CHAPTER 7

EQUITABLE COLORING OF MYCIELSKI’S GRAPH OF GRAPHS

In this chapter, interesting results regarding the equitable chromatic number \( \chi_e \) for the Mycielski’s graph of complete graph \( \mu(K_n) \), the Mycielski’s graph of cycle \( C(C_n) \), the Mycielski’s graph of path \( C(P_n) \), the Mycielski’s graph of Helm graph \( \mu(H_n) \), the Mycielski’s graph of Gear graph \( \mu(G_n) \), the Mycielski’s graph of Wheel graph \( \mu(W_n) \) and the Mycielski’s graph of Complete Bipartite graph \( \mu(K_{m,n}) \) have been obtained.

7.1 PRELIMINARIES

The open neighborhood of a vertex \( v \) in a graph \( G \), denoted by \( N_G(v) \), is the set of all vertices of \( G \), which are adjacent to \( v \). Also, \( N_G[v] = N_G(v) \cup \{v\} \) is called the closed neighborhood of \( v \) in the graph \( G \).

In this chapter, by \( G \) one means a connected graph. From a graph \( G \), by Mycielski’s construction [22, 38, 50], one can get a graph \( \mu(G) \) with \( V(\mu(G)) = V \cup U \cup \{w\} \), where

\[
V = V(G) = \{v_1, \ldots, v_n\}, \quad U = \{u_1, \ldots, u_n\}, \quad \text{and} \\
E(\mu(G)) = E(G) \cup \{u_i v : v \in N_G(v_i) \cup \{w\}, i = 1, \ldots, n\}.
\]

For each \( 0 \leq i \leq n, \ v_i \) and \( u_i \) are called the corresponding vertices of \( \mu(G) \) and
denote \( C(v_i) = u_i, \) \( C(u_i) = v_i. \) Moreover, for subsets \( A \subseteq U, \) \( B \subseteq V, \) one denotes: \( C(A) = \{ C(u_i) : u_i \in A \}, \) \( C(B) = \{ C(v_i) : v_i \in B \}. \) Also, \( x \leftrightarrow y \) is denoted, when \( \{x, y\} \) is an edge.

### 7.2 EQUITABLE COLORING ON MYCIELSKI’S GRAPH OF \( K_n, C_n \) AND \( P_n \)

**Theorem 7.2.1.** The equitable chromatic number of Mycielski’s graph of complete graph, \( \chi_e(\mu(K_n)) = n + 1. \)

**Proof.** Let \( V(K_n) = \{x_i : 1 \leq i \leq n\}. \) By the construction of Mycielski’s graph,

\[
V(\mu(K_n)) = V(K_n) \cup \{y_i : 1 \leq i \leq n\} \cup \{z\} \quad \text{and} \quad E(\mu(K_n)) = E(K_n) \cup \{y_i : x : x \in N_{K_n}(x_i) \cup \{z\}, i = 1, 2, \ldots, n\}.
\]

Since \( z \) is adjacent with each vertex of \( \{y_i : 1 \leq i \leq n\}, \) also \( \mu(K_n) \) contains a \( n \)-clique, \( \chi_e(\mu(K_n)) \geq n \) and hence \( \chi_e(\mu(K_n)) \geq n. \)

![Figure 7.1: Mycielski’s graph of complete graph \( \mu(K_n). \)](image-url)
Assume that $\chi_=(\mu(K_n)) = n$. (i.e.) There exists a partition $P = \{V_1, V_2, \ldots, V_n\}$, since $|V(\mu(K_n))| = 2n + 1$, each $V_i (1 \leq i \leq n)$ contains either 2 or 3 vertices. Since each $x_i$ is adjacent with each $y_j (1 \leq j \leq n)$ where $i \neq j$, $x_i, y_i \in V_i (1 \leq i \leq n)$. Since $z$ is adjacent to each $y_i (1 \leq i \leq n)$, $z \notin V_i$ for every $i, (1 \leq i \leq n)$

\[ \Rightarrow \sum_{i=1}^n |V_i| = 2n \neq 2n + 1 \]

This contradiction shows that $\chi_=(\mu(K_n)) \geq n + 1$.

Now partition the vertex set $V(\mu(K_n))$ as follows,

\[ V_i = \{x_i, y_i : 1 \leq i \leq n\} \]
\[ V_{n+1} = \{z\}. \]

Clearly $V_i (1 \leq i \leq n)$ and $V_{n+1}$ are independent sets, since $|V_i| = 2(1 \leq i \leq n)$ and $|V_{n+1}| = 1$ satisfying the condition $||V_i| - |V_j|| \leq 1$, for any $i \neq j$, $\chi_=(\mu(K_n)) \leq n + 1$.

Hence $\chi_=(\mu(K_n)) = n + 1$.

