CHAPTER V

DIFFERENT KINDS OF CURVES AND LINES IN SASAKIAN HYPERSURFACE

1. INTRODUCTION

In the present chapter, we have generalised certain known results, concerning geodesic curves for union curves, in Sasakian spaces. The analytic representation of the congruence in § 2 is followed in § 3 by the derivation of the differential equations of the union curves in Sasakian spaces, referred to an arbitrary system of co-ordinates on the surface. From the definition of union curvature vector in § 4, it is seen that a union curve on a surface may be defined as a curve, for which the union curvature vector is a null vector at every point of the curve. There appears in § 5 a geometrical interpretation of the union curvature of a curve on a surface, which agrees with the definition of geodesic curvature of the curve for the particular case of the congruence of the normal to the surface.

Further, we have developed the differential equations of the union curves of a hypersurface $V_n$, immersed in a Riemannian space $V_{n+1}$ of $(n+1)$-dimensions. The osculating plane to a curve on a surface is generalized to a totally geodesic surface, the straight lines of which are the geodesic in the space $V_{n+1}$. A formula is given for the union curvature vector of a curve in $V_n$. Also, a Darboux line (or, D-line) on a surface of a 3-dimensional Euclidean space is a curve for which the osculating sphere at each point is tangent to the surface. We have defined and studied the hyper D-lines in Sasakian hypersurface and some properties of these lines have also been investigated. Finally, we have studied union, special and hyperasymptotic curves of a Sasakian space.

A rectilinear congruence in ordinary three dimensional Euclidean space may be defined by specifying the direction of a unique line at each point of a given surface. A union curve on the surface, relative to a given congruence, has the property that its osculating plane at each point of the curve contains the line of the congruence through the point. It is well known that the union curves relative to the congruence of normals to a
surface are the geodesic curves on the surface. The notation of the Eisenhart ( [21]) will be employed for the most part. The notation \( \{ \beta^\alpha_{\gamma} \} \) or \( \Gamma^\alpha_{\beta \gamma} \) will be used here as the christoffel's symbol of the second kind. The Greek indices will vary from the range 1,2,............m, while the latin indices will vary from the range 1,2,........n ( m < n).

A curve on an ordinary surface is a union curve (Sperry [111]), if its osculating plane at each point contains the line of a specified rectilinear congruence through the point. Springer ([113]) has obtained the differential equations of union curves on a metric surface in ordinary space and has exhibited certain generalizations for union curves of known results concerning geodesic curves on a surface.

A Darboux line (or, D-line) on a surface of a 3-dimensional Euclidean space is a curve for which the osculating sphere at each point is tangent to the surface. Some properties of these curves have been studied by Semin ([83], [84]). These curves have been generalized to give the hyper D-lines of a surface (pravanovitch [69]). Singh ([107]) has defined and studied the hyper D-lines of order h in Riemannian subspace. Some properties of union hyper D-lines have also been investigated by him. Further, Singh ([91]) has defined and studied hyper D-lines in a Kaehlerian hypersurface.

The union curve of a Riemannian hypersurface has been defined and studied by Springer ([112]). Mishra ([48]) has investigated the properties of these curves in a subspace of a Riemannian space. Also, Singh and Nautiyal ([95]) have defined and studied special, union and hyperasymptotic curves of a Kaehler hypersurface.

Consider a hypersurface \( C_n \):

\[
z^i = z^i (u^\alpha), \quad \bar{z}^i = \bar{z}^i (\bar{u}^{\bar{\alpha}}),
\]

of an (n+1)-dimensional complex manifold \( C_{n+1} \), whose metric tensor satisfies the Kaehler's condition:

\[
(1.1) \quad \frac{\partial g_{\bar{i}j}}{\partial \bar{z}^k} = \frac{\partial g_{i\bar{k}}}{\partial z^j}.
\]

It has been shown (Yano- [137]) that an analytic hypersurface of a
Kaehler manifold is also a Kaehler manifold. Similarly, the analytic hypersurface of a Sasakian manifold is a Sasakian manifold. The hypersurface and embedding Sasakian space will be denoted by $S_n$ and $S_{n+1}$ respectively.

The metric tensors of $S_n$ and $S_{n+1}$ are related by

$$g_{\alpha \beta}(u, \bar{u}) = g_{ij}(z, \bar{z}) B^i_\alpha B^j_\beta,$$

where

$$B^i_\alpha = \frac{\partial z^i}{\partial u^\alpha}, \quad B^j_\bar{\alpha} = \frac{\partial z^j}{\partial u^{\bar{\alpha}}}.$$

If $(N^i, \bar{N}^i)$ be the components of a vector, normal to the hypersurface, then

$$2 g_{ij} N^i \bar{N}^j = 1,$$

(1.4)

$$g_{ij} N^i B^j_\beta = 0$$

and

(1.5)

$$g_{ij} \bar{N}^j B^i_\alpha = 0$$

and so its complex conjugate.

Consider a curve $C: z^i = z^i(s); \bar{z}^i = \bar{z}^i(s)$ (where $s$ is real) of $S_n$. The components $\frac{dz^j}{ds}$ or, $(\frac{dz^j}{ds})$ and $\frac{du^{\bar{\alpha}}}{ds}$ or, $(\frac{du^{\bar{\alpha}}}{ds})$ of the unit tangent vector of $C$ are given by

(1.6)

$$\frac{dz^j}{ds} = B^i_\alpha \frac{du^i}{ds},$$

(1.7)

$$\frac{dz^j}{ds} = B^i_\bar{\alpha} \frac{du^{\bar{\alpha}}}{ds}.$$
If \((q^i, q^\bar{i})\) and \((p^\alpha, p^\bar{\alpha})\) are the components of the first curvature vectors w.r.t. \(S_{n+1}\) and \(S_n\) respectively, then (Saxena and Bihari [6]):

\[
q^i = B^i_\alpha p^\alpha + K_n N^i, \tag{1.8}
\]
\[
q^\bar{i} = B^\bar{i}_\bar{\alpha} p^\bar{\alpha} + \bar{K}_n N^\bar{i}, \tag{1.9}
\]

where the normal curvature \((K_n, \bar{K}_n)\) of \(S_n\) is given by

\[
K_n = \Omega_{\alpha\beta} \frac{du^\alpha}{ds} \frac{du^\beta}{ds}; \quad \bar{K}_n = \Omega_{\bar{\alpha}\bar{\beta}} \frac{d\bar{u}^\alpha}{ds} \frac{d\bar{u}^\beta}{ds}
\]

and

\[
B^i_{\alpha, \beta} = \Omega_{\alpha\beta} N^i; \quad B^\bar{i}_{\bar{\alpha}, \bar{\beta}} = \Omega_{\bar{\alpha}\bar{\beta}} \bar{N}^\bar{i},
\]

where \(\Omega_{\alpha\beta}\) and \(\Omega_{\bar{\alpha}\bar{\beta}}\) are the components of the second fundamental tensors of the hypersurface \(S_n\).

