CHAPTER II

AN EXACT STATIC SOLUTION OF EINSTEIN’S FIELD EQUATIONS USING COSMOLOGICAL CONSTANT Λ WITH SPECIFIED EQUATION OF STATE *

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"It is true that the theory of relativity, particularly the general theory, has played a rather modest role in the correlation of empirical facts so far, and it has contributed little to atomic physics and our understanding of Quantum phenomena".

.... "A. EINSTEIN".
2.1. **INTRODUCTION:**

The present study deals with the exact solutions of the Einstein's field equations for the perfect fluid with variable gravitational and cosmological “constants” for a spatially homogeneous and anisotropic cosmological model. The Einstein's field equation has two parameters; the cosmological constants $\Lambda$ and the gravitational constant $G$. Cosmological models with a cosmological constant are currently serious candidates to describe the dynamics of the Universe [15, 17, 22]. The rise of interest in the theory of General Relativity as a tool for studying the evolution and behaviour of various cosmological models has been rapid expensive. Since the early 1920’s to the present, the Einstein's theory of relativity has been used extensively as a tool in the prediction and modelling of the cosmos. One reason for the prominence of modern relativity is its success in predicting the behaviour of large scale phenomena where gravitation plays a dominant role [6-8]. Various researcher in theory of relativity have focused their mind to the study of solution of Einstein’s field equation with cosmological constant $\Lambda = 0$ and equation of state $p = \rho$. Solution of Einstein’s field equation of state $p = \rho$ have been obtained by various authors e.g., Latelier [12], Letelier and Tabensky [13], Tabensky, R., et.al.[24] and Yadav[33]. Singh and Yadav [20] have also discussed the static fluid sphere with the equation of the state $p = \rho$. Further study in the line has been done by Yadav and Saini [30], which is more general than one due to Singh and Yadav [20]. Also in this case the relative mass $m$ of a particle in the gravitational field related to its proper mass $m_0$ studied by Narlikar [14]. Schwarzschild [18] considered the perfect fluid spheres with homogeneous density and isotropic pressure in general relativity and obtained the solutions of relativistic field equations. Tolman [26] developed
a mathematical method for solving Einstein's field equations applied to static fluid spheres in such a manner as to provide explicit solutions in terms of known analytic functions. A number of new solutions were thus obtained and the properties of three of them were examined in detail.

No stationary in homogeneous solutions to Einstein's equations for an irrotational perfect fluid have featured equations of state \( p = \rho \) (Letelier [12], Letelier and Tabensky [13] and Singh and Yadav [20]). Solutions to Einstein's equations with a simple equations of state have been found in various cases, e.g. for \( \rho + 3p = \text{constant} \) (Whittaker [29]) for \( \rho = 3p \) (Klein [9]); for \( p = \rho + \text{constant} \) (Buchdahl and Land [4], Allunt [1]) and for \( \rho = (1+a)\sqrt{p} + ap \) (Buchdahl [2]). But if one takes, e.g. polytrophic fluid sphere \( \rho = ap^{\frac{1+1}{n}} \) (Klein [10], Tooper [27], Buchdahl [3]) or a mixture of ideal gas radiation (Suhonen [23]), one soon has to use numerical methods. Yadav and Saini [30] have also studied the static fluid sphere with equation of state \( p = \rho \) (i.e. stiff matter). Davidson [5] has presented a solution a non stationary analog to the case when \( p = \frac{1}{3}\rho \). Tolman [26], Yadav and Purushottam [31], Thomas E Kiess [25], Karmer [11], Singh. et.al.[19], Raychaudhari[16], Walecka[28], Yadav, et.al.[34-35] and Yadav and Singh[32] are some of the authors who have studied various aspects of interacting fields in the framework of Einstein's field equations for the perfect fluid with specified equation of state and general relativity.

In this chapter we have obtained some exact static spherically symmetric solution of Einstein field equation for the static fluid sphere with cosmological constant \( \Lambda = 0 \) and equation of state \( p = \rho \). It has been obtained taking suitable choice of \( g_{11} \) and \( g_{44} \) (e.g. \( e^\beta = l r^n, e^\beta = l r^{n-1} \),
e^\beta = l \ r^{5/4} \text{ or } e^{-\alpha} = a, \text{ where } l \ & a \text{ are constants). For different values of } n \text{ we get many previously known solutions. To overcome the difficulty of infinite density at the centre, it is assumed that distribution has a core of radius } r_0 \text{ and constant density } \rho_0 \text{ which is surrounded by the fluid with the specified equation of state. Many previously known solutions are contained here in as a particular case. Various physical and geometrical properties have been also studied.}

\subsection*{2.2 THE FIELD EQUATIONS:}

We consider the static spherically symmetric metric given by

\begin{equation}
(2.2.1) \quad ds^2 = e^\beta \ dt^2 - e^\alpha \ dr^2 - r^2 \ d\theta^2 - r^2 \sin^2 \theta \ d\varphi^2
\end{equation}

where \( \alpha \) and \( \beta \) are functions of \( r \) only.

