

Relation between Hutton uniformities and Hutton $L$-uniformities.

**Theorem 3.7.1**: Let $D$ be a Hutton uniformity on $X$. Then the mapping $D : \Omega(L^X) \to L$ defined by each $e \in \Omega(L^X)$ $D(e) = \vee\{\alpha \in L|\alpha \wedge e \in D\}$ is a Hutton $L$-uniformity on $X$.

**Proof**: 

$HLU_1)$ Since $D \neq \phi$ there exist $e \in D \subset \Omega(L^X)$ $D(e) = \vee\{\alpha \in L|\alpha \wedge e \in D\} = 1$.

$HLU_2)$ $D(e) \wedge D(h)$

\[ = \vee\{\alpha \in L|\alpha \wedge e \in D\} \wedge \vee\{\beta \in L|\beta \wedge h \in D\} \]

\[ \leq \vee\{\alpha \wedge \beta|(\alpha \wedge \beta) \wedge (e \wedge h) \in D\} \]

\[ \leq \vee\{\gamma|\gamma \wedge (e \wedge h) \in D\} \]

\[ = D(e \wedge h). \]

$HLU_3)$ Let $e \leq h$, $D(e) = \alpha$ then $e \wedge \alpha \in D$ it is trivial $h \wedge \alpha \in D$ then $\alpha \leq \vee\{\beta \in L|h \wedge \beta \in D\}$ then $D(e) \leq D(h)$.

$HLU_4)$ $D(e^{-1}) = \vee\{\alpha \in L|e^{-1} \wedge \alpha \in D\} = \vee\{\alpha \in L|e \wedge \alpha \in D\} = D(e)$.

$HLU_5)$ Let $e \in \Omega(L^X)$, $D(e) = r$ by definition of $D e \wedge r \in D$ by $(HU_5)$ there exist $h \in D$ $hoh \leq e \wedge r$ then $hoh \leq e$ and since $D(h) = 1$ it is trivial $D(h) \geq D(e)$.

**Theorem 3.7.2**: Let $(X, D)$ be an $L$-uniform space. Then for each $r \in L$, the family $D_r = \{e \in \Omega(L^X)|D(e) > r\}$ is a Hutton uniformity on $X$. 

**Proof:**

HU$_1$) It is trivial from (HLU$_1$).

HU$_2$) Let $e, h \in D_r$ then $D(e) \geq r, D(h) \geq r$ from (HLU$_2$)

HU$_3$) Let $e \in D_r, e \leq h$ then $D(e) \geq r$ from (HLU$_3$), $D(h) \geq r$ then $h \in D_r$.

HU$_4$) Let $e \in D_r, D(e) \geq r$ from (HLU$_4$) $D(e^{-1}) \geq r$ then $e^{-1} \in D_r$.

HU$_5$) Let $e \in D_r, D(e) \geq r$ from (HLU$_5$) there exist $h \in \Omega(L^X)$ s.t. $D(h) \geq r$ and $hoh \leq e$ consequently there is $h \in D_r$ and $hoh \leq e$.

Hence, $D_r$ is a Hutton uniformity on $X$ for all $r \in L$.

**Definition 3.7.3**: For each $r \in L$, the family $D_r$ is called $r$-level (Hutton uniformity) on $X$ w.r.t the Hutton $L$-uniformity $D$.

**Relation between Hutton $L$-uniformities and $L$ - uniformities.**

Relation between Hutton L-uniformities and L - uniformities is established in[26]. We are establishing the similar relation for $(L, \leq, \wedge, ')$.

We can define $\Lambda : L^X \times X \rightarrow \Omega(L^X)$ where $\Lambda(d)$ is defined for each $a \in L^X$

by $[\Lambda(d)](a) = \vee_{x \in X} a(x) \wedge d(x, -)$ and for each $\alpha \in L$ and $x \in X$

$$[\Lambda(d)](\alpha \cdot 1_{\{x\}}) = \alpha \wedge d(x, -) = \alpha \wedge [\Lambda(d)](1_{\{x\}})$$