The unit vector \((\xi^i, \xi^\bar{i})\) orthogonal to

\[
\left( \frac{dz^j}{ds}, \frac{dz^\bar{j}}{ds} \right)
\]

is given by

\[
g^-_{ij} \frac{dz^j}{ds} \xi^j + g^-_{i\bar{j}} \frac{dz^\bar{j}}{ds} \xi^\bar{j} = 0 \tag{1.10}
\]

and

\[
2g^-_{ij} \xi^i \xi^j = 1. \tag{1.11}
\]

Two vectors \((x^\alpha, x^\bar{\alpha})\) and \((y^\alpha, y^\bar{\alpha})\) of the hypersurface are said to be conjugate if, the relation:
\[ \Omega_{\alpha\beta} \ x^\alpha \ y^\beta = 0 \]
and its conjugate hold.

2. ANALYTICAL REPRESENTATION OF THE CONGRUENCE

Let the surface S be defined analytically w.r.t. an orthogonal system of co-ordinates by

\[ x^i = x^i \ ( u^{\alpha} ) , \ ( \alpha = 1, 2, \ldots, m ) \]
\[ ( i = 1, 2, \ldots, n ) , \]
where the functions \( x^i \) and their partial derivatives to the second order are understood to be continuous at any point P on S. Let the line L of the congruence at P have direction cosines given by

\[ \lambda^i = \lambda^i \ ( u^{\alpha} ) , \ \lambda^i \lambda^i = 1, \]
where the functions \( \lambda^i \) are continuous at P. If \( X^i \) denote the direction cosine of the normal to S at P, the direction cosine of L at P may be written in the form:

\[ \lambda^i = p^{\alpha} x^{i,\alpha} + q X^i , \ ( q > 0 ) , \]
where, for convenience, the notation of the covariant derivative \( x^{i,\alpha} \) of \( x^i \) w.r.t. the first fundamental tensor \( g_{\alpha\beta} = x^{i,\alpha} x^{i,\beta} \) of S is used instead of \( \frac{\partial x^i}{\partial x^{\alpha}} \). By virtue of the second part of the equation (2.2), there results

\[ \lambda^i \lambda^i \equiv \ ( p^{\alpha} x^{i,\alpha} + q X^i ) \ ( p^{\beta} x^{i,\beta} + q X^i ) \equiv g_{\alpha\beta} p^{\alpha} p^{\beta} + q^2 \]
\[ p^{\beta} + q^2 = 1. \]
i.e., \( ( \lambda^i )^2 = 1. \)
If $\theta$ is the angle between $L$ at $P$ and the normal to $S$ at $P$, it is seen from equation (2.3) that

\[(2.5) \quad \cos \theta = \lambda^i X^i = q ,\]

and from (2.4) that the magnitude of the vector with contravariant components $p^{\alpha^*}$ is $\sin \theta$. It may be noticed also that if $C$ is any curve on $S$ through $P$ represented by $u^{\alpha^*} = u^{\alpha^*}(s)$, where $s$ denotes the arc length, then the cosine of the angle $\phi$ between $L$ and the direction cosine of the tangent to $C$ at $P$ is given by

\[(2.6) \quad \cos \phi = \lambda^i \frac{dx^i}{ds} = \left( p^{\alpha^*} x^i, \alpha^* + q X^i \right) \frac{x^i, \beta}{du^\beta} = g_{\alpha^* \beta} p^{\alpha^*} \]

3. DIFFERENTIAL EQUATIONS OF THE UNION CURVES

The osculating plane to the curve $C$:

$u^{\alpha^*} = u^{\alpha^*}(s)$ on $S$ at $P$, has the determinantal equation

\[(3.1) \quad \delta^1_{ij} \delta^2_{jk} \left( \tilde{x}^j - x^j \right) \frac{dx^j}{ds} \frac{d^2x^k}{ds^2} = 0 ,\]

the $\tilde{x}^i$ being current co-ordinates.

On writing

\[(3.2) \quad \frac{dx^j}{ds} = x^j, \sigma u^\sigma ; \quad \frac{d^2x^k}{ds^2} = x^k, \gamma u^\gamma + \frac{d^2x^k}{du^\alpha du^\beta} u^\alpha u^\beta ,\]

where the primes indicate differentiation w.r.t. $s$ and, in term, the Gauss equation of the surface $S$, namely

\[(3.3) \quad \frac{\partial^2 x^k}{\partial u^\alpha \partial u^\beta} = \{ \alpha^\gamma \beta \} x^k, \gamma + d_{\alpha \beta} X^k ,\]

we see that equation (3.1) takes the form:
\[(3.4) \ \delta^{123}_{ijk} \ (\bar{x}^i - x^i) \ (u'^{\sigma} \ x^{i,\sigma}) \ (\rho^\gamma \ x^k,_{\gamma} + d_{\alpha\beta} \ u'^{\alpha} \ u'^{\beta} \ X^k) = 0,\]

where \(\rho^\gamma\), the components of the curvature vector of \(C\) at \(P\), are given by

\[(3.5) \ \rho^\gamma = u''^\gamma + \{ \alpha^\gamma \beta \} \ u'^{\alpha} \ u'^{\beta}.\]

If the osculating plane to \(C\) at \(P\) contains line \(L\), the co-ordinates \(x^i + t \lambda^i \equiv x^i + t \ (p^\gamma \ x^i,_{\gamma} + qX^i)\) must satisfy equation (3.4) for all \(t\). Thus, the condition:

\[(3.6) \ \delta^{123}_{ijk} \ (p^\gamma \ x^i,_{\gamma} + qX^i) \ (u'^{\sigma} \ x^{i,\sigma}) \ (\rho^\tau \ x^k,_{\tau} + d_{\alpha\beta} \ u'^{\alpha} \ X^k) = 0\]

must exist. The use of the fact that \(\delta^{123}_{ijk} \ x^i,_{\gamma} x^i,_{\sigma} x^k,_{\tau} = 0\) (because \(\gamma, \sigma, \tau\) cannot all be different) and \(\delta^{123}_{ijk} \ X^i X^j X^k \equiv 0\), together with a change of dummy indices (or, umbral indices), allows equation (3.6) to take the form:

\[(3.7) \ (\delta^{123}_{ijk} \ X^i X^j,_{\sigma} X^k,_{\tau}) \ (p^{\sigma} \ u'^{\tau} \ d_{\alpha\beta} \ u'^{\alpha} \ u'^{\beta} + q \ u'^{\sigma} \ \rho^\tau) = 0.\]

It is to be noticed here that \(d_{\alpha\beta} \ u'^{\alpha} \ u'^{\beta}\) is the normal curvature \(K_n\) of the curve \(C\) with the direction \(u'^{\alpha}\) \((\alpha = 1, 2)\) on the surface \(S\). Summing on \(\sigma\) and \(\tau\) in equation (3.7) and neglecting the non-zero determinant \(\delta^{123}_{ijk} \ X^i X^j,_{1} X^k,_{2}\), we get

\[(3.8) \ e_{\sigma\tau} \ (p^{\sigma} \ u'^{\tau} \ K_n + q \ u'^{\sigma} \ \rho^\tau) = 0,\]

where \(e_{12} = 1, \ e_{21} = -1, \ e_{11} = e_{22} = 0.\)