Taking cosmological constant \( \Lambda \) into account, we obtain the field equations

\begin{equation}
(2.2.2a) \quad R^i_j - \frac{1}{2} R \delta^i_j + \Lambda \delta^i_j = -8 \pi T^i_j
\end{equation}

For \( \Lambda = 0 \), (2.2.2a) gives

\begin{equation}
(2.2.2b) \quad R^i_j - \frac{1}{2} R \delta^i_j = -8 \pi T^i_j
\end{equation}

For the metric (2.2.1) are (Tolman [26])
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(2.2.3) \(-8\pi T^1_1 = e^{-\alpha} \left( \frac{\beta'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2}\)

(2.2.4) \(-8\pi T^2_2 = -8\pi T^3_3\)

\[= e^{-\alpha} \left( \frac{\beta''}{2} - \frac{\alpha' \beta'}{4} + \frac{\beta'^2}{4} + \frac{\beta' - \alpha'}{2r} \right)\]

(2.2.5) \(-8\pi T^4_4 = e^{-\alpha} \left( \frac{\alpha^1}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2}\)

where a prime denotes differentiation with respect to \(r\).

Through the investigation, we set velocity of light \(C\) and gravitational constant \(G\) to be unity. A Zeldovich fluid can be regarded as a perfect fluid having the energy momentum tensor.

(2.2.6) \(T^i_j = (\rho + p)u^i u^j - \delta^i_j p\)

Specified by the equation of state

(2.2.7) \(\rho = p\)

we use co-moving co-ordinates so that

\[u^1 = u^2 = u^3 = 0 \text{ and } u^4 = e^{\frac{-\beta}{2}}\]

The non-vanishing components of the energy momentum tensor are
\[ T_1^1 = T_2^2 = T_3^3 = -p \text{ and } T_4^4 = \rho \]

We can then write the field equations:

\[
\begin{align*}
(2.2.8) & \quad 8\pi p = e^{-\alpha} \left( \frac{\beta'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} \\
(2.2.9) & \quad 8\pi p = e^{-\alpha} \left( \frac{\beta''}{2} - \frac{\alpha' \beta'}{4} + \frac{\beta''^2}{4} + \frac{\beta' - \alpha'}{2r} \right) \\
(2.2.10) & \quad 8\pi \rho = e^{-\alpha} \left( \frac{\alpha'}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2}
\end{align*}
\]

### 2.3 Solution of the Field Equations

Using equations (2.2.7), (2.2.8) & (2.2.10), we have

\[
(2.3.1) \quad e^{-\alpha} \left( \frac{\beta'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} = e^{-\alpha} \left( \frac{\alpha'}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2}
\]

From [2.3.1] we see that if \( \beta \) is known, \( \alpha \) can be obtained, so we choose –

\[ \textbf{CASE I} \]

\[
(2.3.2) \quad e^\beta = l \, r^n,
\]

(where \( l \) is a constant)
Using (2.3.2), equation (2.3.1) goes to the –

(2.3.3) \( \frac{d e^{-\alpha}}{dr} + \frac{n+2}{r} e^{-\alpha} = \frac{2}{r} \)

Put \( \tau = e^{-\alpha} \) in the equation (2.3.3) is reduced to

(2.3.4) \( \frac{d \tau}{dr} + \frac{n+2}{r} \tau = \frac{2}{r} \)

This is a linear differential equation whose solution is given by

(2.3.5) \( \tau = \frac{2}{n+2} + \frac{C}{r^{n+2}} \)

or,

(2.3.6) \( e^{-\alpha} = \frac{2}{n+2} + \frac{C}{r^{n+2}} \)

Where \( C \) is integration constant.