$\Lambda$ preserves arbitrary sups

$$[\Lambda(\vee_{i \in J} d_i)](a) = \vee_{i \in J} [\Lambda(d_i)](a)$$

we can define $\gamma : \Omega(L^X) \rightarrow L^X \times X$ where $\gamma$ is defined for each $\phi \in \Omega(L^X)$

by

$$\gamma(\phi) = \vee \{ d \in L^X \times X : \Lambda(d) \leq \phi \}$$
\( \Lambda(d) \leq \phi \iff \forall \alpha \in L, \forall x \in X[\Lambda(d)](\alpha \cdot 1_{\{x\}}) \leq \phi(\alpha \cdot 1_{\{x\}}) \)
\( \iff \forall \alpha \in L, \forall x \in X \alpha \land d(x, -) \leq \phi(\alpha \cdot 1_{\{x\}}) \)
\( \iff \forall x \in X^i \ d(x, -) \leq \land_{\alpha \in L} \alpha' \lor \phi(\alpha \cdot 1_{\{x\}}). \)

So
\[
\gamma(\phi)(x, y) = \land_{\alpha \in L} \alpha' \lor [\phi(\alpha \cdot 1_{\{x\}})](y)
\]
for each \( x, y \in X \).

**Remark 3.7.4**: A join preserving mapping \( \phi \) is completely determined by the collection of \( L \)-sets
\[
\{ \phi(\alpha.1_{\{x\}}) : \alpha \in L, x \in X \}
\]
since any element \( \phi \in \Omega(L^X) \) satisfies the equality \( \phi(a) = \lor \{\phi(a(x)1_{\{x\}}) : x \in X\} \) for each \( a \in L^X \).

**Proposition 3.7.5 (Properties of \( \Lambda \))**: Let \( d, d_1, d_2 \in L^{X \times X} \) and \( \alpha \in L \) then

1. \( \Lambda(1_{\phi}) = \phi_0, \phi_0(a) = 1_{\phi} \quad \Lambda(1_{X \times X}) = \phi_1 \quad \phi_1(a) = 1_X. \)
2. \( \Lambda(d_1 \land d_2) \geq \Lambda(d_1) \land \Lambda(d_2) \)
3. \( \Lambda(1_\Delta) = 1_{L^X} \)
4. \( [\Lambda(d)]^{-1} = \Lambda(d^{-1}) \)
5. \( \Lambda(d_2od_1) = \Lambda(d_2)o\Lambda(d_1) \)
6. \( \Lambda(\alpha \land d) = \alpha \land \Lambda(d) \)

**Proof**: (1) Let \( \alpha \in L, x \in X \) then
\[
\Lambda(1_{\phi})(a \cdot 1_{\{x\}}) = \alpha \land 1_{\phi}(x, -) = 1_{\alpha} = \phi_0(\alpha \cdot 1_{\{x\}})
\]
\( \Lambda(1 \times X)(\alpha \cdot 1_{\{x\}}) = \alpha \wedge 1_{X} = \phi_{1}(\alpha \cdot 1_{\{x\}}). \)

(2) \[ \Lambda(d_{1} \land d_{2})(a_{1} \land a_{2}) = \forall_{x \in X}(a_{1} \land a_{2})(x) \land (d_{1} \land d_{2})(x, -) \]
\[ \leq \forall_{x \in X}(a_{1}(x) \land d_{1}(x, -)) \land (a_{2}(x) \land d_{2}(x, -)) \]
\[ \leq (\forall_{x \in X}a_{1}(x) \land d_{1}(x, -)) \land (\forall_{x \in X}a_{2}(x) \land d_{2}(x, -)) \]
\[ = [\Lambda(d_{1})](a_{1}) \land [\Lambda(d_{2})](a_{2}) \quad \text{and so} \quad \Lambda(d_{1} \land d_{2}) \leq \Lambda(d_{1}) \land \Lambda(d_{2}). \]

(3) Let \( \alpha \in L \) and \( x \in X \)
\[ [\Lambda(1_{\Delta})](\alpha \cdot 1_{\{x\}}) = \alpha \land 1_{\Delta}(x, -) = \alpha \cdot 1_{\{x\}} = 1_{L_{X}}(\alpha \cdot 1_{\{x\}}). \]

(4) Let \( \alpha \in L, x \in X \)
\[ [\Lambda(d)^{-1}](\alpha \cdot 1_{\{x\}}) = \wedge \{a \in L^{X} ||\Lambda(d)|| (a') \leq (\alpha \cdot 1_{\{x\}}') \} \]
\[ = \wedge \{a \in L^{X} | \forall_{z \in X} a'(z) \land d(z, -) \leq (\alpha \cdot 1_{\{x\}}') \} \]
\[ = \wedge \{a \in L^{X} | \alpha \land a'(z) \land d(z, x) = 0 \ \forall_{z \in X} \} \]
\[ = \wedge \{a \in L^{X} | \alpha \land d(-, x) \leq a \} \]
\[ = \alpha \land d(-, x) \]
\[ = \alpha \land d^{-1}(x, -) = [\Lambda(d^{-1})](\alpha \cdot 1_{\{x\}}). \]

