3.1 Introduction

Recently various authors have introduced several $q$-type distributions such as $q$-exponential, $q$-Weibull, $q$-logistic and various pathway models in the context of information theory, statistical mechanics, reliability modeling etc. The Weibull distribution, including the exponential distribution plays an important role in reliability theory, survival analysis and modeling time series data of non-negative random variables such as hydrological data, wind velocity data etc. [see Barlow and Proschan (1981), Brown et al (1984)]. Such data need long tailed marginal distributions. Marshall and Olkin (1997) introduced a new family of survival functions which is constructed by adding a new parameter to an existing distribution. The introduction of a new parameter will result in flexibility in the distribution.

Some results in this chapter were presented in the 7th World Congress in Probability and Statistics, National University of Singapore during 14-19 July 2008. See also Jose et al (2008).
Jose and Alice (2001), Ghitany et al (2005) conducted detailed studies of Marshall-Olkin Weibull distribution. Different first order autoregressive (AR(1)) time series models can be developed using the Marshall-Olkin Weibull distribution which are very useful in the present scenario to study global warming due to ozone depletion and the catastrophes associated with these problems. It is also useful to model time series data on mortality rates due to the spreading of diseases and unexpected natural calamities. Now we shall extend the $q$-Weibull distributions to obtain a more general class called Marshall-Olkin $q$-Weibull distribution.

In this chapter, we introduce the Marshall-Olkin $q$-Weibull distribution. The shape properties of the probability density functions and the hazard rate functions are discussed. We fit the Marshall-Olkin $q$-Weibull distribution to data on cancer remission times and compare the results with respect to the $q$-Weibull distribution. It is shown that the new distribution is a good fit. We develop two different types of AR(1) minification models and max-min processes and the sample path properties are studied [Jose et al (2008b), Naik et al (2008)]. A case study is reported in which the new process is applied to a data on daily discharge of Neyyar river in Kerala, India.

### 3.2 Marshall-Olkin $q$-Weibull Distribution

Let $\bar{F}(x)$ be the survival function of a given distribution. Then the Marshall-Olkin distribution is obtained by introducing a parameter $p$ such that the new survival function is

$$
\bar{G}(x) = \frac{p\bar{F}(x)}{1 - (1-p)\bar{F}(x)}, \quad -\infty < x < \infty, \quad 0 < p < \infty.
$$

Clearly when $p = 1$ we get the standard form of the distribution functions. Corresponding density function is given by

$$
g(x) = \frac{p f(x)}{[1 - (1-p)\bar{F}(x)]^2}, \quad -\infty < x < \infty, \quad 0 < p < \infty
$$

and the hazard rate function is given by

$$
\lambda(x) = \frac{p f(x)}{[1 - (1-p)\bar{F}(x)]^2}, \quad -\infty < x < \infty, \quad 0 < p < \infty
$$
\[ h(x) = \frac{h_F(x)}{1 - (1 - p)\bar{F}(x)}, \quad -\infty < x < \infty, \quad 0 < p < \infty \]

where \( h_F(x) \) denote the hazard rate function of the original model with distribution function \( F \). Now consider the two parameter \( q \)-Weibull distribution (discussed in Section 2.2) with the survival function \( \bar{F}_1(x) = [1 + (q - 1)(\lambda x)^{\alpha}]^{\frac{q-2}{q-1}}, \quad x > 0, \ 1 < q < 2, \ \alpha, \lambda > 0 \). Substituting \( \bar{F}_1(x) \) in \( \bar{G}(x) \), we get a new family of distributions called Marshall-Olkin \( q \)-Weibull distribution (MO-\( q \)-W), whose survival function is given by

\[ \bar{G}_1(x; p, q, \alpha, \lambda) = p \frac{\bar{F}_1(x)}{1 - (1 - p)\bar{F}(x)} = \frac{p [1 + (q - 1)(\lambda x)^{\alpha}]^{\frac{q-2}{q-1}}}{1 - (1 - p)[1 + (q - 1)(\lambda x)^{\alpha}]^{\frac{q-2}{q-1}}}. \quad (3.2.1) \]

Then the p.d.f. corresponding to \( G_1(\cdot) \) is given by

\[ g_1(x; p, q, \alpha, \lambda) = \frac{p \alpha \lambda^{\alpha} (2 - q) x^{\alpha-1} [1 + (q - 1)(\lambda x)^{\alpha}]^{-\frac{1}{q-1}}}{1 - (1 - p)[1 + (q - 1)(\lambda x)^{\alpha}]^{\frac{q-2}{q-1}}}, \quad x > 0, \ 1 < q < 2. \]