Equation (3.8) is the differential equation of the second order of the union curves on the surface \(S\) through any point \(P\) on \(S\), the parametric curves being any whatever. Lane has given the differential equation of the union curves on a metric surface for the case in which the lines of curvature on
the surface are taken as parametric. From equation (3.8), one may conclude that if \( q = 0 \) (which means that \( L \) lies in the tangent plane to \( S \) at \( P \)), then the only union curves are those in the direction given by \( p' \, du^2 - p^2 \, du' = 0 \) and by \( K_n \equiv d_{\alpha \beta} \, u'_{\alpha} \, u'_{\beta} = 0 \), the asymptotic directions. Suppose, henceforth, that \( q \neq 0 \) and let \( e^\sigma \) be written for \( p^\sigma / q \). Then, if \( \sigma \) and \( \tau \) be interchanged in the second term of (3.8), the differential equation of the union curves on \( S \) through \( P \) becomes

\[
(3.9) \quad e_{\sigma \tau} \, u'_{\sigma} \left( \rho^{\tau} - K_n \, e^{\tau} \right) = 0.
\]

If the components of the curvature vector of the curve are zero, the curve is a geodesic. From (3.9), it can be seen that the geodesic and union curves on a surface coincide in three directions of the asymptotic curves on the surface. Moreover, it may be seen from equ. (3.9) that if congruence is normal to the surface (\( e^\sigma = 0 \)), the union curves are geodesic curvature on \( S \).

4. UNION CURVATURE OF A CURVE ON A SURFACE

Equation (3.9) is a single differential equation of the second order. It will be shown to be equivalent to a pair of differential equations of the second order.

The curvature vector with congruence \( \rho^{\tau} \) is orthogonal to the direction \( u'_{\alpha} \) of \( C \) on \( S \). Hence, we have.

\[
(4.1) \quad g_{\alpha \beta} \, \rho^\alpha \, u'_{\beta} = 0.
\]

If (4.1), multiplied by \( u'_{1} \) and (3.9), multiplied by \( g_{2 \beta} \, u'_{\beta} \), are subtracted, these results, by use of \( g_{\alpha \beta} \, u'_{\alpha} \, u'_{\beta} = 1 \), the first of the following differential equations of the union curves on \( S \) are given by

\[
(4.2) \quad \eta^1 = \rho^1 - K_n \, g_{2 \beta} \, u'_{\beta} \cdot e_{\sigma \tau} \, e^\gamma \, u'_{\tau} = 0.
\]
\[ \eta^2 = \rho^2 + K_n g_{1\beta} u^{i\beta} \cdot e_{\sigma \tau} e^{\sigma} u^{i\tau} = 0 \]

and the second equation is obtained in a similar manner. The vector with components \( \eta^\alpha \), given by (4.2), lies in the tangent plane to S at P. It may be called the union curvature vector of the curve C on S at P. It may be conclude from (4.2) that the union curvature vector is a zero vector at each point of a union curve.

The geodesic curvature \( K_g \) of the curve C at P is given by

\[ (4.3) \quad K_g \equiv \epsilon_{\alpha \beta} u^{i\alpha} \rho^{i\beta}, \]

where

\[ \epsilon_{\alpha \beta} = g^{\frac{1}{2}} e_{\alpha \beta}. \]

Therefore, it appears appropriate to define the union curvature \( K_u \) of C at P by

\[ (4.4) \quad K_u \equiv \epsilon_{\alpha \beta} u^{i\alpha} \eta^{i\beta}, \]

which may be written, by use of equations (4.1) and (4.2), in the form

\[ (4.5) \quad K_u \equiv \epsilon_{\sigma \tau} u^{i\sigma} (\rho^{i\tau} - K_n e^{i\tau}) = K_g - K_n \epsilon_{\sigma \tau} u^{i\sigma} l^{i\tau}. \]

It may be observed from (4.5) that the geodesic curvature along a union curve (\( K_u \equiv 0 \)) is given by

\[ (4.6) \quad K_g = K_n \epsilon_{\sigma \tau} u^{i\sigma} e^{i\tau}. \]

From (3.9), it is seen that the co-ordinate curves on the surface are union curves iff, the congruence is specified by

\[ (4.7) \quad e^1 = \Gamma_{22}' / d_{22}, \quad e^2 = (\Gamma_{11}' / d_{11}). \]
For this particular congruence, the geodesic curvature of any union curve on \( S \) at \( P \) is given by

\[
K_g = g^{\frac{1}{2}} \left( \Gamma^2_{11} u'^1/d_{11} - \Gamma^2_{22} u'^2/d_{22} \right) d_{\alpha\beta} u'^\alpha u'^\beta.
\]

If \( K_{g1} \) and \( K_{g2} \) denote the geodesic curvatures of the curves through \( P \) represented by \( du^2 = 0 \), \( du^1 = 0 \), it can be seen from (4.8) that the sum of the geodesic curvatures of the co-ordinates curves on a surface is given by

\[
K_{g1} + K_{g2} = g^{\frac{1}{2}} \left[ \Gamma^2_{11} (u'^1)^3 - \Gamma^2_{22} (u'^2)^3 \right].
\]

When the asymptotic curves are taken as parametric, the directions of "Segre" at the point \( P \) on the surface are given by

\[
\left[ \Gamma^2_{11} (u'^1)^3 - \Gamma^2_{22} (u'^2)^3 \right] = 0.
\]

Hence, it can be conclude that the directions of "Segre" are those in which the geodesic curvature of the asymptotic curves differ in sign. This may be compared with the result that the torsions of the asymptotic curves through any point differ in sign.

### 5. Geometrical Interpretation of Union Curvature

The geodesic curvature of a curve \( C \) at \( P \) on the surface is the curvature of the curve obtained by projecting the curve \( C \) orthogonally onto the tangent plane to the surface at \( P \). Let the curve \( C \) be projected onto the tangent plane to the surface at \( P \) in the direction of the line \( L \) of the congruence. It will be shown that the curvature of the plane curve \( C' \), thus obtained is given by the expression in equation (4.5).