**Theorem 7.2.2.** The equitable chromatic number of Mycielski's graph of cycle

\[ \chi_=(\mu(C_n)) = \begin{cases} 
3 & \text{if } n = 4, 6, 8 \\
4 & \text{if } n \geq 10, n \text{ is even} \\
4 & \text{if } n \text{ is odd} 
\end{cases} \]

**Proof.** Let $V(C_n) = \{x_i : 1 \leq i \leq n\}$ be the set of vertices of $C_n$ taken in cyclic order. By the construction of Mycielski's graph,

\[ V(\mu(C_n)) = V(C_n) \cup \{y_i : 1 \leq i \leq n\} \cup \{z\} \text{ and} \]
\[ E(\mu(C_n)) = E(C_n) \cup \{y_ix : x \in N_{C_n}(x_i) \cup \{z\}, i = 1, 2, \ldots, n\}. \]

Now partition the vertex set of $V(\mu(C_n))$ as follows:
Case 1: \((n = 4, 6, 8)\)

\[
V_1 = \left\{ x_{2i-1} : 1 \leq i \leq \frac{n}{2} \right\} \cup \{z\}
\]
\[
V_2 = \left\{ x_{2i} : 1 \leq i \leq \frac{n}{2} \right\} \cup \{y_i : i = 4, 6\}
\]
\[
V_3 = \{y_1, y_2, y_3, y_5, y_7, y_8\}.
\]

It is clear that \(V_i \cap V_j = \phi\) for \(i \neq j\).

Since there exists a 5-cycle \(x_1x_2y_3zy_2x_1\), \(\chi(\mu(C_n)) \leq 3\) and hence \(\chi_(\mu(C_n)) = 3\).

Case 2: \(n \geq 10\), \(n\) is even and \(n\) is odd

Since there exist a 5-cycle as shown in the previous case, it is clear that \(\chi_(\mu(C_n)) \geq 3\).

Assume that \(\chi_(\mu(C_n)) = 3\) and let \(V = \{V_1, V_2, V_3\}\) be the corresponding equitable partition of the color class. Since \(|V(\mu(C_n))| = 2n + 1|\),

\(|V_i| = \left\lfloor \frac{2n+1}{3} \right\rfloor\) or \(\left\lfloor \frac{2n+1}{3} \right\rfloor + 1, 1 \leq i \leq 3\).

If \(2n + 1 \equiv 0 \mod 3\), \(|V_i| = \left\lfloor \frac{2n+1}{3} \right\rfloor, 1 \leq i \leq 3\)

\(2n + 1 \equiv 1, 2 \mod 3\), \(|V_i| = \left\lfloor \frac{2n+1}{3} \right\rfloor\) or \(\left\lfloor \frac{2n+1}{3} \right\rfloor + 1\)
Since \( \{x_{2i} : 1 \leq i \leq \frac{n}{2} \} \cup \{z\} \) is an independent set, any one of the \( V_i \)'s contains exactly these vertices.

(i.e.) \(|V_i| = \frac{n}{2} + 1\) for some \( i \), \( 1 \leq i \leq 3 \)

Hence the sum of the number of vertices of other two partition is \((2n + 1) - \left( \frac{n}{2} + 1 \right) = \frac{3n}{2}\)

(i.e.) \(|V_j| = \left\lfloor \frac{3n}{4} \right\rfloor\) for \( j \neq i \)

But \( \sum_{i=1}^{n} |V_i| \neq 2n + 1\) if \( n \geq 10\) and \( n \) is odd.

This contradiction shows that \( \chi = (\mu (C_n)) \geq 4\).

Now the following partition gives the equitable coloring of \( \mu (C_n) \) for even \( n \geq 10\) and odd \( n \).

1. If \( n \geq 10 \) is even:

\[
\begin{align*}
V_1 &= \{x_{2i-1} : 1 \leq i \leq \frac{n}{2}\} \cup \{z\} \\
V_2 &= \{x_{2i} : 1 \leq i \leq \frac{n}{2}\} \\
V_3 &= \{y_{2i-1} : 1 \leq i \leq \frac{n}{2}\} \\
V_4 &= \{y_{2i} : 1 \leq i \leq \frac{n}{2}\}
\end{align*}
\]

2. If \( n \) is odd:

\[
\begin{align*}
V_1 &= \{x_{2i-1} : 1 \leq i \leq \frac{n-1}{2}\} \cup \{z\} \\
V_2 &= \{x_{2i} : 1 \leq i \leq \frac{n-1}{2}\} \\
V_3 &= \{y_{2i-1} : 1 \leq i \leq \frac{n-1}{2}\} \cup \{x_n\} \\
V_4 &= \{y_{2i} : 1 \leq i \leq \frac{n-1}{2}\} \cup \{y_n\}
\end{align*}
\]