Similarly when \( q < 1 \), the p.d.f. of the MO-\( q \)-W distribution is given by

\[ g_2(x; p, q, \alpha, \lambda) = \frac{p \alpha \lambda^{\alpha} (2 - q) x^{\alpha-1} [1 - (1 - q)(\lambda x)^{\alpha}]^{-\frac{1}{q-1}}}{1 - (1 - p)[1 - (1 - q)(\lambda x)^{\alpha}]^{\frac{2-q}{2-q}}}, \quad 0 < x < (\lambda(1 - q)^{\frac{1}{\alpha}})^{-1}. \]

Now we examine the shape properties of the p.d.f. \( g_1 \). Let

\[ s(x) = \alpha - 1 + (\alpha q - 2\alpha - q + 1)\lambda^{\alpha} x^{\alpha} + (1 - p) [(\alpha q - 2\alpha + q - 1)\lambda^{\alpha} x^{\alpha} - \alpha + 1] \psi_1(x)^{\frac{q-2}{q-1}}, \]

where \( \psi_1(x) = 1 + (q - 1)\lambda^{\alpha} x^{\alpha} \). Then the p.d.f. \( g_1 \) has the following properties.

1. If \( \alpha > 1 \), then \( g_1 \) is a unimodal function with mode at \( x_0 \), where \( x_0 \) is the solution of the equation \( s(x_0) = 0 \). Also, \( g_1(0) = g_1(\infty) = 0 \).
2. If $\alpha < 1$ and $s$ has two real roots $x_1 < x_2$, then $g_1$ decreases at $(0, x_1] \cup (x_2, \infty)$ and increases at $(x_1, x_2]$. Furthermore, $g_1(0) = \infty$ and $g_1(\infty) = 0$.

3. If $\alpha < 1$ and $s < 0$, then $g_1$ is a decreasing function with $g_1(0) = \infty$ and $g_1(\infty) = 0$.

Figure 3.1 shows possible shapes of the p.d.f. $g_1$.

The corresponding hazard rate function for $1 < q < 2$ is given by

$$h_1(x) = \frac{\alpha \lambda^\alpha (2 - q) x^{\alpha - 1}[1 + (q - 1)\lambda x^\alpha]^\alpha}{1 - (1 - p)[1 + (q - 1)\lambda x^\alpha]^\alpha}.$$

The possible shapes of the hazard rate function $h_1$ for various combinations of the parameter values are given in Figure 3.2.

Let us consider the shape properties of the hazard rate function $h_1$. Let

$$u(x) = \frac{(\alpha - \psi_1(x))(q - 1) - \alpha(1 - p)(2q - 3)\psi_1(x)^{q - 2}}{1 - (1 - p)(q - 1 + \alpha q - 2\alpha)\psi_1(x)^{q - 1}}.$$
and

$$v(x) = (\alpha - \psi_2(x))(1 - q) + (\alpha q - 2\alpha - q + 1)(1 - p)\psi_2(x)^{-\frac{1}{1-q}}$$

$$+ \alpha(1 - p)\psi_2(x)^{\frac{q-2}{1-q}}.$$ 

The following cases are possible for $h_1$:

1. If $\alpha < 1$ and $u < 0$, then $h_1$ is a decreasing function with $h_1(0) = \infty$ and $h_1(\infty) = 0$.

2. If $\alpha < 1$ and $u$ has two real roots $x_1 < x_2$, then $h_1$ decreases at $(0, x_1] \cup (x_2, \infty)$ and increases at $(x_1, x_2]$. Furthermore, $h_1(0) = \infty$ and $h_1(\infty) = 0$.

3. If $\alpha > 1$ and $u$ has unique root $x_0$, then $h_1$ increases at $(0, x_0]$ and decreases at $(x_0, \infty)$ subject to the condition $h_1(0) = h_1(\infty) = 0$.

4. If $\alpha > 1$ and $u$ has three real roots $x_1 < x_2 < x_3$, then $h_1$ increases at $(0, x_1] \cup (x_2, x_3]$ and decreases at $(x_1, x_2] \cup (x_3, \infty)$ with $h_1(0) = h(\infty) = 0$.

**Theorem 3.2.1.** Let $\{X_i, i \geq 1\}$ be a sequence of i.i.d. random variables with common survival function $\bar{F}(x)$ and let $N$ be a geometric random variable independently distributed of $\{X_i\}$ such that $P[N = n] = \theta(1 - \theta)^{n-1}$, $n = 1, 2, \ldots, 0 < \theta < 1$, which is for all $i \geq 1$. Let $U_N = \min_{1 \leq i \leq N} X_i$. Then $\{U_N\}$ is distributed as MO-$q$-W iff $\{X_i\}$ follows $q$-Weibull distribution.