(5) Let \( \alpha \in L \) and \( x \in X \), then
\[ [\Lambda(d_{2} \circ d_{1})](\alpha \cdot 1_{\{x\}}) = \alpha \land (d_{2} \circ d_{1})(x, -) \]
\[ = \forall_{z \in X} \alpha \land d_{1}(x, z) \land d_{2}(x, z) \]
\[ = \forall_{z \in X} [\alpha \land d_{1}(x, -)](z) \land d_{2}(z, -) \]
\[ = [\Lambda(d_{2})][\Lambda(d_{1})](\alpha \cdot 1_{\{x\}}) \]
\[ = [\Lambda(d_{2}) \circ \Lambda(d_{1})](\alpha \cdot 1_{\{x\}}). \]
(6) Let $\alpha, \beta \in L, x \in X$ and $d \in L^{X \times X}$ then

$$[\Lambda(\alpha \wedge d)](\beta \cdot 1_{\{x\}}) = \beta \wedge ((\alpha \wedge d)(x, -)) = \alpha \wedge [\Lambda(d)](\beta \cdot 1_{\{x\}}).$$

**Proposition 3.7.6 (Properties of $\gamma$):** Let $\phi, \phi_1, \phi_2 \in \Omega(L^X)$ and $\alpha \in L$, then

1. $\gamma(\phi_1) = 1_{X \times X}$ and $\gamma(\phi_2) = 1_{\phi}$
2. $\gamma(\phi_1) \wedge \gamma(\phi_2) \leq \gamma(\phi_1 \wedge \phi_2)$
3. $\gamma(1_L) = 1_{\Delta}$
4. $[\gamma(\phi)]^{-1} = \gamma(\phi^{-1})$
5. $\gamma(\phi_2) \circ \gamma(\phi_1) \leq \gamma(\phi_2 \circ \phi_1)$
6. $\alpha \wedge \gamma(\phi) \leq \gamma(\alpha \wedge \phi)$.

**Proof:**

1. $\Lambda(d) \leq \phi_1$ for each $d \in L^{X \times X}$ and $\Lambda(d) \leq \phi_0$ if and only if $d = 1_{\phi}$.
2. Let $d_1, d_2 \in L^{X \times X}$ such that $\Lambda(d_1) \leq \phi_1, \Lambda(d_2) \leq \phi_2$. It follows from previous proposition $\Lambda(d_1 \wedge d_2) \leq \Lambda(d_1) \wedge \Lambda(d_2) \leq \phi_1 \wedge \phi_2$.

Since $\wedge$ distributes over arbitrary sups

$$\gamma(\phi_1) \wedge \gamma(\phi_2) = \vee\{d_1 \wedge d_2|\Lambda(d_1) \leq \phi_1 \text{ and } \Lambda(d_2) \leq \phi_2\}$$

$$\leq \vee\{d \in L^{X \times X}|\Lambda(d) \leq \phi_1 \wedge \phi_2\} = \gamma(\phi_1 \wedge \phi_2).$$

3. It follows from previous proposition.

4. 

$$[\gamma(\phi)]^{-1} = (\vee\{d \in L^{X \times X}|\Lambda(d) \leq \phi\})^{-1}$$

$$= \vee\{d \in L^{X \times X}|\Lambda(d^{-1}) \leq \phi\}$$
\[
\begin{align*}
\forall \{ d \in L^{X \times X} | \Lambda(d)^{-1} \leq \phi \} \\
\forall \{ d \in L^{X \times X} | \Lambda(d) \leq \phi^{-1} \} \\
= \gamma(\phi^{-1}).
\end{align*}
\]

(5) Let \( \phi_1, \phi_2 \in \Omega(L^X) \) by previous proposition

\[
\Lambda(\gamma(\phi_2) \circ \gamma(\phi_1)) = \Lambda(\gamma(\phi_2)) \circ \Lambda(\gamma(\phi_1)) \leq \phi_2 \circ \phi_1
\]

and so \( \gamma(\phi_2) \circ \gamma(\phi_1) \leq \gamma(\phi_2 \circ \phi_1) \).