Let the cylindrical surface of projection of curve \( C \) onto the tangent plane at \( P \) in the direction of line \( L \) at \( P \) be denoted by \( S^1 \). If \( \frac{1}{R} \) is the nor-
mal curvature of the curve C at P and normal to the plane of L and the tangent line to C at P, then by the theorem of Meusnier, we get

\[ \frac{e}{R} = \rho^{-1} \cos \alpha, \]

where \( e = \pm 1 \) and \( \frac{1}{\rho} \) is the curvature of C at P. Further, if \( \frac{1}{R} \) is the curvature of the curve C' at P and if \( \beta \) is the angle between the tangent plane to S at P and the plane through the tangent line to C' normal to the plane of L and this line, then again by the theorem of Meusnier, we have

\[ \frac{1}{R} = \gamma^{-1} \cos \beta. \]

Therefore, by equations (5.1) and (5.2), we have

\[ \frac{e}{\gamma} = \cos \alpha / \rho \cos \beta. \]

Substitution of the analytical expressions for \( \cos \alpha \) and \( \cos \beta \) into (5.3) yields, after some simplification, the following:

\[ \frac{e}{\gamma} = e_{\sigma \tau} u^{\sigma}_{\gamma} ( \rho \gamma - K_n l^\gamma ), \]

which is the union curvature \( K_u \) given in (4.5). Therefore, the union curvature of a curve C at a point P on a surface S, relative to a given congruence, is the curvature of the curve, obtained by projecting C onto the tangent plane to S at P in the direction of the line L of the congruence at P.

6. VECTOR FIELD IN \( V_n \)

If \( y^\alpha ( \alpha = 1, 2, \ldots \ldots \ldots n+1 ) \) denotes the co-ordinates of a point in \( V_{n+1} \) and \( x^i ( i = 1, 2, \ldots \ldots \ldots n ) \) the co-ordinates of a point in \( V_n \), then \( y^\alpha \) may be written in the form:

\[ y^\alpha = y^\alpha ( x^1, x^2, \ldots \ldots \ldots, x^n ), \]
for points in $V_n$, the fundamental matrix $\| \frac{\partial x^i}{\partial y^\alpha} \|$ is of rank $n$. Let the metric in $V_n$ be denoted by $g_{ij} \ dx^i \ dx^j$ and that of $V_{n+1}$ by $a_{\alpha\beta} \ dy^\alpha \ dy^\beta$. These metrics are assumed to be positive definite. It follows that

\[ (6.2) \quad a_{\alpha\beta} \ y^\alpha, i \ y^\beta, j = g_{ij}, \]

where $y^\alpha, i$ denotes the covariant derivative of $y^\alpha$ with respect to $x^i$ (Greek indices always have the range 1,2, .......... $n +1$ and latin indices the range 1,2, ..........n ). If $N^\alpha$ denote the components of a unit vector in $V_{n+1}$ normal to $V_n$, then

\[ (6.3) \quad a_{\alpha\beta} \ y^\alpha, i \ N^\beta = 0 \]

and

\[ (6.4) \quad a_{\alpha\beta} \ N^\beta = 1. \]

If a vector field in $V_n$ has components $U^\alpha$ in the $y^s$ and components $u^i$ in the $x^s$ then the relation:

\[ (6.5) \quad U^\alpha = y^\alpha, i \ u^i \]

must be obtained. If $q^\alpha$ are the contravariant components in $y^s$ of derived vector relative to $V_{n+1}$ of a vector of the field along a curve C in $V_n$ and if $p^i$ are the contravariant components in the $x^s$ of the derived vector relative to $V_n$ of the same vector along C, it can be shown that (Weatherburn [135]):

\[ (6.6) \quad q^\alpha = \Omega_{ij} \ u^i \ \frac{dx^j}{ds} \ N^\alpha + y^\alpha, i \ p^i, \]

where $\Omega_{ij}$ is the second fundamental form for $V_n$.

7. TOTALLY GEODESIC SURFACE IN $V_{n+1}$

As an analogue for the osculating plane, in ordinary space, a totally geodesic surface in $V_{n+1}$ is introduced. It is determined the tangent to the curve C with equations
\[ x^i = x^i (s) \]
in \( V_n \), \( s \) denoting arc length by the first curvature in \( V_{n+1} \) of the curve \( C \). Let \( \lambda^\alpha \) be the contravariant components in the \( y^i \)'s of a unit vector in the direction of a curve of congruence of curves, one curve of which passes through each point of \( V_n \). The vector with components \( \lambda^\alpha \) is, in general, not normal to \( V_n \) and may be specified by

\[ (7.1) \quad \lambda^\alpha = t^i y^{\alpha, i} + \gamma N^\alpha, \]

where \( t^i \) and \( \gamma \) are parameters. Because \( \lambda^\alpha \) represent a unit vector \( a_{\alpha\beta} \lambda^\alpha \lambda^\beta = 1 \) and it follows by use of equations (6.3), (6.4) and (7.1) that

\[ t_i t^i = 1 - \gamma^2. \]

If the geodesic in \( V_{n+1} \) in the direction of the curve of the congruence with direction \( \lambda^\alpha \) is to be a geodesic, of the totally geodesic surface, then it is necessary that \( \lambda^\alpha \) be a linear combination of \( y^{\alpha, i} u^i \) and \( q^\alpha \). Hence, we get

\[ (7.2) \quad t^i y^{\alpha, i} + \gamma N^\alpha = v y^{\alpha, i} u^i + w q^\alpha, \]

where \( v \) and \( w \) are to be determined, the \( u^i \) of equation (6.5) are now \( \frac{\mathrm{d}x^i}{\mathrm{d}s} \) and \( q^\alpha \) are given by

\[ (7.3) \quad q^\alpha = \Omega_{ij} \frac{\mathrm{d}x^j}{\mathrm{d}s} \frac{\mathrm{d}x^i}{\mathrm{d}s} N^\alpha + y^{\alpha, i} p^i. \]

and \( p^i \) are given by

\[ (7.4) \quad p^i = \frac{\mathrm{d}^2 x^i}{\mathrm{d}s^2} + \left\{ j \begin{array}{c} i \\ k \end{array} \right\} \frac{\mathrm{d}x^j}{\mathrm{d}s} \frac{\mathrm{d}x^k}{\mathrm{d}s}. \]

If \( K_n \) is written for \( \Omega_{ij} \frac{\mathrm{d}x^j}{\mathrm{d}s} \frac{\mathrm{d}x^i}{\mathrm{d}s} \), which is the normal component of the curvature vector of the curve \( C \) in \( V_{n+1} \), equation (7.2) take the form:

\[ (7.5) \quad t^i y^{\alpha, i} + \gamma N^\alpha = v y^{\alpha, i} \frac{\mathrm{d}x^i}{\mathrm{d}s} + w \left( K_n N^\alpha + y^{\alpha, i} p^i \right). \]

Multiplication of equation (7.5) by \( a_{\alpha\beta} y^{\beta, j} \), summation with respect to
\( \alpha \) and use of equations (6.2), (6.3) yields the \( n \) equations

\[
(7.6) \quad g_{ij} t^i = v g_{ij} \frac{dx^j}{ds} + w g_{ij} p^i .
\]

If equations (7.5) are multiplied by \( a_{\alpha \beta} N^\alpha \), summation on \( \alpha \) and use of (6.4) give the relation

\[
(7.7) \quad \gamma = wK_n .
\]

The solution of (7.6) for \( v \) is effected by multiplying by \( \frac{dx^j}{ds} \) and summation on \( j \). Because \( g_{ij} p^i \frac{dx^j}{ds} = 0 \), it follows that

\[
(7.8) \quad v = g_{ij} t^i \frac{dx^j}{ds} .
\]

Therefore, on using the values of \( v \) and \( w \) from (7.7) and (7.8), the \( n \)-equations (7.6) take the form:

\[
(7.9) \quad g_{ij} t^i = g_{ij} \frac{dx^j}{ds} \ g_{lm} t^l \frac{dx^m}{ds} + \frac{\gamma}{K_n} g_{ij} p^i .
\]

Multiplication of equation (7.9) by \( g^{jk} \), summation on \( j \) and replacement of \( t^k / \gamma \) by \( l^k \), lead to

\[
(7.10) \quad p^k - K_n ( l^k - g_{im} t^i \frac{dx^m}{ds} \frac{dx^k}{ds} ) = 0. \ (k = 1, 2, \ldots n),
\]

where \( p^k \) are given by equation (7.4).