**Proof:** The survival function of the random variable $U_N$ is

$$\bar{H}(x) = P(U_N > x) = \theta \sum_{n=1}^{\infty} [\bar{F}(x)]^n(1 - \theta)^{n-1} = \frac{\theta \bar{F}(x)}{1 - (1 - \theta) \bar{F}(x)}.$$ 

If $X_i$ has the survival function of the $q$-Weibull distribution given by (3.2.1), then $U_N$ has the survival function of the MO-$q$-W distribution. The converse easily follows.
Figure 3.2: The hazard rate function of the Marshall-Olkin \(q\)-Weibull distribution for \(\lambda = 1\) and (a) \(p = 0.1, q = 1.1, \alpha = 2\), (b) \(p = 0.2, q = 1.6, \alpha = 1.2\), (c) \(p = 3, q = 1.2, \alpha = 0.9\), (d) \(p = 2, q = 1.2, \alpha = 0.2\).

Remark 3.2.1. When \(1 < q < 2\), the log likelihood function for a sample of size \(n\), say \((x_1, x_2, \ldots, x_n)\) can be obtained as

\[
\log L(p, q, \alpha, \lambda) = n \log p + n \log \alpha + n \log \lambda + n \log(2 - q) + (\alpha - 1) \sum_{i=1}^{n} \log x_i
\]

\[
-\frac{1}{q-1} \sum_{i=1}^{n} \log \psi_1(x_i) - 2 \sum_{i=1}^{n} \log \left(1 - (1 - p)\psi_1(x_i)^{\frac{q-2}{q-1}}\right).
\]

Then the normal equations become

\[
\frac{\partial \log L}{\partial p} = \frac{n}{p} - 2 \sum_{i=1}^{n} \frac{\psi_1(x_i)}{\psi_1(x_i)^{\frac{q-2}{q-1}}} = 0
\]

\[
\frac{\partial \log L}{\partial \alpha} = \frac{n}{\alpha} + n \log \lambda + \sum_{i=1}^{n} \log x_i - \lambda^n \sum_{i=1}^{n} \frac{x_i^{\alpha} \log(\lambda x_i)}{\psi_1(x_i)}
\]

\[
+ 2(1 - p)(q - 2)\lambda^n \sum_{i=1}^{n} \frac{x_i^{\alpha} \log(\lambda x_i) \psi_1(x_i)^{-\frac{1}{q-1}}}{1 - (1 - p)\psi_1(x_i)^{\frac{q-2}{q-1}}} = 0
\]
\[ \frac{\partial \log L}{\partial \lambda} = \frac{n\alpha}{\lambda} - \alpha \lambda^{\alpha - 1} \sum_{i=1}^{n} \frac{x_i^\alpha}{\psi_1(x_i)} + 2(1-p)(q-2)\alpha \lambda^{\alpha - 1} \times \sum_{i=1}^{n} \frac{x_i^\alpha \psi_1(x_i)^{-\frac{1}{q-1}}}{1 - (1-p)\psi_1(x_i)^{\frac{1}{q-1}}} = 0 \]

\[ \frac{\partial \log L}{\partial q} = \frac{n}{q - 2} + \frac{1}{(q-1)^2} \sum_{i=1}^{n} \log \psi_1(x_i) - \frac{\lambda^{\alpha}}{q - 1} \sum_{i=1}^{n} \frac{x_i^\alpha}{\psi_1(x_i)} + 2(1-p) \sum_{i=1}^{n} \frac{\psi_1(x_i)^{\frac{2q-3}{q-1}}}{1 - (1-p)\psi_1(x_i)^{\frac{1}{q-1}}} \left\{ \frac{\log \psi_1(x_i)}{(q-1)^2} + \frac{(q-2)\lambda^{\alpha}x_i^\alpha}{(q-1)\psi_1(x_i)} \right\} = 0. \]

The normal equation can be derived similarly for \( q < 1 \).

The maximum likelihood estimates of the unknown parameters \( p, q, \alpha \) and \( \lambda \) can be obtained by using the function nlm in the statistical software R.

**Remark 3.2.2.** When \( q < 1 \), similar results can be obtained by proceeding with \( \bar{F}_2(x) \) instead of \( \bar{F}_1(x) \).

### 3.3 Data Analysis

In this section we analyze a data set (Table 3.1) and compare Marshall-Olkin \( q \)-Weibull distributions with \( q \)-Weibull distributions.