(6) Let \( \phi \in \Omega(L^X) \) and \( \alpha \in L \)

\[
\Lambda(\alpha \land \gamma(\phi)) = \alpha \land \Lambda(\gamma(\phi)) \leq \alpha \land \phi.
\]

**Theorem 3.7.7**: Let \( \xi \) be an \( L \)-uniformity on \( X \). Then the mapping \( \mathcal{D}^\xi : \Omega(L^X) \to L \) defined by \( \mathcal{D}^\xi = (\gamma)^{-1}(\xi) = \xi \circ \gamma \) is a Huttan \( L \)-uniformity on \( X \).

**Proof**: \( \Omega(L^X) \xrightarrow{\gamma} L^{X \times X} \xrightarrow{\xi} L \).

**HLU_1**

\[
\mathcal{D}^\xi(\phi_0) = \xi(\gamma(\phi_0)) = \xi(1_\emptyset) = 0
\]

\[
\mathcal{D}^\xi(\phi_1) = \xi(\gamma(\phi_1)) = \xi(1_{X \times X}) = 1.
\]

**HLU_2** Let \( \phi_1, \phi_2 \in \Omega(L^X) \)

\[
\mathcal{D}^\xi(\phi_1) \land \mathcal{D}^\xi(\phi_2) = \xi(\gamma(\phi_1)) \land \xi(\gamma(\phi_2)) \\
\leq \xi(\gamma(\phi_1) \land \gamma(\phi_2)) \\
\leq \xi(\gamma(\phi_1 \land \phi_2)) \\
= \mathcal{U}^\xi(\phi_1 \land \phi_2).
\]
It follows from \((LU_3)\)

\[ D^\xi(\phi) = \xi(\gamma(\phi)) \leq \xi([\gamma(\phi)]^{-1}) = \xi(\gamma(\phi^{-1})) = D^\xi(\phi^{-1}) \]

\(HLU_5\)

\[ D^\xi(\phi) = \xi(\gamma(\phi)) \leq \vee\{\xi(d)|d \in L^{X \times X}, d \odot d \leq \gamma(\phi)\} \]

\[ = \vee\{D^\xi(\Lambda(d))|d \in L^{X \times X}, \Lambda(d) \odot \Lambda(d) \leq \phi\} \]

\[ = \vee\{D^\xi(\Lambda(d))|d \in L^{X \times X}, \Lambda(d) \odot \Lambda(d) \leq \phi\} \]

\[ \leq \vee\{D^\xi(\phi_1)|\phi_1 \in \Omega(L^X), \phi_1 \phi_2 \leq \phi\}. \]

**Theorem 3.7.8**: Let \((X, \xi)\) and \((X', \xi')\) be two \(L\)-uniform spaces. A mapping \(\phi: (X, \xi) \to (X', \xi')\) is \(L\)-uniformly continuous iff \(\phi: (X, D^\xi) \to (X', D^{\xi'})\) is \(L\)-uniformly continuous.

**Proof**: Let \(\phi: (X, \xi) \to (X', \xi')\) be \(L\) - uniformly continuous and \(\psi \in \Omega(L^X)\) then for all \(x_1, x_2 \in X\)

\[ [\gamma(\phi^{-}\psi)](x_1, x_2) = \land_{\alpha \in L} \lor [\phi^{-}\psi(\alpha \cdot 1_{x_1})](x_2) \]

\[ = \land_{\alpha \in L} \lor [\phi(\psi(\phi(\alpha \cdot 1_{x_1})))](x_2) \]

\[ = \land_{\alpha \in L} \lor [\psi(\alpha \cdot 1_{\phi(x_1)})](\phi(x_2)) \]

\[ = [\gamma(\psi)](\phi(x_1), \phi(x_2)) \]

\[ = [\gamma(\psi)](\phi(x_1), \phi(x_2)). \]

Consequently

\[ \gamma(\phi^{-}\psi) = (\phi \times \phi)^{-1}(\gamma(\psi)) \]

and

\[ D^\xi(\phi^{-}\psi) = \xi(\gamma(\phi^{-}\psi)) = \xi(\phi \times \phi)^{-1}(\gamma(\psi)) \geq \xi'(\gamma(\psi)) = D^{\xi'}(\psi). \]
**Theorem 3.7.9**: Let \( \mathcal{D} \) be a Hutton \( L \)-uniformity on \( X \) satisfying \( \mathcal{D}(\Lambda(\gamma(\phi))) = \mathcal{D}(\phi) \) for each \( \phi \in \Omega(L^X) \) then mapping \( \xi^D : L^{X \times X} \to L \) defined by \( \xi^D = \mathcal{D} \circ \Lambda \) is an \( L \)-uniformity on \( X \).