8. UNION CURVES IN \( V_n \)

For a congruence specified by the parameters \( l^k \), the solutions of the \( n \)-equations (7.10) determine the union curves in \( V_n \) relative to that congruence. The parameter \( \gamma \) cannot vanish under the assumption that the direction \( \lambda^\alpha \) is not in the \( V_n \). The left members of the equations (7.10)
may be denoted by $\eta^k$, of the union curvature vector in $V_{n+1}$. A union curve of $V_n$ with respect to a congruence determined by the parameters $l^k$ may therefore be defined as a curve along which the union curvature vector is a null vector.

By use of (7.4) and the fact that $g_{ij} \, dx^i \, dx^j = ds^2$, equations (7.10) can be written in the form:

$$(8.1) \quad \eta^k = p^k - K_n \, v^k = 0,$$

where the vector $v^k$ is defined by

$$(8.2) \quad v^k = g_{ij} \frac{dx^i}{ds} \left( l^k \frac{dx^j}{ds} - l^j \frac{dx^k}{ds} \right).$$

From equations (8.1), it follows that if the curve $C$ is an asymptotic curve in $V_n$, in which case $K_n = 0$ along the curve, then for a union curve ($\eta^k = 0$), $p^k = 0$ and the curve is a geodesic.

Furthermore, if a union curve is a geodesic, then it either an asymptotic curve, or the vector of components $v^k$ is a null vector.

The magnitude $K_U$ of the vector $\eta^k$ is given by

$$K_U^2 = g_{ij} \eta^i \eta^j.$$

From equations (8.1), it is seen that the angle $\phi$ between the vector $\lambda^\alpha$ and $N^\alpha$ in $V_{n+1}$ is given by

$$\cos \phi = \gamma$$

and because

$$t^k / \gamma = l^k$$

and

$$t_i t^i = 1 - \gamma^2,$$

it follows that

$$g_{ij} \, l^i \, l^j = \tan^2 \phi.$$
The angle \( \alpha \) between the vector \( l^k \) and the tangent vector \( C \) is given by

\[
\cos \alpha = g_{ik} \frac{dx^k}{ds}.
\]

In terms of \( \phi \) and \( \alpha \), the magnitude \( K_U \) of the union curvature vector can be shown to be given by

\[
K_U = Kg - Kn \tan \phi \cdot \sin \alpha,
\]

where \( Kg \) is the geodesic curvature of the curve \( C \) in \( V_n \). It is to be observed that if \( \phi = 0 \), the union curve is a geodesic.

### 9. HYPER DARBOUX LINES

Consider a curve \( C : z^i = z^i(s) ; \ z^i = z^i(s) \) (not a geodesic of the enveloping space) of \( S_n \). The components

\[
( \eta^i(0) , \ \tilde{\eta}^i(0)) \equiv ( \frac{dz^i}{ds} , \ \frac{d\tilde{z}^i}{ds}) , \ ( \eta^i(1) , \ \tilde{\eta}^i(1)) \quad \text{and} \quad ( \eta^i(2) , \ \tilde{\eta}^i(2))
\]

of unit tangent, unit principal normal vector and unit first binormal vector define an orthogonal system of unit vectors at every point of the curve. Assuming that \( \frac{\delta}{\delta s} \) is the usual covariant differential along \( C \), we have the first two Frenet's formulae of a curve in \( S_n \) given by

\[
\frac{\delta \eta^i(0)}{\delta s} = K(1) \ \eta^i(1) ,
\]

(9.1)

\[
\frac{\delta \tilde{\eta}^i(0)}{\delta s} = \tilde{K}(1) \ \tilde{\eta}^i(1) ,
\]

(9.2)

and

\[
\frac{\delta \eta^i(1)}{\delta s} = -K(1) \ \eta^i(0) + K(2) \ \eta^i(2) ,
\]

(9.3)

\[
\frac{\delta \tilde{\eta}^i(1)}{\delta s} = -\tilde{K}(1) \ \tilde{\eta}^i(0) + \tilde{K}(2) \ \tilde{\eta}^i(2) ,
\]

(9.4)
where the scalars

\[( K_{(1)}, \bar{K}_{(1)} ) \equiv ( \frac{1}{R_{(1)}}, \frac{1}{R_{(1)}} )\]

and \[( K_{(2)}, \bar{K}_{(2)} ) \equiv ( \frac{1}{R_{(2)}}, \frac{1}{R_{(2)}} ) ,\]

are the first and second curvatures of the curve in the embedding space \( S_{n+1} \).

Consider a congruence of curves given by the vector field \( \vec{\lambda} = ( \lambda^i, \bar{\lambda}^\bar{i} ) \), such that through each point of \( S_{n+1} \), there passes exactly one curve of the congruence. At the points of the hypersurface, we have

\[
(9.5) \quad \lambda^i = t^\alpha B_\alpha^i + C N^i ,
\]

\[
(9.6) \quad \bar{\lambda}^\bar{i} = \bar{t}^\bar{\alpha} B_{\bar{\alpha}}^\bar{i} + \bar{C} N^\bar{i} ,
\]

where \( t^\alpha \) and \( C \) are parameters and \( \bar{t}^\bar{\alpha} \), \( \bar{C} \) are their complex conjugate.

The curve \( C \) is said to be hyper \( D \)-line of the hypersurface if, the surface spanned by the vectors \( ( \eta^{(o)}_i, \eta^{(o)}_{\bar{i}} ) \) and \( ( R_{(1)} \eta^{(1)}_i + R_{(2)} \frac{dR_{(1)}}{ds} \eta^{(1)}_{\bar{i}} ) \), \( \eta^{(2)}_i, R_{(1)} \eta^{(1)}_i + R_{(2)} \frac{dR_{(1)}}{ds} \eta^{(2)}_{\bar{i}} \) contains the vector \( ( \lambda^i, \bar{\lambda}^{\bar{i}} ) \).