**Data set:** The ordered remission times (in months) of a random sample of 142 bladder cancer patients (Lee and Wang (2003)) reveals that the \( q \)-Weibull distribution gives a better fit for the survival data. Four distributions namely two \( q \)-Weibull distributions with parameters \( q, \alpha \) and \( \lambda \) and two Marshall-Olkin \( q \)-Weibull distributions with parameters \( p, q, \alpha \) and \( \lambda \) are fitted to the data. The results are presented in Table 3.2. The QQ and PP plots given in Figure 3.3 also revealed that the Marshall-Olkin \( q \)-Weibull distributions are more appropriate models for the data set.
Table 3.1: Remission times (months) of 142 bladder cancer patients.

| Time (months) | 0.08 | 0.20 | 0.40 | 0.50 | 0.51 | 0.81 | 0.87 | 0.90 | 1.05 | 1.19 | 1.26 | 1.35 | 1.40 | 1.46 | 2.46 | 2.54 | 2.62 | 2.64 | 2.69 | 2.75 | 2.83 | 2.87 | 3.02 | 3.25 | 3.31 | 3.36 | 3.36 | 3.48 | 3.52 | 3.57 | 3.64 | 3.70 | 3.82 | 3.88 | 4.18 | 4.23 | 4.26 | 4.33 | 4.33 | 4.34 | 4.40 | 4.50 | 4.51 | 4.65 | 4.70 | 4.87 | 4.98 | 5.06 | 5.09 | 5.17 | 5.32 | 5.32 | 5.34 | 5.41 | 5.41 | 5.49 | 5.62 | 5.71 | 5.85 | 6.25 | 6.54 | 6.76 | 6.93 | 6.94 | 6.97 | 7.09 | 7.26 | 7.32 | 7.39 | 7.59 | 7.62 | 7.28 | 7.32 | 7.39 | 7.59 | 7.62 | 7.63 | 7.66 | 7.87 | 7.93 | 8.26 | 8.37 | 8.53 | 8.60 | 8.65 | 8.66 | 9.02 | 9.22 | 9.47 | 9.74 | 10.06 | 10.34 | 10.66 | 10.75 | 10.86 | 11.25 | 11.64 | 11.79 | 11.98 | 12.02 | 12.03 | 12.07 | 12.69 | 13.11 | 13.29 | 13.80 | 14.24 | 14.76 | 14.77 | 14.83 | 15.96 | 16.62 | 17.12 | 17.14 | 17.36 | 18.10 | 19.13 | 19.36 | 20.28 | 21.73 | 22.69 | 23.63 | 24.80 | 25.74 | 25.82 | 26.31 | 32.15 | 34.26 | 36.66 | 43.01 | 46.12 | 79.05 |

Table 3.2: Estimated values, log-likelihood, Kolmogorov-Smirnov statistics and p-value for data set.

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Estimates</th>
<th>(-\log L)</th>
<th>K-S</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>q-Weibull, (1 &lt; q &lt; 2)</td>
<td>(\hat{q} = 1.3285), (\hat{\alpha} = 1.4718), (\hat{\lambda} = 0.1805)</td>
<td>452.6269</td>
<td>0.0426</td>
<td>0.9590</td>
</tr>
<tr>
<td>q-Weibull, (0 &lt; q &lt; 1)</td>
<td>(\hat{q} = 0.0434), (\hat{\alpha} = 1.4718), (\hat{\lambda} = 0.0873)</td>
<td>452.6271</td>
<td>0.0424</td>
<td>0.9602</td>
</tr>
<tr>
<td>Marshall-Olkin q-Weibull, (1 &lt; q &lt; 2)</td>
<td>(\hat{p} = 1.6954), (\hat{\alpha} = 1.4452), (\hat{\lambda} = 0.2508), (\hat{q} = 0.9621)</td>
<td>452.5417</td>
<td>0.0422</td>
<td>0.9621</td>
</tr>
<tr>
<td>Marshall-Olkin q-Weibull, (0 &lt; q &lt; 1)</td>
<td>(\hat{p} = 14.7251), (\hat{\alpha} = 1.1100), (\hat{\lambda} = 0.4179), (\hat{q} = 0.1278)</td>
<td>452.4391</td>
<td>0.0421</td>
<td>0.9634</td>
</tr>
</tbody>
</table>
Figure 3.3: QQ and PP plots for $q$-Weibull and MO-$q$-$W$. 
3.4 AR(1) models with $q$-Weibull Marginal Distribution

Tavares (1977), (1980) introduced two stationary Markov processes with similar structural forms which he had found useful in hydrological applications. Lewis and McKenzie (1991) discuss various aspects on first order autoregressive minification processes. In this section we develop autoregressive minification processes of order one and order $k$ with minification structures where MO-$q$-W distribution is the stationary marginal distribution. We call the process as MO-$q$-W AR(1) process. Now we have the following theorem.