This implies \( \chi = (\mu (C_n)) = 4\). ☐
Theorem 7.2.3. The equitable chromatic number of Mycielski’s graph of path

\[\chi_{\text{e}}(\mu(P_n)) = \begin{cases} 
2 & n = 1 \\
3 & \text{if } n \leq 11, \ n \neq 10 \\
4 & \text{if } n \geq 10, \ n \neq 11 
\end{cases}\]

Proof. Let \(V(P_n) = \{x_i : 1 \leq i \leq n\}\) be the set of vertices of \(P_n\). By the construction of Mycielski’s graph,

\[\begin{align*}
V(\mu(P_n)) &= V(P_n) \cup \{y_i : 1 \leq i \leq n\} \cup \{z\} \text{ and} \\
E(\mu(P_n)) &= E(P_n) \cup \{y_i x : x \in N_{P_n}(x_i) \cup \{z\}, \ i = 1, 2, \ldots, n\}.
\end{align*}\]

Now partition the vertex set of \(V(\mu(P_n))\) as follows:

Case 1: \(n = 1\)

\[\begin{align*}
V_1 &= \{x_1, z\} \\
V_2 &= \{y_1\}
\end{align*}\]

It is obvious.
Case 2: By this partitions $\chi = (\mu (P_n)) = 3$

1. For $2 \leq n \leq 4$

\[
V_1 = \{x_1, x_3, z\} \\
V_2 = \{x_2, x_4, y_2\} \\
V_3 = \{y_1, y_3, y_4\}
\]

2. For $5 \leq n \leq 9$

\[
V_1 = \{x_{2i-1} : 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil \} \cup \{z\} \\
V_2 = \{x_{2i} : 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \} \cup \{y_2, y_4\} \\
V_3 = \{y_i : 5 \leq i \leq n\} \cup \{y_1, y_3\}
\]

3. For $n = 11$

\[
V_1 = \{x_{2i-1} : 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil \} \cup \{z\} \\
V_2 = \{x_{2i} : 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \} \cup \{y_2, y_4, y_6\} \\
V_3 = \{y_i : 7 \leq i \leq n\} \cup \{y_1, y_3\}
\]

It is clear that $V_i \cap V_j = \emptyset$ for $i \neq j$. Since there exists a 5-cycle $x_1x_2y_3zy_2x_1$, $\chi (\mu (P_n)) \leq 3$ and hence $\chi = (\mu (P_n)) = 3$.

Case 3: By this partition $\chi = (\mu (P_n)) = 4$ for $n \geq 10$, $n \neq 11$.

Since there exist a 5-cycle as shown in the previous case, it is clear that $\chi = (\mu (K_n)) \geq 3$.

Assume that $\chi = (\mu (P_n)) = 3$ and let $V = \{V_1, V_2, V_3\}$ be the corresponding equitable partition of the color class. Since $|V (\mu (P_n))| = 2n + 1$,

$|V_i| = \left\lceil \frac{2n + 1}{3} \right\rceil$ or $\left\lfloor \frac{2n + 1}{3} \right\rfloor + 1$, $1 \leq i \leq 3$.

If $2n + 1 \equiv 0 \mod 3$, $|V_i| = \left\lceil \frac{2n + 1}{3} \right\rceil$, $1 \leq i \leq 3$

If $2n + 1 \equiv 1, 2 \mod 3$, $|V_i| = \left\lfloor \frac{2n + 1}{3} \right\rfloor$ or $\left\lceil \frac{2n + 1}{3} \right\rceil + 1$
Since \( \{ x_{2i} : 1 \leq i \leq \frac{n}{2} \} \cup \{ z \} \) is an independent set, any one of the \( V_i \)'s contains exactly these vertices.

(i.e.) \( |V_i| = \frac{n}{2} + 1 \) for some \( i \), \( 1 \leq i \leq 3 \)

Hence the sum of the number of vertices of other two partition is \( (2n + 1) - \left( \frac{n}{2} + 1 \right) = \frac{3n}{2} \)

(i.e.) \( |V_j| = \left\lceil \frac{3n}{4} \right\rceil \) for \( j \neq i \)

But \( \sum_{i=1}^{n} |V_i| \neq 2n + 1 \) if \( n \geq 10, \ n \neq 11 \).

This contradiction shows that \( \chi = (\mu(P_n)) \geq 4 \).

Now the following partition gives the equitable coloring of \( \mu(P_n) \) for \( n \geq 10, \ n \neq 11 \).

\[
V_1 = \{ x_{2i} : 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \} \cup \{ z \}
\]

\[
V_2 = \{ x_{2i-1} : 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil \}
\]

\[
V_3 = \{ y_{2i-1} : 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \}
\]

\[
V_4 = \{ y_{2i} : 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil \}
\]

This implies \( \chi = (\mu(P_n)) = 4 \).

\[\square\]

### 7.3 EQUITABLE COLORING ON MYCIELSKI’S GRAPH

OF \( H_n, G_n \) AND \( W_n \)