From \((9.1)\) and \((9.2)\), we have

\[
(9.7) \quad \frac{\delta^2 \eta^{(o)}_i}{\delta s^2} = - K_{(1)}^2 \eta^{(o)}_i + \frac{d}{ds} K_{(1)} \eta^{(1)}_i + K_{(1)} K_{(2)} \eta^{(2)}_i
\]

and

\[
(9.8) \quad \frac{\delta^2 \eta^{(o)}_i}{\delta s^2} = - \bar{K}_{(1)}^2 \eta^{(o)}_i + \frac{d}{ds} \bar{K}_{(1)} \eta^{(1)}_i + \bar{K}_{(1)} \bar{K}_{(2)} \eta^{(2)}_i .
\]

These equation give

\[
(9.9) \quad g_{ij} \left( R_{(1)} \eta^{(1)}_i + R_{(2)} \frac{dR_{(1)}}{ds} \eta^{(2)}_i \right) \left( \frac{\delta^2 \eta^{(o)}_j}{\delta s^2} \right) = 0
\]
and

(9.10) \[ g_{ij} \left( \tilde{R}(1) \eta^i_{(1)} + \tilde{R}(2) \frac{d\tilde{R}(1)}{ds} \eta^i_{(2)} \right) \left( \frac{\delta^2 \eta^j_{(0)}}{\delta s^2} \right) = 0. \]

For the hyper D-line of the hypersurface, we get

(9.11) \[ \lambda^i = u \left[ R(1) \eta^i_{(1)} + R(2) \frac{d}{ds} R(1) \eta^i_{(2)} \right] + v \eta^i_{(0)} \]
and

(9.12) \[ \bar{\lambda}^i = \bar{u} \left[ \bar{R}(1) \bar{\eta}^i_{(1)} + \bar{R}(2) \frac{d}{ds} \bar{R}(1) \bar{\eta}^i_{(2)} \right] + \bar{v} \bar{\eta}^i_{(0)}. \]

We define the following:

(9.13) \[ \lambda_{(h)} = 2 g_{ij} \lambda^i \eta^j_{(h)} \]

and

(9.14) \[ \bar{\lambda}_{(h)} = 2 g_{ij} \bar{\lambda}^i \eta^j_{(h)}. \]

Multiplying equation (9.11) by

(9.15) \[ 2 g_{ij} \left( \frac{\delta^2 \eta^j_{(0)}}{\delta s^2} \right) \lambda^i = 2 g_{ij} \left( \frac{\delta^2 \bar{\eta}^j_{(0)}}{\delta s^2} \right) \lambda_{(0)} \eta^i_{(0)}. \]

From (9.7), (9.8) and (9.15), we deduce

(9.16) \[ \left( \frac{dK_{(1)}}{ds} \right) \lambda_{(1)} + K(1) K(2) \lambda_{(2)} = 0. \]

Similarly, with the help of equations (9.10), (9.12) and (9.14), we may deduce
The equations (9.16) and (9.17) represent the hyper D-line of $S_n$ relative to $\lambda$.

We, now, have the following:

THEOREM (9.1): If the congruence $\lambda$ is along the principal normal (i.e., along the vector $(\eta^i_1, \eta^i_1)$ of a curve) then the necessary and sufficient condition that it be a hyper D-line of the hypersurface is that it be a curve of constant first curvature.

PROOF: The proof follows immediately from the equations (9.13), (9.14), (9.16), (9.17) and the fact that the vector $(\eta^i_1, \eta^i_1)$ constitute an orthogonal system.

In view of theorem (9.1), a hyper D-line satisfies the following:

THEOREM (9.2): If the congruence $\lambda$ lies along the first binormal, then the necessary and sufficient condition that it be a hyper D-line is that it be the curve of second curvature.

PROOF: The proof follows from (9.16), (9.17) and the fact that the curve is not a geodesic of $S_{n+1}$.

We are familiar with the well known relations:

\[(9.18) \quad \frac{\delta n^i_0}{\delta s} = p^\alpha B^i_\alpha + \Omega_{\alpha\beta} \left( \frac{du^\alpha}{ds} \right) \left( \frac{du^\beta}{ds} \right) N^i \]

and

\[(9.19) \quad \frac{\delta \bar{n}^i_0}{\delta s} = p^\alpha B^i_\bar{\alpha} - \Omega^i_{\alpha\bar{\beta}} \left( \frac{du^\bar{\alpha}}{ds} \right) \left( \frac{du^\bar{\beta}}{ds} \right) \bar{N}^i.\]

Writing the covariant differential of the equation (9.18) and using (Mishra [46]):

\[\bar{N}^i, \gamma = - \Omega_{\beta\gamma} g^{\beta\alpha} B^i_\alpha + \theta^i_{\gamma} N^i,\]
we have
\[
\delta^2 \eta^i_{(0)} = \left\{ \frac{\delta p^\alpha}{\delta s} - \sum_{\mu, \nu} \mu \nu \frac{du^\mu}{ds} \frac{du^\nu}{ds} \Omega_{\mu \nu} \gamma \alpha \frac{du^\gamma}{ds} \right\} B^i_{\alpha} + [ 3 \Omega_{\alpha \beta} \frac{du^\beta}{ds} P^\alpha + \Omega_{\alpha \beta \gamma} \frac{du^\beta}{ds} \frac{du^\gamma}{ds} + \Omega_{\alpha \beta} \left( \frac{du^\beta}{ds} \frac{du^\gamma}{ds} \right) \theta_{\gamma} ] N^i.
\]

Similarly, writing the covariant differential of (9.19) and using (Mishra [46]):
\[
N^i_{\alpha \gamma} = - \Omega_{\beta \gamma} \alpha \beta \gamma \frac{\partial}{\partial s} B^i_{\alpha} + \theta_{\gamma} \lambda \gamma N^i,
\]
we obtain the conjugate of (9.20) given by
\[
\delta^2 \eta^i_{(0)} = \left\{ \frac{\delta p^\alpha}{\delta s} - \sum_{\mu, \nu} \mu \nu \frac{du^\mu}{ds} \frac{du^\nu}{ds} \Omega_{\mu \nu} \gamma \alpha \frac{du^\gamma}{ds} \right\} B^i_{\alpha} + [ 3 \Omega_{\alpha \beta} \frac{du^\beta}{ds} P^\alpha + \Omega_{\alpha \beta \gamma} \frac{du^\beta}{ds} \frac{du^\gamma}{ds} + \Omega_{\alpha \beta} \left( \frac{du^\beta}{ds} \frac{du^\gamma}{ds} \right) \theta_{\gamma} ] N^i.
\]

Making use of (9.20) in (9.15) and using the fact that \((\eta^i_{(0)}, \eta^i_{(0)})\) is orthogonal to \((N^i, N^i)\), we get
\[
(9.22) \left( \frac{\delta p^\alpha}{\delta s} \right) t_\alpha - \sum_{\alpha, \beta} \alpha \beta \frac{du^\alpha}{ds} \frac{du^\beta}{ds} \Omega_{\gamma \nu} t^\gamma \frac{du^\nu}{ds} + C \left[ 3 \Omega_{\alpha \beta} P^\alpha \left( \frac{du^\beta}{ds} \right) + \Omega_{\alpha \beta \gamma} \frac{du^\alpha}{ds} \frac{du^\beta}{ds} \frac{du^\gamma}{ds} + \Omega_{\alpha \beta} \left( \frac{du^\beta}{ds} \frac{du^\gamma}{ds} \right) \theta_{\gamma} \right] + \lambda \left( \Omega_{\alpha \beta} \frac{du^\alpha}{ds} \frac{du^\beta}{ds} \right)^2 - g_{\alpha \beta} \left( \frac{\delta p^\alpha}{\delta s} \right) \frac{du^\beta}{ds} = 0.
\]