Theorem 3.4.1. Consider an AR(1) structure given by

$$X_n = \begin{cases} 
\epsilon_n, & \text{w.p. } p_1 \\
\min(X_{n-1}, \epsilon_n), & \text{w.p. } 1 - p_1 
\end{cases}$$

where w.p. denotes ‘with probability’, $0 < p_1 < 1$ and $\{\epsilon_n\}$ is a sequence of i.i.d. random variables independently distributed of $X_n$. Then $\{X_n\}$ is a stationary Markovian AR(1) process with MO-$q$-W marginal if and only if $\{\epsilon_n\}$ is distributed as $q$-Weibull distribution.

Proof: From the given structure it follows that

$$\bar{F}_{X_n}(x) = p_1 \bar{F}_{\epsilon_n}(x) + (1 - p_1) \bar{F}_{X_{n-1}}(x) \bar{F}_{\epsilon_n}(x).$$

Under stationary equilibrium, it reduces to

$$\bar{F}_X(x) = \frac{p_1 \bar{F}_\epsilon(x)}{1 - (1 - p_1) \bar{F}_\epsilon(x)}. \quad (3.4.1)$$

On substituting the survival function $\bar{F}_\epsilon(x)$ of $\epsilon$, for $1 < q < 2$, we get

$$\bar{F}_{1X}(x) = \frac{p_1 [1 + (q - 1)(\lambda x)^{q-1}]}{1 - (1 - p_1)[1 + (q - 1)(\lambda x)^{q-1}].}$$
which resembles the survival function $\bar{G}_1(\cdot)$ of the MO-$q$-W distribution. Conversely, if we take the survival function of the above form, we get the survival function of $\epsilon_n$ as the $q$-Weibull distribution under stationary equilibrium. Similar is the case when $q < 1$.

The behavior of the MO-$q$-W AR(1) process can be studied using the sample path. For $\alpha = 0.5, q = 0.5, p_1 = 0.1, 0.5, 0.9$ it is given in Figure 3.4, 3.5 and 3.6.

**Figure 3.4:** Sample paths of MO-$q$-W AR(1) process for $\alpha = 0.5, q = 0.5, p_1 = 0.1, 0.5, 0.9$.

**Figure 3.5:** Sample paths of MO-$q$-W AR(1) process for $\alpha = 1, q = 0.9, p_1 = 0.1, 0.5, 0.9$.

**Figure 3.6:** Sample paths of MO-$q$-W AR(1) process for $\alpha = 2, q = 1.9, p_1 = 0.1, 0.5, 0.9$. 

50
### Table 3.3: \( P(X_n < X_{n-1}) \) for the MO-\( q \)-W process where \( \lambda = 1, \alpha = 5 \) and \( q = 0.91 \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( p_1 )</th>
<th>( 200 )</th>
<th>( 400 )</th>
<th>( 600 )</th>
<th>( 800 )</th>
<th>( 1000 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.7605171 (0.002001369)</td>
<td>0.7523953 (0.003661458)</td>
<td>0.7573626 (0.2899318)</td>
<td>0.745249 (0.002795277)</td>
<td>0.7589137 (0.002238917)</td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>0.6740114 (0.005064845)</td>
<td>0.6597776 (0.003381987)</td>
<td>0.6564568 (0.003264189)</td>
<td>0.6621319 (0.002337129)</td>
<td>0.6654536 (0.001765257)</td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td>0.6253248 (0.0551385)</td>
<td>0.5887756 (0.002164813)</td>
<td>0.5930888 (0.0259571)</td>
<td>0.5941391 (0.002407355)</td>
<td>0.5913183 (0.003051416)</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.5027496 (0.005699338)</td>
<td>0.4992585 (0.004394413)</td>
<td>0.5230972 (0.003465077)</td>
<td>0.5237219 (0.001724348)</td>
<td>0.5153204 (0.003295175)</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.4097895 (0.004380096)</td>
<td>0.4390949 (0.0034124351)</td>
<td>0.4486389 (0.003437423)</td>
<td>0.4265652 (0.003071052)</td>
<td>0.4378709 (0.00083392)</td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>0.3228981 (0.005855759)</td>
<td>0.355286 (0.005029095)</td>
<td>0.3523585 (0.003365158)</td>
<td>0.3440362 (0.00241289)</td>
<td>0.357975 (0.000300952)</td>
<td></td>
</tr>
<tr>
<td>0.7</td>
<td>0.2695281 (0.005099691)</td>
<td>0.2676458 (0.003165013)</td>
<td>0.2646377 (0.003349412)</td>
<td>0.2773244 (0.003459127)</td>
<td>0.2731059 (0.003940234)</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>0.1873956 (0.005808668)</td>
<td>0.207557 (0.006566941)</td>
<td>0.1761628 (0.002695993)</td>
<td>0.1863542 (0.0033855295)</td>
<td>0.2003661 (0.003308759)</td>
<td></td>
</tr>
<tr>
<td>0.9</td>
<td>0.1175264 (0.007946155)</td>
<td>0.1194025 (0.006232362)</td>
<td>0.1119517 (0.005194928)</td>
<td>0.1012818 (0.002785095)</td>
<td>0.1130184 (0.003507806)</td>
<td></td>
</tr>
</tbody>
</table>