**Theorem 7.3.1.** The equitable chromatic number of Mycielski’s graph of Helm graph

\[
\chi = (\mu(H_{n+1})) = \begin{cases} 
4 & \text{if } n \text{ is even} \\
5 & \text{if } n \text{ is odd}
\end{cases}
\]
Proof. Let
\[ V(H_{n+1}) = \{x_i : 0 \leq i \leq n\} \cup \{x_i' : 1 \leq i \leq n\} \]
and \[ E(H_{n+1}) = \{x_0x_i : 1 \leq i \leq n\} \cup \{x_ix_{i+1} : 1 \leq i \leq n-1\} \]
\[ \cup \{x_nx_1\} \cup \{x_ix_i' : 1 \leq i \leq n\} . \]

By Mycielski’s construction,
\[ V(\mu(H_{n+1})) = V(H_{n+1}) \cup \{y_i : 0 \leq i \leq n\} \cup \{y_i' : 1 \leq i \leq n\} \cup \{z\} . \]

In \( \mu(H_{n+1}) \), \( y_0 \) is adjacent with each of \( \{x_i : 1 \leq i \leq n\} \), each \( y_i \) is adjacent with the neighbour of \( x_i \) \((0 \leq i \leq n)\), \( z \) is adjacent with each of \( \{y_i : 0 \leq i \leq n\} \) and \( \{y_i' : 1 \leq i \leq n\} \). Also each \( y_i' \) is adjacent with \( x_i \) \((1 \leq i \leq n)\)

Since \( x_0 \) and \( y_0 \) are non-adjacent vertices, we use same color to color \( x_0 \) and \( y_0 \) let it be 1. Since \( y_0 \) is adjacent with each vertex of \( \{x_i : 1 \leq i \leq n\} \), the color 1 cannot be assigned to any vertex \( \{x_i : 1 \leq i \leq n\} \).

**Case (i) \((n \text{ is even})\)**

Since the set of vertices \( \{x_i : 1 \leq i \leq n\} \) induce a cycle of length \( n \) in \( \mu(H_n) \), we use two colors 2 and 3 to color all the vertices of this cycle. Also same color cannot be assigned to color the vertices of \( \{y_i : 1 \leq i \leq n\} \), since \( z \) is adjacent with each vertex of \( \{y_i : 0 \leq i \leq n\} \), \( z \) cannot be assigned with any of the three colors and let it be assigned with color 4. Hence \( \chi_e(\mu(H_n)) \geq 4 \).

we use the following partition to color \( \mu(H_n) \) equitably with 4 given colors.

\[ V_1 = \{x_0, y_0, y_i' : 3 \leq i \leq n\} \]
\[ V_2 = \{x_{2i-1}, y_{2i-1} : 1 \leq i \leq \frac{n}{2}\} \cup \{y_2'\} \]
\[ V_3 = \{x_{2i}, y_{2i} : 1 \leq i \leq \frac{n}{2}\} \cup \{y_1'\} \]
\[ V_4 = \{z, x_i' : 1 \leq i \leq n\} . \]

Clearly \( ||V_i| - |V_j|| \leq 1 \) for \( i \neq j \).
Hence $\chi(\mu(H_n)) \leq 4$ and so $\chi(\mu(H_n)) = 4$.

Case (ii) ($n$ is odd)

Since the vertices $\{x_i : 1 \leq i \leq n\}$ induce a cycle of odd length $n$, in $\mu(H_n)$, we use three colors (2, 3 and 4) to color the vertices of this cycle. Now $z$ is colored with a new color 5. Hence $\chi(\mu(H_n)) \geq 5$

We use the following partition to color the vertices of $\mu(H_n)$ equitable in the following cases,

Case (ii)a $n = 6k - 3$

$$V_1 = \{x_0, y_0\} \cup \left\{x_i' : \frac{2n + 6}{3} \leq i \leq n\right\} \cup \left\{y_i' : \frac{n - 3}{3} \leq i \leq n\right\}$$

$$V_2 = \left\{x_{3i-2}, y_{3i-2} : 1 \leq i \leq \frac{n}{3}\right\} \cup \left\{y_{3i} : 1 \leq i \leq \frac{n - 3}{3}\right\}$$

$$V_3 = \left\{x_{3i-2}, y_{3i-1} : 1 \leq i \leq \frac{n}{3}\right\} \cup \left\{y_{3i-2} : 1 \leq i \leq \frac{n - 3}{3}\right\}$$

$$V_4 = \left\{x_{3i}, y_{3i} : 1 \leq i \leq \frac{n}{3}\right\} \cup \left\{y_{3i-1} : 1 \leq i \leq \frac{n - 3}{3}\right\}$$

$$V_5 = \left\{x_i' : 1 \leq i \leq \frac{2n + 3}{3}\right\} \cup \{z\}$$

Case (ii)b $n = 6k - 1$

$$V_1 = \{x_0, y_0\} \cup \left\{x_i' : \frac{2n + 2}{3} \leq i \leq n\right\} \cup \left\{y_i' : \frac{n + 1}{2} \leq i \leq n\right\}$$