Similarly, the conjugate of the equation (9.22) can be obtained. We have the first two Frenet's formulae for \(S_n\) given by
\[
p^\alpha = \frac{\delta}{\delta s} \left( \frac{du^\alpha}{ds} \right) = K(1) \xi^\alpha (1),
\]
\[
\frac{\delta \xi^\alpha (1)}{\delta s} = - K(1) \left( \frac{du^\alpha}{ds} \right) + K(2) \xi^\alpha (2),
\]
with its conjugate. \(\xi^\alpha_1\) and \(\xi^\alpha_2\) are the components of the principal and binormal vectors with respect to \(S_n\). These equation give

\[
\frac{\delta p^\alpha}{\delta s} = -K^2(1) \left( \frac{du^\alpha}{ds} \right) + \frac{dK(1)}{ds} \xi^\alpha_1 + K(1) K(2) \xi^\alpha_2.
\]

Substituting the value of \(\frac{\delta p^\alpha}{\delta s}\) in (9.22) and using the fact

\[
\lambda_0 = 2 g_i - \lambda^i \eta^j(0) = 2 g_{\alpha \beta} t^\alpha \frac{du^\beta}{ds} = t(0),
\]

we have

\[
(9.24) \left( \frac{d}{ds} K(1) \right) t(1) + K(1) K(2) t(2) - \Omega_{\alpha \beta} \frac{du^\alpha}{ds} \frac{du^\beta}{ds}
\]

\[
\Omega_{\gamma \nu} t^\gamma \left( \frac{du^\nu}{ds} \right) + 3 C \Omega_{\alpha \beta} p^\alpha \left( \frac{du^\beta}{ds} \right) + C \Omega_{\alpha \beta, \gamma} \left( \frac{du^\alpha}{ds} \right)
\]

\[
\left( \frac{du^\beta}{ds} \right) \left( \frac{du^\gamma}{ds} \right) + C \Omega_{\alpha \beta} \frac{du^\alpha}{ds} \frac{du^\beta}{ds} \theta_{\gamma} \frac{du^\gamma}{ds} + t(0) \left( \Omega_{\alpha \beta} \frac{du^\gamma}{ds} \right)
\]

\[
\frac{du^\beta}{ds} \right)^2 = 0,
\]

where \(t(1)\) and \(t(2)\) are the projection of \(t^\alpha\) in the direction of \(\xi^i_1\) and \(\xi^i_2\) respectively. In the same way, we may obtain the conjugate of (9.24) given by

\[
(9.25) \left( \frac{d}{ds} \bar{K}(1) \right) \bar{t}(1) + \bar{K}(1) \bar{K}(2) \bar{t}(2) - \bar{\Omega}_{\alpha \beta} \frac{d\bar{u}^\alpha}{ds} \frac{d\bar{u}^\beta}{ds}
\]

\[
\bar{\Omega}_{\bar{\gamma} \bar{\nu}} \bar{t}^\bar{\gamma} \left( \frac{d\bar{u}^\bar{\nu}}{ds} \right) + 3 C \bar{\Omega}_{\bar{\alpha} \bar{\beta}} \bar{p}^\bar{\alpha} \left( \frac{d\bar{u}^\bar{\beta}}{ds} \right) + C \bar{\Omega}_{\bar{\alpha} \bar{\beta}, \bar{\gamma}} \left( \frac{d\bar{u}^\bar{\alpha}}{ds} \right)
\]

\[
\left( \frac{d\bar{u}^\bar{\beta}}{ds} \right) \left( \frac{d\bar{u}^\bar{\gamma}}{ds} \right) + C \bar{\Omega}_{\bar{\alpha} \bar{\beta}} \frac{d\bar{u}^\bar{\alpha}}{ds} \frac{d\bar{u}^\bar{\beta}}{ds} \bar{\theta}_{\bar{\gamma}} \frac{d\bar{u}^\bar{\gamma}}{ds} + \bar{t}(0) \left( \bar{\Omega}_{\bar{\alpha} \bar{\beta}} \frac{d\bar{u}^\bar{\gamma}}{ds} \right)
\]
The equations (9.24) and (9.25) represent the hyper D-line of the hypersurface.

The equations have been represented in terms of the second fundamental tensors and the curvatures (of the curve) with respect to the hypersurface.

10. UNION AND SPECIAL CURVES

Consider two congruences \( \lambda \) and \( \mu \) of curves at any point of \( S_n \) by

\[
\lambda^i = t^\alpha B^i_\alpha + CN^i
\]

and its conjugate,

where \( t^\alpha \) and \( C \) are parameters and \( t^{\bar{\alpha}}, \bar{C} \) are their complex conjugate.

Since \( (\lambda^i, \lambda^{\bar{i}}) \) represents a unit vector, 

\[
2 g_{ij} \lambda^i \lambda^{\bar{j}} = 1
\]

and it follows by use of equations (1.3), (1.4), (1.5) and (10.1) that

\[
2 g_{\alpha \beta} t^\alpha \bar{t}^\beta = 1 - |C|^2,
\]

\[
\mu^i = s^\alpha B^i_\alpha + DN^i
\]

and its conjugate,

where \( s^\alpha \) is a real parameter.

Let \( \Gamma \) be a special curve relative to the congruence \( \lambda \) and an union curve relative to the congruence \( \mu \). A curve of the congruence is said to be a special curve relative to the congruence \( \lambda \), if the vector \( (\lambda^i, \lambda^{\bar{i}}) \) lies in a variety spanned by the vectors \( (p^\alpha B^i_\alpha, p^{\bar{\alpha}} B^{\bar{i}}_{\bar{\alpha}}) \) and \( (q^i, q^{\bar{i}}) \) and a curve of the congruence is said to be union curve relative to the congruence \( \mu \), if the vector \( (\mu^i, \mu^{\bar{i}}) \) lies in a variety spanned by the vectors \( \left( \frac{dz^i}{ds}, \frac{dz^{\bar{i}}}{ds} \right) \).
\( \frac{dz^i}{ds} \) and \(( q^i, q^i)\). In other words,

\[
(\lambda^i, \lambda^i) = x ( p^\alpha B^i_{\alpha}, p^\tilde{\alpha} B^i_{\tilde{\alpha}}) + \omega (q^i, q^i)
\]

and

\[
(\mu^i, \mu^i) = u ( \frac{dz^i}{ds}, \frac{dz^i}{ds}) + z (q^i, q^i),
\]

which yield

\[
(10.3) \quad \lambda^i = xp^\alpha B^i_{\alpha} + wq^i
\]

and its conjugate,

\[
(10.4) \quad \mu^i = u \frac{dz^i}{ds} + zq^i
\]

and its conjugate.