It can be seen that as \( p_1 \) increases, there is less chance for smaller values of \( X_n \) compared to \( X_{n-1} \). These observations can also be verified by referring to Table 3.3 showing \( P(X_n < X_{n-1}) \). These probabilities are obtained through a Monte Carlo simulation procedure. Sequences of 100, 300, 500, 700, 900 observations from MO-\( q \)-W process are generated repeatedly for ten times and for each sequence the probability is estimated. A table of such probabilities is provided with the average from ten trials along with an estimate of standard error in brackets. (See Table 3.3). In the following theorem, we consider a more general structure which allows a probabilistic selection of the process values, innovations and a combination of both.
Theorem 3.4.2. Consider an AR(1) structure given by

\[ X_n = \begin{cases} 
X_{n-1}, & \text{w.p. } p_2 \\
\epsilon_n, & \text{w.p. } p_1(1-p_2) \\
\min(X_{n-1}, \epsilon_n), & \text{w.p. } (1-p_1)(1-p_2)
\end{cases} \]

where \( \{\epsilon_n\} \) is a sequence of i.i.d. random variables independently distributed of \( X_n \).

Then \( \{X_n\} \) is a stationary Markovian AR(1) process with MO-q-W marginal if and only if \( \{\epsilon_n\} \) is distributed as q-Weibull distribution.

**Proof:** From the given structure it follows that

\[ \bar{F}_{X_n}(x) = p_2 \bar{F}_{X_{n-1}}(x) + p_1(1-p_2) \bar{F}_{\epsilon_n}(x) + (1-p_1)(1-p_2) \bar{F}_{X_{n-1}}(x) \bar{F}_{\epsilon_n}(x). \]

On simplification we get, the same expression as in equation (2) under stationarity. Then the result is obvious. The following theorem generalizes the results to a \( k \)th order autoregressive structure.

Theorem 3.4.3. Consider an AR(k) structure given by

\[ X_n = \begin{cases} 
\epsilon_n, & \text{w.p. } p_0 \\
\min(X_{n-1}, \epsilon_n), & \text{w.p. } p_1 \\
\min(X_{n-2}, \epsilon_n), & \text{w.p. } p_2 \\
\vdots & \vdots \\
\min(X_{n-k}, \epsilon_n), & \text{w.p. } p_k
\end{cases} \]

where \( \{\epsilon_n\} \) is a sequence of i.i.d. random variables independently distributed of \( X_n \), 

\( 0 < p_i < 1, \quad p_1 + p_2 + \cdots + p_k = 1 - p_0 \). Then the stationary marginal distribution of \( \{X_n\} \) is MO-q-W if and only if \( \{\epsilon_n\} \) is distributed as q-Weibull distribution.

**Proof:** From the given structure the survival function is given as follows:
\[ F_{X_n}(x) = p_0 \bar{F}_{\epsilon_n}(x) + p_1 \bar{F}_{X_{n-1}}(x)\bar{F}_{\epsilon_n}(x) + \cdots + p_k \bar{F}_{X_{n-k}}(x)\bar{F}_{\epsilon_n}(x). \]

Under stationary equilibrium, this yields

\[ \bar{F}_X(x) = p_0 \bar{F}_\epsilon(x) + p_1 \bar{F}_X(x)\bar{F}_\epsilon(x) + \cdots + p_k \bar{F}_X(x)\bar{F}_\epsilon(x). \]

This reduces to

\[ \bar{F}_X(x) = \frac{p_0 \bar{F}_\epsilon(x)}{1 - (1 - p_0)\bar{F}_\epsilon(x)}. \]

Then the theorem easily follows by similar arguments as in Theorem 3.4.2.