$$V_2 = \left\{x_{3i-2}, y_{3i-2} : 1 \leq i \leq \frac{n + 1}{3}\right\} \cup \left\{y_{3i} : 1 \leq i \leq \frac{n + 1}{3}\right\}$$

$$V_3 = \left\{x_{3i-2}, y_{3i-1} : 1 \leq i \leq \frac{n + 1}{3}\right\} \cup \left\{y_{3i-2} : 1 \leq i \leq \frac{n + 1}{3}\right\}$$

$$V_4 = \left\{x_{3i}, y_{3i} : 1 \leq i \leq \frac{n - 2}{3}\right\} \cup \left\{y_{3i-1} : 1 \leq i \leq \frac{n + 1}{6}\right\}$$

$$V_5 = \left\{x_i' : 1 \leq i \leq \frac{2n - 1}{3}\right\} \cup \{z\}$$
Case (ii)c \( n = 6k + 1 \)

\[
V_1 = \{x_0, y_0\} \cup \{x_i : \frac{5n+1}{6} \leq i \leq n\} \cup \{y'_i : \frac{n+11}{3} \leq i \leq n\}
\]

\[
V_2 = \{x_{3i-2}, y_{3i-2} : 1 \leq i \leq \frac{n+2}{3}\} \cup \{y'_{3i+2} : 1 \leq i \leq \frac{n-7}{6}\}
\]

\[
V_3 = \{x_{3i-1}, y_{3i-1} : 1 \leq i \leq \frac{n+1}{3}\} \cup \{y'_{3i+3} : 1 \leq i \leq \frac{n-7}{6}\} \cup \{y'_1, y'_3\}
\]

\[
V_4 = \{x_{3i}, y_{3i} : 1 \leq i \leq \frac{n-1}{3}\} \cup \{y'_{3i+4} : 1 \leq i \leq \frac{n-7}{6}\} \cup \{y'_2, y'_4\}
\]

\[
V_5 = \{x'_i : 1 \leq i \leq \frac{5n-5}{6}\} \cup \{z\}
\]

Clearly in all the cases \(||V_i| - |V_j|| \leq 1\) for \( i \neq j \).

Hence \( \chi_{=}(\mu(H_n)) \leq 5 \) and so \( \chi_{=}(\mu(H_n)) = 5 \) for \( n \) is odd.

**Theorem 7.3.2.** The equitable chromatic number of Mycielski’s graph of Gear graph

\[
\chi_{=}(\mu(G_n)) = \begin{cases} 
3 & \text{if } 3 \leq n \leq 5 \\
4 & \text{if } n \geq 6 
\end{cases}
\]

**Proof.** Let

\[
V(G_n) = \{x_0\} \cup \{x_i : 1 \leq i \leq n\} \cup \{x'_i : 1 \leq i \leq n\}
\]

and

\[
E(G_n) = \{x_0x_i : 1 \leq i \leq n\} \cup \{x_ix'_i : 1 \leq i \leq n\} \\
\cup \{x'_ix_{i+1} : 1 \leq i \leq n-1\} \cup \{x_nx_1\}.
\]

By the construction of Mycielski’s graph,

\[
V(\mu(G_n)) = V(G_n) \cup \{y_i : 1 \leq i \leq n\} \cup \{y'_i : 1 \leq i \leq n\} \cup \{z\}.
\]

In \( \mu(G_n) \), \( y_0 \) is adjacent with each vertex of \( \{x_i : 1 \leq i \leq n\} \), each \( y_i \) is adjacent with the neighbour of \( x_i \) (\( 1 \leq i \leq n\)) and each \( y'_i \) is adjacent with the neighbours of \( x'_i \) (\( 1 \leq i \leq n\)), \( z \) is adjacent with each vertex of \( \{y_i : 1 \leq i \leq n\} \cup \{y'_i : 1 \leq i \leq n\} \cup \{y_0\} \).
Case (i) \((3 \leq n \leq 5)\)

Since the set of vertices \(\{x_i : 1 \leq i \leq n\} \cup \{x_i' : 1 \leq i \leq n\}\) induces an even cycle in \(\mu(G_n)\), it requires at least two colors to color to the vertices of this cycle (say 1 and 2). If we assign the same color 1 and 2 to color the vertices of \(\{y_i : 1 \leq i \leq n\} \cup \{y_i' : 1 \leq i \leq n\}\), then \(z\) will receive a new color other than 1 and 2. Hence \(\chi=(\mu(G_n)) \geq 3.\)

\[V_1 = \{x_i' : 1 \leq i \leq n\} \cup \{z\}\]
\[V_2 = \{x_i : 1 \leq i \leq n\} \cup \{y_1, y_3\}\]
\[V_3 = \{y_0, y_2, y_4, y_5\}\]

Clearly \(||V_i| - |V_j|| \leq 1\) for \(i \neq j\).