From equations (1.8), (10.1), (10.2), (10.3), (10.4) and their conjugates, we have

\[
(10.5) \quad t^\alpha = (x+w)p^\alpha, \quad C = wK_n
\]

\[
(10.6) \quad s^\alpha = n \frac{du^\alpha}{ds} + z p^\alpha, \quad D = zK_n
\]

and their conjugate relations.

We define

\[
(10.7) \quad R^2 = 2 g_{\alpha \beta} t^\alpha t^\beta, \quad S^2 = 2 g_{\alpha \beta} s^\alpha s^\beta
\]

and their conjugate relations.

\[
K^2_{(1)} = 2 g_{\alpha \beta} p^\alpha p^\tilde{\beta}
\]

\[
\cos \omega = (g_{\alpha \beta} t^\alpha s^\beta + g_{\alpha \beta} t^\alpha s^\beta)/2 \sqrt{g_{\alpha \beta} t^\alpha s^\beta} \sqrt{g_{\alpha \beta} t^\alpha s^\beta},
\]
where $K_{(1)}$ is the first curvature of the curve in $S_n$. Using (1.10), (10.5), (10.6) and their conjugates, we get

(10.8) \[ K_n S \cos \omega = D k_{(1)} \]
and its conjugate.

From this equation and the equations

(10.9) \[ 1 = 2 g_{i j} \mu^i \mu^j = S^2 + D, \]

(10.10) \[ K^2_{(1)} = 2 g_{i j} q^i q^j, \]

where

\[ K^2_{(1)} = k^2_{(1)} + K_n \tilde{K}_n, \]

we have

(10.11) \[ K_n = \frac{eK_{(1)}(1-S^2)^{\frac{1}{2}}}{(1-S^2 \sin^2 \omega)^{\frac{1}{2}}} \]

and

(10.12) \[ K_{(1)} = \frac{eK_{(1)} s \cos \omega}{(1-s^2 \sin^2 \omega)^{\frac{1}{2}}}, \]

where $e = \pm 1$ in order that $e \cos \omega$ be non-negative.

We, now, have the following:

THEOREM (10.1): If a special curve relative to a fixed congruence $\lambda$ is an union curve relative to another fixed congruence $\mu$, then the modulus of the normal and first curvatures with respect to $S_n$ at a given point of the curve are proportional to its first curvature with respect to $S_{n+1}$.
THEOREM (10.2). If the components of the vector field $\lambda$ and $\mu$ to the hypersurface are in the same direction, then the ratio of the two first curvatures is equal to the magnitude of the tangential (to the hypersurface) component of $\mu$.

PROOF: $w = 0$, proves both the theorems.

11. HYPER ASYMPTOTIC CURVES

A curve of hypersurface is said to be a hyperasymptotic curve relative to its congruence $\mu$, if the vectors $(\mu^i, \tilde{\mu}^i)$ lies in the variety spanned by the vectors

\[
(\eta^i(0), \tilde{\eta}^i(0)) \text{ and } (\eta^i(2), \tilde{\eta}^i(2)).
\]

In another words, $(\mu^i, \tilde{\mu}^i)$ and $u(1)(\eta^i(0), \tilde{\eta}^i(0)) + z(1)(\eta^i(2), \tilde{\eta}^i(2))$,

which yields,

\[
(11.1) \quad \mu^i = u(1)\eta^i(0) + z(1)\eta^i(2)
\]

and its conjugate.

From equation (9.1) and (9.2), we deduce

\[
(11.2) \quad \frac{\delta g^i}{\delta s} = -K^2(1)\eta^i(1) + \frac{d}{ds}\log K(1)q^i + K(1)K(2)\eta^i(2)
\]

and its conjugate.

Another expression for $(\frac{\delta q^i}{\delta s}, \frac{\delta \tilde{q}^i}{\delta s})$ will be obtained from (1.8) and its conjugate after using

\[
(11.3) \quad \frac{\delta N^i}{\delta s} = \frac{1}{2} \Omega_{\beta \gamma} g^{\beta \alpha} B^i \alpha \frac{du^\gamma}{ds}
\]
and its conjugate.
From equations (1.8), (9.1), (11.1), (11.2) and their conjugates, we have

\[(11.4) \quad s^\alpha = u_1 \frac{du^\alpha}{ds} + \nu \left( \frac{\delta p^\alpha}{\delta s} - \frac{K_n}{2} \Omega^{\beta\gamma}_{\beta\gamma} g^\beta \alpha \frac{du^\gamma}{ds} - p^\alpha \right)\]

\[\frac{d}{ds} \log K_1 + K_2 \left( \frac{du^\alpha}{ds} \right),\]

\[(11.5) \quad D = \nu \left( \Omega^{\alpha\beta} p^\alpha \frac{du^\beta}{ds} + \frac{dK_n}{ds} - K_n \frac{d}{ds} \log k_1 \right)\]

and their conjugates, where

\[\nu = \frac{Z_1}{K_1 K_2} .\]

Let \((\xi^\alpha_0, \bar{\xi}^\alpha_0), (\xi^\alpha_1, \bar{\xi}^\alpha_1)\) and \((\xi^\alpha_2, \bar{\xi}^\alpha_2)\) be the unit tangent, unit principal normal vector and unit first binormal vectors, \(k_1\) and \(k_2\) be the first and second curvatures of the curve with respect to the hypersurface.

We obtain from Frenet's formulae with respect to \(S_n\),

\[(11.6) \quad \frac{\delta p^\alpha}{\delta s} = - k^2 \left( \frac{du^\alpha}{ds} \right) + p^\alpha \frac{d}{ds} \log k_1 + k_1 k_2 \xi^\alpha_2\]

and its conjugate.

Equations (11.4), (11.5), (11.6), their conjugates and the definition

\[\text{Cos} \phi = \left( g_{\alpha\beta} - s^\alpha \frac{du^\alpha}{ds} + g_{\bar{\alpha}\bar{\beta}} s^\alpha \frac{du^\bar{\beta}}{ds} \right) / S\]

gives

\[(11.7) \quad S \text{Cos} \phi = u_1\]

and
\((11.8) \quad (s^\alpha - s \cos \phi \frac{du^\alpha}{ds}) \left[ \Omega_{\gamma\beta} p^\gamma \frac{du^\beta}{ds} + \frac{dK_n}{ds} - K_n \frac{d}{ds} \right] \log k^{(2)} = D \left[ K_n \overline{K}_n \frac{du^\alpha}{ds} + p^{\alpha} \frac{d}{ds} \log \frac{k^{(1)}}{k^{(2)}} + k^{(1)} k^{(2)} \right] \)

\[ \xi^2_{(2)} - \frac{K_n}{2} \Omega_{\beta\gamma} g^{\alpha\beta} \frac{du^\gamma}{ds} \]

and its conjugate.

A hyperasymptotic curve relative to \(\mu\) is characterized by this equation.