**Proof**: 

1. \( LU_1 \) \quad \( \xi^D(1_o) = \mathcal{D}(\Lambda(1_o)) = \mathcal{D}(\phi_0) = 0 \)

2. \( LU_2 \) \quad \( \xi^D(1_{X \times X}) = \mathcal{D}(\Lambda(1_{X \times X})) = \mathcal{D}(\Lambda(\phi_1)) = \mathcal{D}(\phi_1) = 1 \)

3. \( LU_3 \) \quad Let \( d_1, d_2 \in L^{X \times X} \) we have

   \[
   \xi^D(d_1) \land \xi^D(d_2) = \mathcal{D}(\Lambda(d_1)) \land \mathcal{D}(\Lambda(d_2)) \\
   \leq \mathcal{D}(\Lambda(d_1) \land \Lambda(d_2)) \\
   = \mathcal{D}(\Lambda(d_1 \land d_2)) \\
   = \xi^D(d_1 \land d_2).
   \]

4. \( LU_4 \) \quad It follows from \( GU_3 \).

5. \( LU_5 \) \quad \( \xi^D(d) = \mathcal{D}(\Lambda(d)) \leq \mathcal{D}([\Lambda(d)]^{-1}) = \mathcal{D}(\Lambda(d^{-1})) = \xi^D(d^{-1}). \)

6. \( LU_6 \) \quad \( \xi^D(d) = \mathcal{D}(\Lambda(d)) \leq \vee\{\mathcal{D}(\phi) | \phi \in \Omega(L^X), \phi \circ \phi \leq \Lambda(d)\} \)

   \[
   = \vee\{\mathcal{D}(\Lambda(\gamma(\phi))) | \phi \in \Omega(L^X), \phi \circ \phi \leq \Lambda(d)\} \\
   = \vee\{\xi^D(\gamma(\phi)) | \phi \in \Omega(L^X), \gamma(\phi \circ \phi) \leq d\} \\
   \leq \vee\{\xi^D(\gamma(\phi)) | \phi \in \Omega(L^X), \gamma(\phi) \circ \gamma(\phi) \leq d\} \\
   \leq \vee\{\xi^D(d_1) | d_1 \in L^{X \times X}, d_1 \circ d_1 \leq d\}.
   \]
Theorem 3.7.10: Let $(X, D)$ and $(X', D')$ be two Hutton $L$-uniform spaces. A mapping $\phi : (X, D) \to (X', D')$ is $L$-uniformity continuous then $\phi : (X, \xi^D) \to (X', \xi^{D'})$ is $L$-uniformly continuous.

Proof: Let $\phi : (X, D) \to (X', D')$ be $L$-uniformly continuous and $e \in L^{X \times X'}$. Then for each $\alpha \in L, x_1, x_2 \in X$

\[
[\phi^\succeq(\Lambda(e))][\alpha \cdot 1_{\{x_1\}}](x_2) = [\phi^\prec(\Lambda(e))](\phi^\prec(\alpha \cdot 1_{\{x_1\}}))(x_2) \\
= [\Lambda(e)](\alpha \cdot 1_{\{\phi(x_1)\}})(\phi(x_2)) = \alpha \land e(\phi(x_1), \phi(x_2)) \\
= \alpha \land [(\phi \times \phi)^{-1}(e)](x_1, x_2) = [[\Lambda((\phi \times \phi)^{-1}(e))](\alpha \cdot 1_{\{x_1\}})](x_2).
\]

Consequently,

\[
\Lambda((\phi \times \phi)^{-1}(e)) = \phi^\succeq(\Lambda(e)) \\
\xi^D((\phi \times \phi)^{-1}(e)) = D(\Lambda((\phi \times \phi)^{-1}(e))) \\
= D(\phi^\succeq(\Lambda(e))) \geq D'(\Lambda(e)) = \xi^{D'}(e).
\]