Hence \(\chi=(\mu(G_n)) \leq 3\) and hence \(\chi= (\mu (G_n)) = 3.\)

Case (ii) \((n \geq 6)\)

We know that \(|V (\mu (G_n))| = 4n + 3\). Out of these \(4n + 3\), the vertices of \(\{x_i' : 1 \leq i \leq n\} \cup \{x_0\} \cup \{z\}\) can be assigned with one color. i.e., \(n + 2\) vertices can be colored with one color. Hence remaining number of vertices to be colored is 
\((4n + 3) - (n + 2) = 3n + 1.\)

If \(\chi=(\mu(G_n)) = 3\), these \(3n + 1\) vertices should be divided into two sets, each containing \(\frac{3n}{2}\) and \(\frac{3n+1}{2}\) vertices.

Since \(\frac{3n}{2} \neq n + 2\) and \(\frac{3n+1}{2} \neq n + 2\) for \(n \geq 6\), we conclude that the partition of \(V (\mu (G_n))\) into three sets satisfying \(||V_i| - |V_j|| \leq 1\) for \(i \neq j\) is not possible and hence \(\chi= (\mu(G_n)) \geq 4.\)

\[V_1 = \{x_i' : 1 \leq i \leq n\} \cup \{x_0\}\]
\[V_2 = \{x_i : 1 \leq i \leq n\} \cup \{z\}\]
\[V_3 = \{y_i' : 1 \leq i \leq n\} \cup \{y_0\}\]
\[V_4 = \{y_i : 1 \leq i \leq n\}\]
Clearly $||V_i| - |V_j|| \leq 1$ for $i \neq j$.

Hence $\chi = \mu(G_n) \leq 4$ and so $\chi = \mu(G_n) = 4$.

**Theorem 7.3.3.** The equitable chromatic number of Mycielskian of Wheel graph

\[
\chi = (\mu(W_{n+1})) = \begin{cases} 
5 & \text{if } n = 3 \\
\left\lceil \frac{2n + 4}{3} \right\rceil & \text{if } n \geq 4 
\end{cases}
\]

**Proof.** Let $V(W_{n+1}) = \{x_i : 0 \leq i \leq n\}$ and $E(W_{n+1}) = \{x_0x_i : 1 \leq i \leq n\} \cup \{x_ix_{i+1} : 1 \leq i \leq n-1\} \cup \{x_nx_1\}$. By Mycielski’s construction, $V(\mu(W_n)) = V(W_n) \cup \{y_i : 0 \leq i \leq n\} \cup \{z\}$. In $\mu(W_{n+1})$, each $y_i (0 \leq i \leq n)$ is adjacent with each vertex of $N_{W_{n+1}}(x_i)$, and $z$ is adjacent with each vertex of $\{y_i : 0 \leq i \leq n\}$.

**Case (i) :** $n = 3$

Since $\{x_0, x_1, x_2, x_3\}$ is $K_4$, each of these vertices receives distinct colors $c_1, c_2, c_3, c_4$ respectively. The same color can be used to color the vertices of $y_0, y_1, y_2$ and $y_3$ in the same order. Since $z$ is adjacent with each of $y_0, y_1, y_2$ and $y_3$, there exists a new color to color $z$. Hence $\chi = \mu(W_{n+1}) = 5$ for $n = 3$.

**Case (ii) :** $n \geq 4$

By the definition of Mycielskian, $x_0$ is adjacent with each of $\{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n\}$. The color which is assigned to $x_0$ cannot be assigned to any vertex of $\{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n\}$ and hence it can be assigned at most two times (possibly to $z$) in $\mu(W_{n+1})$. By the definition of equitable coloring, the number of vertices of $\mu(W_{n+1})$, receiving the same color, is 2 or 3. Therefore $\chi = \mu(W_{n+1})) \geq \left\lceil \frac{2n + 1}{3} \right\rceil + 1 = \left\lceil \frac{2n + 4}{3} \right\rceil$.

We use the following partition to color the vertices of $\mu(W_{n+1})$ equitably in the following cases,
Case (ii)a : \( n = 4 \)

\[
\begin{align*}
V_1 &= \{x_0, y_0\} \\
V_2 &= \{x_1, x_3, z\} \\
V_3 &= \{x_2, y_2, x_4\} \\
V_4 &= \{y_1, y_3, y_4\}
\end{align*}
\]

Case (ii)b : \( n = 5 \)

\[
\begin{align*}
V_1 &= \{x_0, y_0\} \\
V_2 &= \{x_1, x_4, y_1\} \\
V_3 &= \{x_2, y_2, y_4\} \\
V_4 &= \{x_3, y_3, y_5\} \\
V_5 &= \{x_5, z\}
\end{align*}
\]

Case (ii)c : \( n = 6 \)

\[
\begin{align*}
V_1 &= \{x_0, y_0\} \\
V_2 &= \{x_1, y_1, y_5\} \\
V_3 &= \{x_2, y_2, y_6\} \\
V_4 &= \{x_3, y_3\} \\
V_5 &= \{x_4, y_4, x_6\} \\
V_6 &= \{x_5, z\}
\end{align*}
\]
Case (ii)d: \( n \geq 7 \)

\[
\begin{align*}
V_1 &= \{x_0, y_0\} \\
V_{i+1} &= \left\{ x_i : 1 \leq i \leq \left\lfloor \frac{2n+1}{3} \right\rfloor \right\} \cup \left\{ y_i : 1 \leq i \leq \left\lfloor \frac{2n+1}{3} \right\rfloor \right\} \\
&\quad \cup \left\{ x_{\frac{2n+4}{3}} : 1 \leq i \leq n - \left\lfloor \frac{2n+4}{3} \right\rfloor \right\} \\
&\quad \cup \left\{ y_{\frac{4n+2}{3}} : n+2 - \left\lfloor \frac{2n+4}{3} \right\rfloor \leq i \leq \left\lfloor \frac{2n-1}{3} \right\rfloor \right\} \\
V_{\frac{2n+4}{3}} &= \left\{ x_{\frac{2n+4}{3}}, x_n, z \right\}
\end{align*}
\]

Clearly \(|V_i - V_j| \leq 1\) for \( i \neq j \). Therefore \( \chi^=(_Wn) \leq \left\lfloor \frac{2n+4}{3} \right\rfloor \). Hence
\[
\chi^=(_Wn) = \left\lfloor \frac{2n+4}{3} \right\rfloor \text{ for } n \geq 4.
\]

\section{Equitable Coloring on Mycielski’s Graph of Bipartite Graph}

**Theorem 7.4.1.** The equitable chromatic number of Mycielski’s graph of Bipartite graph

\[
\chi^=(_Km,n) = \begin{cases} 
3 & \text{if } m = n \leq 4 \\
4 & \text{if } m = n \geq 5 \\
3 & \text{if } 2m = n \\
\left\lfloor \frac{m+n}{\min (m,n)} \right\rfloor , \text{ otherwise}
\end{cases}
\]

**Proof.** Let \( V(_Km,n) = \{x_i : 1 \leq i \leq m\} \cup \{y_j : 1 \leq j \leq n\} \) and \( E(_Km,n) = \bigcup_{i=1}^{m} \{e_{ij} \} \)
\[ = x_iy_j : 1 \leq j \leq n \}. \]  By the construction of Mycielski’s graph, \( V(\mu(K_{m,n})) = V(K_{m,n}) \cup \{x'_i : 1 \leq i \leq m\} \cup \{y'_j : 1 \leq j \leq n\} \cup \{z\} \).

Case 1: \( m = n \leq 4 \)

Since \( \langle\{x_i : 1 \leq i \leq 4\}\rangle, \langle\{y_i : 1 \leq i \leq 4\}\rangle \) and \( \langle\{y'_i : 1 \leq i \leq 4\}\rangle \) are totally disconnected, each of these subgraphs can be assigned with single color.

Assign the color \( c_1 \) to each of \( \{x_i : 1 \leq i \leq 4\} \), \( c_2 \) to each of \( \{y_i : 1 \leq i \leq 4\} \) and \( c_3 \) to each of \( \{y'_i : 1 \leq i \leq 4\} \). Assign \( c_1 \) to \( x'_1 \), \( c_2 \) to \( z \) and \( c_3 \) to \( x'_2 \). By this process of assigning color, each color is assigned exactly 4 times. To color the remaining vertices \( x'_3 \) and \( x'_4 \), we can use \( c_1 \) and \( c_3 \) without affecting the equitable condition. Hence \( \chi(\mu(K_{m,n})) = 3 \) for \( m = n \leq 4 \).

Case 2: \( m = n \geq 5 \)

By applying the above said procedure to color equitably, let us color each vertices of \( \{y_i : 1 \leq i \leq n\} \cup \{z\} \) by \( c_2 \). Hence the remaining number of vertices to be colored is \( 3n \). These \( 3n \) vertices can be divided by into two sets, each containing \( \frac{3n}{2} \) and \( \frac{3n}{2} + 1 \) vertices.

Since \( \frac{3n}{2} \neq n + 1 \) and \( \frac{3n}{2} + 1 \neq n + 1 \) for \( n \geq 5 \), now conclude that the partition of \( V(\mu(K_{m,n})) \) into three sets satisfying \( ||V_i| - |V_j|| \leq 1 \) for \( i \neq j \) is not possible and hence \( \chi(\mu(K_{m,n})) \geq 4 \). Use the following partition to color the vertices of \( \mu(K_{m,n}) \) for \( m = n \geq 5 \) equitably,

\[
V_1 = \{x_i : 1 \leq i \leq n\} \cup \{z\} \\
V_2 = \{y_i : 1 \leq i \leq n\} \\
V_3 = \{x'_i : 1 \leq i \leq n\} \\
V_4 = \{y'_i : 1 \leq i \leq n\}
\]

Clearly \( ||V_i| - |V_j|| \leq 1 \) for \( i \neq j \).

Hence \( \chi(\mu(K_{m,n})) \leq 4 \) and hence \( \chi(\mu(K_{m,n})) = 4 \).

Case 3: \( 2m = n \)

Since there exist an odd cycle \( x_1y'_1zx'_1y_1x_1 \) in \( \mu(K_{m,n}) \), for \( 2m = n \),
\[ \chi(\mu(K_{m,n})) \geq 3 \]

\[
V_1 = \{ x_i, x'_i : 1 \leq i \leq m \} \\
V_2 = \{ y_i : 1 \leq i \leq n \} \cup \{ z \} \\
V_3 = \{ y'_i : 1 \leq i \leq n \}
\]

Clearly \( |V_i| - |V_j| \leq 1 \) for \( i \neq j \).

Hence \( \chi(\mu(K_{m,n})) \leq 3 \) and so \( \chi(\mu(K_{m,n})) = 3 \).

**Case 4:**

\[
\left\lfloor \frac{m + n}{\min(m,n)} \right\rfloor
\]

\(|V(\mu(K_{m,n}))| = 2m + 2n + 1\) without loss of generality assume that \( m < n \).

Then by the construction of equitable coloring, all the vertices of \( X = \{ x_i : 1 \leq i \leq m \} \cup \{ x'_i : 1 \leq i \leq m \} \) receive the colors according to any one of the following cases.

1. If \( X \) receives the same color 1. In this case the color 1, appears at the maximum of \( 2m \) times. The other colors 2, 3, \ldots are used at the maximum of \( 2m+1 \) times to color the vertices of \( \{ y_j : 1 \leq j \leq n \} \cup \{ y'_j : 1 \leq j \leq n \} \cup \{ z \} \).

Therefore each color appear atmost \( 2m + 1 \) times and at least \( 2m \) times.

2. If \( X \) receives the same color 1 with an another color 3, then the color 1, appears at the maximum of \( 2m - 1 \) times and other colors 2, 3, \ldots are used at the maximum of \( 2m \) times to color the vertices of \( \{ y_j : 1 \leq j \leq n \} \cup \{ y'_j : 1 \leq j \leq n \} \cup \{ z \} \). Therefore in this case each color appear atmost \( 2m \) times and at least \( 2m - 1 \) times.

Therefore from both the subcases the minimum number of colors to be used to get an equitable coloring of \( \mu(K_{m,n}) \) is either \( \left\lfloor \frac{2m + 2n + 1}{2m} \right\rfloor \) or \( \left\lfloor \frac{2m + 2n + 1}{2m + 1} \right\rfloor \). (i.e.) \( \left\lfloor \frac{m+n}{m} \right\rfloor \) from the subcases 1 and 2.

In general \( \left\lfloor \frac{m+n}{\min(m,n)} \right\rfloor \).

In such a coloring, out of \( \left\lfloor \frac{m+n}{\min(m,n)} \right\rfloor \) colors,
\[(2m + 2n + 1) - \left(\left\lfloor \frac{2m + 2n + 1}{m + n} \right\rfloor \times \left\lfloor \frac{m + n}{\min(m, n)} \right\rfloor \right)\] colors appear

\[\left\lfloor \frac{2m + 2n + 1}{m + n} \right\rfloor + 1\] times

and

\[\left(\left\lfloor \frac{2m + 2n + 1}{m + n} \right\rfloor + 1\right) \times \left\lfloor \frac{m + n}{\min(m, n)} \right\rfloor - (2m + 2n + 1)\] colors appear

\[\left\lfloor \frac{2m + 2n + 1}{m + n} \right\rfloor\] times.

Hence \(\chi(\mu(K_{m,n})) = \left\lfloor \frac{m + n}{\min(m, n)} \right\rfloor\).
CONCLUSIONS

The area of equitable coloring is wide open and this thesis has barely scratched the surface of it. There are other directions which could lead to fruitful research. However one wants to stress the significance of the Equitable $\Delta$–Coloring Conjecture again. The settlement of this conjecture presumably will reveal profound nature of the equitable coloribility.

Although some of the equitable coloring results discussed here can be extended to disconnected graphs. A comprehensive study of the disconnected case has not been achieved. The issue of efficient algorithms for these special classes of graphs, how to design efficient algorithms to color them as even as possible is largely unexplored area.

This thesis establishes an optimal polynomial time solution to the equitable chromatic number of different families of graph like corona product of graphs, Knödel graphs, Wheel graph families, Mycielski’s graph of graphs and as a by-product ECC to these classes of graphs are confirmed.