\( \textbf{CASE. II} \)

(2.3.7) \( e^\beta = l r^{n-1} \)

(where \( l \) is constant)

Using (2.3.7), in equation (2.3.1) we get

(2.3.8) \( \frac{d e^{-\alpha}}{dr} + \frac{n+1}{r} e^{-\alpha} = \frac{2}{r} \)
Put $\tau = e^{-\alpha}$ in equation (2.3.8) is reduced to

$$\frac{d\tau}{dr} + \frac{n+1}{r} \tau = \frac{2}{r}$$

(2.3.9)

Solution of this linear differential equation is

$$\tau = \frac{2}{n+1} + \frac{C}{r^{n+1}}$$

(2.3.10)

or

$$e^{-\alpha} = \frac{2}{n+1} + \frac{C}{r^{n+1}}$$

(2.3.11)

So we get a generalised value for this (i.e. $e^\beta = l^r$) :-

$$e^{-\alpha} = \frac{2}{k} + \frac{C}{r^k}$$

(2.3.12)

or

$$e^{\alpha} = \frac{k r^k}{2r^k + kC}$$

(2.3.13)

(\text{where } k = n + 2, \ n = \text{power of } r \ \text{and } C = \text{integral constant}).

Hence, using the equation (2.3.6) the metric (2.2.1) yields:-
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(2.3.14) \[ ds^2 = lr^n dt^2 - \left( \frac{2}{n+2} + \frac{C}{r^{n+2}} \right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \]

Absorbing the constant \( l \) in the co-ordinates differentials \( dt \) and putting \( C = 0 \), the metric (2.3.14) goes to the form:-

(2.3.15a) \[ ds^2 = r^n dt^2 - \frac{n+2}{2} dr^2 - r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \]

Or

(2.3.15b) \[ ds^2 = r^n dt^2 - \frac{k}{2} dr^2 - r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \]

The non-zero components of Reimann-Christoffel curvature tensor \( R_{hijk} \) for the metric [2.3.15] are

(2.3.16) \[ \sin^2 \theta R_{2424} = R_{3434} = \frac{n+2}{2} r^n \sin^2 \theta = \frac{k}{2} r^n \sin^2 \theta = R_{2323} \]

We see that \( R_{hijk} \rightarrow 0 \) as \( r \rightarrow \infty \)

Hence it follows that the space time is asymptotically Homaloidal.

For the metric [3.15] the fluid velocity \( v' \) is given by

(2.3.17) \[ v^1 = v^2 = v^3 = 0 ; \quad v^4 = r^{-n/2} = \frac{1}{r^{n/2}} \]

The scalar of expansion \( \Theta = v^i_i \) is identically zero (i.e. \( \Theta = 0 \)).

The non-vanishing components of the tensor of rotation \( \omega_{ij} \) is defined by
(2.3.18) \[ \omega_{ij} = v_{i,j} - v_{j,i} \]

(2.3.19) \[ \omega_{14} = -\omega_{41} = - \frac{n}{2} r^{n/2-1} = - \frac{n}{2} r^{n-2} \]

The components of the shear tensor \( \sigma_{ij} \) defined by

(2.3.20) \[ \sigma_{ij} = \frac{1}{2} (v_{ij} + v^{ij}) - \frac{1}{3} H_{ij} \]

With the projection tensor

(2.3.21) \[ H_{ij} = g_{ij} - v_{i} v_{j} \]

are

(2.3.22) \[ \sigma_{14} = \sigma_{41} = \frac{n}{2} r^{n-2} = \frac{n}{2} r^{n-2} \]

- **(PARTICULAR CASE):**

If we choose

(2.3.23) \[ e^{\beta} = lr^{5/4} \]

(where \( l \) is constant)
Using (2.3.23) in equation (2.3.1) goes to form:-

\[ (2.3.24) \frac{d e^{-\alpha}}{dr} + \frac{13}{4r} e^{-\alpha} = \frac{2}{r} \]

Substituting \( \tau = e^{-\alpha} \), the equation (2.3.24) is reduced to,

\[ (2.3.25) \frac{d \tau}{dr} + \frac{13}{4r} \tau = \frac{2}{r} \]

which is a linear differential equation whose solution is given by:-

\[ (2.3.26) \tau = \frac{8}{13} + \frac{C}{r^{13/4}} \]

or

\[ (2.3.26a) e^{-\alpha} = \frac{8}{13} + \frac{C}{r^{13/4}} \]

where \( C \) is integration constant.