### 3.5 The Max-min AR(1) Processes

Next we introduce a new model called the max-min process which incorporates both maximum and minimum values of the process. This has wide applications in atmospheric and oceanographic studies. The structure is given as follows.

**Theorem 3.5.1.** Consider an AR(1) structure given by

\[ X_n = \begin{cases} 
\max(X_{n-1}, \epsilon_n), & \text{w.p. } p_1 \\
\min(X_{n-1}, \epsilon_n), & \text{w.p. } p_2 \\
X_{n-1}, & \text{w.p. } 1 - p_1 - p_2 
\end{cases} \]

subject to the conditions \(0 < p_1, p_2 < 1, p_2 < p_1 \text{ and } p_1 + p_2 < 1,\) where \(\{\epsilon_n\}\) is a sequence of i.i.d. random variables independently distributed of \(X_n.\) Then \(\{X_n\}\) is a stationary Markovian AR(1) max-min process with stationary marginal distribution \(\bar{F}_X(x)\) if and only if \(\{\epsilon_n\}\) follows Marshall-Olkin distribution.

**Proof:** From the given structure it follows that

\[ P(X_n > x) = p_1 P(\max(X_{n-1}, \epsilon_n) > x) + p_2 P(\min(X_{n-1}, \epsilon_n) > x) \\
+ (1 - p_1 - p_2) P(X_{n-1} > x) \]
On simplification we get,

\[ P(X_n > x) = p_1 \left[ 1 - (1 - \bar{F}_{X_{n-1}}(x))(1 - \bar{F}_\epsilon(x)) \right] + p_2 \bar{F}_{X_{n-1}}(x) \bar{F}_\epsilon(x) \\
+ (1 - p_1 - p_2) \bar{F}_{X_{n-1}}(x). \]

Under stationary equilibrium,

\[ \bar{F}_\epsilon(x) = \frac{p_2 \bar{F}_X(x)}{p_1 + (p_2 - p_1)\bar{F}_X(x)} = \frac{p'\bar{F}_X(x)}{1 - (1 - p')\bar{F}_X(x)} \quad (3.5.1) \]

where \( p' = \frac{p_2}{p_1} \). This has the same functional form of Marshall-Olkin survival function.

The converse can be proved by mathematical induction, assuming that \( \bar{F}_{X_{n-1}}(x) = \bar{F}_X(x) \).

### 3.5.1 The Max-min Process with Weibull Marginal Distribution

Now consider the standard Weibull distribution with cumulative distribution function \( F(x) = 1 - e^{-(\theta x)^\alpha} \). To obtain the Weibull max-min process, consider the max-min AR(1) structure and substitute the c.d.f. we get,

\[ \bar{F}_\epsilon(x) = \frac{p'}{e^{(\theta x)^\alpha} - (1 - p')} \]

which is the survival function of the Marshall-Olkin Weibull distribution where \( p' = \frac{p_2}{p_1}, p_2 < p_1 \) and \( p_1 + p_2 < 1 \). When \( \alpha = 1 \) it gives a max-min process with exponential marginal distribution.

### 3.5.2 The Max-min Process with q-Weibull Marginal Distribution

Now consider the q-Weibull distribution with survival function \( \bar{F}(x) = [1 + (q - 1)(\lambda x)^\alpha]^{\frac{q-2}{q-1}} \) for \( 1 < q < 2 \). To obtain the q-Weibull max-min process, consider the max-min AR(1) structure and substitute the c.d.f. we get,

\[ \bar{F}_\epsilon(x) = \frac{p'[1 + (q - 1)(\lambda x)^\alpha]^{\frac{q-2}{q-1}}}{1 - (1 - p')[1 + (q - 1)(\lambda x)^\alpha]^{\frac{q-2}{q-1}}}, 1 < q < 2 \]

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which is the MO-\(q\)-W survival function where \(p' = \frac{p_2}{p_1}, p_2 < p_1\) and \(p_1 + p_2 < 1\).

Figures 3.7, 3.8 and 3.9 show the sample path of the max-min AR(1) process with \(q\)-Weibull stationary marginal distribution, when \(\alpha = 0.5, \lambda = 1, p' = 0.3, q = 1.9, 1.1, 0.5\).

Now consider a more general autoregressive structure which includes maximum, minimum as well as the innovations and the process values.

**Theorem 3.5.2.** Consider an AR(1) structure given by

\[
X_n = \begin{cases} 
\max(X_{n-1}, \epsilon_n), & \text{w.p. } p_1 \\
\min(X_{n-1}, \epsilon_n), & \text{w.p. } p_2 \\
\epsilon_n, & \text{w.p. } p_3 \\
X_{n-1}, & \text{w.p. } 1 - p_1 - p_2 - p_3
\end{cases}
\]

with the condition that \(0 < p_1, p_2, p_3 < 1, p_2 < p_1 \) and \(0 < p_1 + p_2 + p_3 < 1\), where \(\{\epsilon_n\}\) is a sequence of i.i.d. random variables independently distributed of \(X_n\). Then \(\{X_n\}\) is a stationary Markovian AR(1) max-min process with stationary marginal distribution \(\bar{F}_X(x)\) if and only if \(\{\epsilon_n\}\) follows Marshall-Olkin distribution.

**Proof:** From the given structure it follows that

\[
P(X_n > x) = p_1 P(\max(X_{n-1}, \epsilon_n) > x) + p_2 P(\min(X_{n-1}, \epsilon_n) > x) + p_3 P(\epsilon_n > x) + (1 - p_1 - p_2 - p_3) P(X_{n-1} > x)
\]

\[
= p_1 \left[ 1 - (1 - \bar{F}_{X_{n-1}}(x))(1 - \bar{F}_{\epsilon_n}(x)) \right] + p_2 \bar{F}_{X_{n-1}}(x)\bar{F}_{\epsilon_n}(x) + p_3 \bar{F}_{\epsilon_n}(x) + (1 - p_1 - p_2 - p_3) \bar{F}_{X_{n-1}}(x).
\]

Under stationary equilibrium, this reduces to

\[
\bar{F}_{\epsilon}(x) = \frac{(p_2 + p_3) \bar{F}_X(x)}{p_1 + p_3 + (p_2 - p_1) \bar{F}_X(x)} = \frac{\beta \bar{F}_X(x)}{1 - (1 - \beta) \bar{F}_X(x)}
\]

where \(\beta = \frac{p_2 + p_3}{p_1 + p_3}\) which has the Marshall-Olkin survival function.
Figure 3.7: Sample paths of max-min process for $\alpha = 0.5, \lambda = 1, p' = 0.3, q = 1.9, 1.1, 0.5$.

Figure 3.8: Sample paths of max-min process for $\alpha = 1, \lambda = 1, p' = 0.5, q = 1.9, 1.1, 0.5$.

Figure 3.9: Sample paths of max-min process for $\alpha = 2, \lambda = 1, p' = 0.9, q = 1.9, 1.1, 0.5$. 
3.6 Case Study

![Image](image.jpg)

**Figure 3.10:** Observed Sample paths of total daily weighted discharge of Neyyar river from July to December 1993.

In this section, we illustrate the application of the MO-$q$-W process in modeling a hydrological data as a case study. The data consists of total daily weighted discharge (in mm$^3$) of Neyyar river in Kerala at the location Amaravilla (near Amaravilla bridge) during 1993. Neyyar is one of the west flowing rivers in Kerala, located in the Southern most part. It originates from Agasthyamala at an elevation of about 1,860 m above mean sea level. From there it flows down rapidly along steep slopes in its higher reaches and then winds its way through flat land in the lower reaches. In the initial stages the course is in a Southwestern direction but at Ottasekharamangalam the river turns and flows west. It again takes a Southwestern course from Valappallikanam upto its fall. The Neyyar is 56 Km. long and has a total drainage area of 497 sq. Km. It is a main source of irrigation in Southern Kerala and the Neyyar Dam is a main source of Hydroelectric power.

The arithmetic mean of the data is 0.81. The estimates are obtained as $\hat{p}_1 = 0.5$ and $\hat{\alpha} = 0.7$ for fixed $q$, ($q = 0.5$). The calculated value of $\chi^2$ is 0.626, which is significantly less than the tabled value. Hence MO-$q$-W distribution is a good model in this situation. The observed sample paths of total daily weighted discharge are given in Figure 3.10. It is found that the simulated MO-$q$-W process has close resemblance to the actual data.
3.7 Conclusion

In this chapter we have introduced a new probability model known as Marshall-Olkin \(q\)-Weibull distribution. We have fitted the Marshall-Olkin \(q\)-Weibull distribution to a data on cancer remission times and compared the results with respect to the \(q\)-Weibull distribution. It is shown that the new distribution is a good fit. We developed different types of AR(1) minification models and max-min processes, and the sample path properties were studied. A case study was done in which the new process is applied to a data on daily discharge of Neyyar river in Kerala, India. The models developed in this chapter can be applied in various contexts such as ecological studies, modeling of hydrological data, wind velocity speeds, climate studies, etc.

References


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butions with application to the exponential and Weibull families, *Biometrika*, 84, 641-652.