Hence the metric (2.2.1) yields

\[ (2.3.27) \quad ds^2 = l r^{5/4} dt^2 - \left( \frac{8}{13} + \frac{C}{r^{13/4}} \right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta \, d\varphi^2) \]

Absorbing the constant \( l \) in the co-ordinate differential \( dt \) and put \( C = 0 \) the metric (2.3.27) goes to the form –
(2.3.28) \[ ds^2 = r^{5/4}dt^2 - \frac{13}{8}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \]

The non-zero component of Reimann-christoffel curvature tensor \( R_{hijk} \) for the metric (2.3.28) is

(2.3.29) \[ \sin^2\theta R_{2424} = R_{3434} = \frac{13}{8} r^{13/4} \sin^2\theta = R_{2323} \]

For the metric [2.3.28] the fluid velocity \( v' \) is given by

(2.3.30) \[ v^1 = v^2 = v^3 = 0, \quad v^4 = r^{-5/8} = \frac{1}{r^{5/8}} \]

In the usual notation, we have the rotation and shear tensor same as equation (2.3.18, 2.3.20, 2.3.21 and 2.3.22) which gives results for metric (2.3.28) as:-

(2.3.31) \[ \theta = 0, \quad \omega_{14} = -\omega_{41} = \frac{-5}{8} r^{-3/8} = \frac{-5}{8r^{3/8}} \]

and

(2.3.32) \[ \sigma_{14} = \sigma_{41} = \frac{5}{8} r^{-3/8} = \frac{5}{8r^{3/8}}. \]

From (2.3.1) we see that if \( \alpha \) is known, then \( \beta \) can be obtained, so we choose-
\section*{Case III}

(2.3.33) \( e^{-\alpha} = a \),

(where \( a \) is constant)

Using (2.3.33), equation (2.3.1) goes to the-

(2.3.34) \( \beta' - \alpha' + \frac{2}{r} \left[ 1 - \frac{1}{a} \right] = 0 \)

Since \( e^{-\alpha} = a \), is constant, then \( \alpha' = 0 \) and hence (2.3.34) reduces to

(2.3.35) \( \beta' + \frac{2}{r} \left[ 1 - \frac{1}{a} \right] = 0 \)

Now (2.3.35) integrate w.r.t \( r \) we get-

(2.3.36) \( e^\beta = Ar^{2\left(1 - \frac{1}{a}\right)} \)

where \( A \) is a integration constant.

If we consider \( a = 3 \), then we get-

(2.3.37) \( e^\beta = Ar^{4/3} \)

Hence, using the equation (2.3.37) the metric (2.3.1) yields:-
(2.3.38) \[ ds^2 = Ar^{4/3}dt^2 - 1/3(dr^2) - r^2(d\theta^2 + \sin^2\theta\,d\varphi^2) \]

Absorbing the constant A in the co-ordinates differentials \( dt \) the metric (2.3.38) goes to the form:

(2.3.39) \[ ds^2 = r^{4/3}dt^2 - 1/3(dr^2) - r^2(d\theta^2 + \sin^2\theta\,d\varphi^2) \]

The non-zero components of Reimann-christoffel curvature tensor \( R_{hijk} \) for the metric (2.3.39) are:

(2.3.40) \[ \sin^2\theta R_{2424} = R_{3434} = -\frac{1}{2}r^2\sin^2\theta = R_{2323} \]

we see that \( R_{hijk} \to 0 \) as \( r \to \infty \)

Hence it follows that the space-time is asymptotically Homaloidal.

For the metric (2.3.39) the fluid velocity \( v' \) is given by

(2.3.41) \[ v^1 = v^2 = v^3 = 0; \quad v^4 = \frac{1}{r} = r^{-1} \]

The scalar of expansion \( \Theta = v^1 \) is identically zero (i.e., \( \Theta = 0 \)). The non-vanishing components of the tensor of rotation \( \omega_{ij} \) is defined by-

\[ \omega_{ij} = v_{ij} - v_{ji} \]

we get

(2.3.42) \[ \omega_{14} = -\omega_{41} r = r^0 = 1 \]
The components of the shear tensor \( \sigma_{ij} \) defined by \( \sigma_{ij} = \frac{1}{2} (v_{ij} + v^{ij}) - \frac{1}{3} H_{ij} \), with the projection tensor \( H_{ij} = g_{ij} - v_i v_j \) are

\[
(2.3.43) \quad \sigma_{14} = \sigma_{41} = \frac{1}{2} r^0 = \frac{1}{2},
\]

with other components are zero.

2.4. \textbf{SOLUTION FOR THE PERFECT FLUID CORE:}

Pressure and density for the metric (2.3.14-15a, 2.3.28) are

\[
(2.4.1) \quad 8\pi p = 8\pi \rho = \frac{n+1}{r^2} \left[ \frac{2}{n+2} + \frac{C}{r^{n+2}} \right] - \frac{1}{r^2}
\]

If we consider \( C = 0 \), then equation (2.4.1) reduces to

\[
(2.4.2) \quad 8\pi p = 8\pi \rho = \frac{n+1}{r^2} \left[ \frac{2}{n+2} \right] - \frac{1}{r^2}
\]

\[
(2.4.3) \quad 8\pi p = 8\pi \rho = \frac{18}{13r^2}
\]

It follows from (2.4.1-2.4.3) that the density of the distribution tends to infinity as \( r \) tends to zero. In order to get rid of singularity at \( r=0 \) in the density we visualize that the distribution has a core of radius \( r_0 \) and
constant \( \rho_o \). The field inside the core is given by Schwarzschild internal solution.

\[
(2.4.4a) \quad e^{-\lambda} = 1 - \frac{r^2}{R^2}
\]

\[
(2.4.4b) \quad e^{\nu} = \left[ L - M \left( 1 - \frac{r^2}{R^2} \right) \right]^2
\]

\[
(2.4.4c) \quad 8\pi p = \frac{1}{R^2} \left[ \frac{3M \left( 1 - \frac{r^2}{R^2} \right) - L}{L - M \left( 1 - \frac{r^2}{R^2} \right)^2} \right]
\]

where \( L, M \) are constants and \( R^2 = \frac{3}{8\pi \rho} \).

The continuity condition for the metric (2.3.14) and (2.4.4a-4b-4c) at the boundary gives

\[
(2.4.5a) \quad R^2 = \frac{r_0^2}{\left( \frac{n}{n+2} - \frac{c}{r_0^{n+2}} \right)}
\]

\[
(2.4.5b) \quad L = r_0^{n/2} + \frac{nR^2}{2r_o^{2-n/2}} \left( 1 - \frac{r_0^2}{R^2} \right)
\]

\[
(2.4.5c) \quad M = \frac{nR^2}{2r_o^{2-n/2}} \left( 1 - \frac{r_0^2}{R^2} \right)^{1/2}
\]
(2.4.5d) \[ C = r_0^{n+2} \left( \frac{n}{n+2} - \frac{r_0^2}{R^2} \right) \]

and the density of the core

(2.4.6) \[ \rho_o = \frac{3}{8\pi r^2} \left( \frac{n}{n+2} - \frac{C}{r_0^{n+2}} \right) \]

which complete the solution for the perfect fluid core of radius \( r_o \) surrounded by considered fluid. The energy condition \( T_{ij} u^i u_j > 0 \) and the Hawking and Penrose condition (Hawking and Penrose, 1970).

\[ (T_{ij} - \frac{1}{2} g_{ij} T) u^i u_j > 0, \]

Both reduces to \( \rho > 0 \), which is obviously satisfied.

For different value of \( n \), solution obtained above in case I and case II provide many previously known solutions. For \( n=2 \) and by suitable adjustment of constant we get the solution due to Singh and Yadav [20] and Yadav and Saini [30]. Also for \( n = 3 \) we get solution due to Yadav et.al [33].

2.5. DISCUSSION:

In this chapter we have obtained some exact static spherical solution of Einstein’s field equation with cosmological constant \( \Lambda = 0 \) and equation of state \( p = \rho \). We have shown that when cosmological constant \( \Lambda = 0 \), then
in the absence of electromagnetic field pressure and density become equal and conversely if pressure and density are equal there is no electromagnetic field. Our assumption is $e^\beta = l r^n$ and $e^{-\alpha} = a$, which investigate a generalised value of $e^\alpha$ and $e^\beta$. It describe several important cases, e.g.- relativistic model, fluid velocity, rotation, shear tensor, scalar of expansion. It also investigates solution for the perfect fluid core.

This paper provide casual limit for ideal gas has also for $p = \rho$ [Zeldovich and Novicove (36)] and if in addition its motion (for Fluid ) is irrotational, then such source has the same stress energy tensor as that of mass less scalar field.

### 2.6 LIMITATION:

If we consider $n = -2$ for equations (2.3.12-13), then the value of $e^{-\alpha}$ is undefined and $e^\alpha$ become a constant value.

If we consider $a = 0$ for equations (2.3.36), then the value of $e^\beta$ becomes undefined but for $a = 1$ it gives a constant value.
2.7 REFERENCES:


