“When we understand string theory, we will know how the universe began. It won’t have much effect on how we live, but it is important to understand where we come from and what we can expect to find as we explore.”

.... “Stephen Hawking”
4.1 **INTRODUCTION**:  

We review recent progress in string cosmology where string dualities are applied so as to obtain complete cosmological evolution. String theory has several potential applications to cosmology, covering the entire history of the cosmos. One line of inquiry focus on the initial singularity, what happen when the universe was squeezed to a size close to the Plank scale[15, 32]. In recent years, there has been considerable interest is string cosmology, as string are believed to have played an important role during early stages of the universe [30, 16] and can generate density fluctuations which lead to galaxy formation [11, 40]. It is still a problem to know the exact physical situation at the very early stages of the formation of the universe. It appears that after the big bang the universe may have undergone a series of phase transitions as its temperature lowered down. During the phase transition, the symmetry of the universe is broken spontaneously. It can give rise to topologically stable defects, vacuum domain walls, strings and monopoles [11]. Out of these three topological characteristics, only strings can lead to very interesting cosmological consequences. They are believed to give rise to density perturbation leading to the formation of galaxies [34, 40].

As these strings possess stress-energy and are coupled to the gravitational field, that arise from strings. In fact, the general relativistic treatment of strings was initiated by Letelier [14] and Stachel [26].This model has been used as a source for Bianchi type-I and Kantowski-Sachs cosmologies by Letelier [14]. After wards, Krori et.al. [12, 13] and Wang
[31,36, 38] have discussed the solutions of Bianchi types-II, VI, VIII and IX for a cloud string. Tikekar and Patel [28], Chakraborty and Chakraborty [9], Singh et al. [25], Turyaev [29] and Vilenkin [30] have presented the exact solutions of Bianchi type -III and spherically symmetric cosmology respectively for a cloud string.

On the other hand, the matter distribution is satisfactorily described by a perfect fluid due to large-scale distribution of galaxies in our universe. However, a realistic treatment of the problem requires the consideration of material distribution other than the perfect fluid [7, 10]. It is well known that when neutrino decoupling occurs, the matter behaves as a viscous fluid in an early stage of the universe. Viscous fluid cosmological models in early universe have been widely discussed in literature [13, 33-37]. Recently Bali and Dave [4] have presented Bianchi type-III string cosmological model with bulk viscosity, where the constant coefficient of bulk viscosity is considered. However, it is known that the coefficient of bulk viscosity is not constant but decreases as the universe expands [1, 3, 8]. Arbab [2], Bali and Tinker [6], Pradhan et al. [17-19], Ray and Mukhopadhay [20], Singh and Singh [21], Singh and Pradhan [22], Singh and Kumar [23-24], Yadav and Pradhan [39] are some of the authors who have studied various aspects of interacting fields in the framework of Bianchi type-III string cosmological model with bulk viscosity in general relativity.

In this chapter, we study the Bianchi type-III string cosmological model with bulk viscosity. To obtain a determinate solution, we assume that the coefficient of the bulk viscosity is a power function of
the scalar of expansion $\xi = k\theta^{a+bm}$ and the shear scalar is proportional to scalar of expansion $\sigma \propto \theta$, which leads to the relation between metric potentials $\beta = a + b\gamma^{\mu+1}$. The physical and geometric features of the model are also discussed.

4.2 **THE FIELD EQUATIONS AND THEIR SOLUTIONS:**

The Bianchi type-III space-time metric we considered here is [5]

$$ds^2 = -dt^2 + \alpha^2 dx^2 + \beta^2 e^{2x} dy^2 + \gamma^2 dz^2$$

where $\alpha, \beta, \gamma$ are only the functions of time $t$.

The energy-momentum tensor for a cloud of string with bulk viscosity is [5]

$$T_{ij} = \rho u_i u_j - \lambda \delta_{ij} - \xi \gamma^i u_j + \eta^i_{ij}$$

where $\rho = \rho_p + \lambda$, is the rest energy density of the cloud of strings with particles attached to them, $\rho_p$ is the rest energy density of particles, $\lambda$ is the tension density of the cloud of strings, $\theta = u_i^{\mu} u_{ij}$, is the scalar of expansion, and $\xi$ is the coefficient of bulk viscosity. According to Letelier [14] the energy density for the coupled system $\rho$ and $\rho_p$ is restricted to be positive, while the tension density $\lambda$ may be positive or negative. The
vector \( u^i \) describes the cloud four-velocity and \( X^i \) represents a direction of anisotropy, i.e. the direction of string. They satisfy the standard relations [14].

\[
(4.2.3) \quad u^i u_j = -X^i X_j = -1, \quad u^i X_i = 0
\]

The expressions for scalar of expansion and shear scalar are (kinematical parameters)

\[
(4.2.4) \quad \theta = u^i_{;i} = \frac{\ddot{\alpha}}{\alpha} + \frac{\dot{\beta}}{\beta} + \frac{\dot{\gamma}}{\gamma}
\]

\[
(4.2.5) \quad \sigma^2 = \frac{1}{2} \sigma_{ij} \sigma^{ij}
\]

\[
= \frac{1}{3} \left( \frac{\dddot{\alpha}^2}{\alpha^2} + \frac{\dddot{\beta}^2}{\beta^2} + \frac{\dddot{\gamma}^2}{\gamma^2} - \frac{\ddot{\alpha} \dddot{\beta}}{\alpha \beta} - \frac{\ddot{\beta} \dddot{\gamma}}{\beta \gamma} - \frac{\ddot{\alpha} \dddot{\gamma}}{\alpha \gamma} \right)
\]

Einstein’s equation we consider here is

\[
(4.2.6) \quad R_{ij} - \frac{1}{2} R g_{ij} = T_{ij}
\]

where we have choose the units such that \( c = 1 \) and \( 8\pi G = 1 \). In the comoving coordinates \( u^i = \delta_0^i \) and \( u^i = -\delta_i^0 \), and with the help of Eqs. (4.2.1)- (4.2.3), the Einstein equation (4.2.6) can be written as [5]
(4.2.7) \[ \frac{\ddot{\beta}}{\beta} + \frac{\ddot{y}}{y} + \frac{\dot{\beta} \dot{y}}{\beta y} = \xi \theta \]

(4.2.8) \[ \frac{\dot{\alpha}}{\alpha} + \frac{\ddot{y}}{y} + \frac{\dot{\alpha} \dot{y}}{\alpha y} = \xi \theta \]

(4.2.9) \[ \frac{\ddot{\alpha}}{\alpha} + \frac{\ddot{\beta}}{\beta} + \frac{\dot{\alpha} \dot{\beta}}{\alpha \beta} - \frac{1}{\alpha^2} = \lambda + \xi \theta \]

(4.2.10) \[ \frac{\dot{\alpha} \dot{\beta}}{\alpha \beta} + \frac{\ddot{\gamma}}{\beta \gamma} + \frac{\dot{\alpha} \dot{\gamma}}{\alpha \gamma} - \frac{1}{\alpha^2} = \rho \]

(4.2.11) \[ \frac{\dot{\alpha}}{\alpha} - \frac{\dot{\beta}}{\beta} = 0 \]

where the dot denotes the differentiation with respect to time t. From Eq. (4.2.11), we have

(4.2.12) \[ \alpha = H \beta \]

where \( H \) is the constant of integration. In order to obtain a more general solution, we assume

(4.2.13) \[ \xi = k \theta^{a+b+m} \]

where \( a, b, k \) and \( m \) are the positive constants.
We note that the five independent equations (4.2.8)-(4.2.11) and (4.2.13) connect six unknown variables \((\alpha, \beta, \gamma, \lambda, \rho, \xi)\). Thus, one more relation connecting these variables is needed to solve these equations. In order to obtain explicit solutions, one additional relation is needed and we adopt an assumption that the shear scalar is proportional to the scalar of expansion \(\sigma \propto \theta\), which leads to

\[
(4.2.14) \quad \beta = a + b\gamma^\mu + 1
\]

where \(\mu\) is a constant.

Now we consider \(a = 0\) and \(b = 1\), then Equation (4.2.13) and (4.2.14) reduces to-

\[
(4.2.15) \quad \xi = k\theta^m
\]

\[
(4.2.16) \quad \beta = \gamma^{\mu + 1}
\]

Substituting Eq. (4.2.16) into Eq. (4.2.4) and using Eq. (4.2.15) we have

\[
(4.2.17) \quad \theta = (2\mu + 3)\frac{\dot{\gamma}}{\gamma}
\]

\[
(4.2.18) \quad \xi\theta = K\frac{\dot{\gamma}^{m+1}}{\gamma^{m+1}}
\]

where
(4.2.19) \[ \mathcal{K} = k(2\mu + 3)^{m+1} \]

with the help of Equations (4.2.16) and (4.2.18), Eq. (4.2.7) reduces to

(4.2.20) \[ \frac{\ddot{\gamma}}{\gamma} + \frac{(\mu+1)^2}{\mu+2} \frac{\dot{\gamma}^2}{\gamma^2} = \frac{\mathcal{K}}{\mu+2} \frac{\dot{\gamma}^{m+1}}{\gamma^{m+1}} \]

To solve Eq. (4.2.20), we denote \( \dot{\gamma} = \vartheta \), then \( \ddot{\gamma} = \vartheta \frac{d\vartheta}{dy} \), and the Eq. (4.2.20) can be reduced to the first-order differential equation in the following form:

(4.2.21) \[ \frac{d\vartheta}{dy} + l \frac{\vartheta}{\gamma} = \frac{\mathcal{K}}{\mu+2} \frac{\vartheta^m}{\gamma^m} \]

where

(4.2.22) \[ l = \frac{(\mu+1)^2}{\mu+2} \]

Equation (4.2.21) can be written as (for \( m \neq 1 \))

(4.2.23) \[ \frac{d}{dy} \left( \vartheta^{1-m} \gamma^{(1-m)l} \right) = \frac{(1-m)\mathcal{K}}{\mu+2} \gamma^{(1-m)l-m} \]

Thus the solution of Eq. (4.2.21) can easily be obtained:
(4.2.24) \[ \theta = \left[ \frac{\mathcal{K} \gamma^{1-m}}{(\mu+2)(l+1)} + D \gamma^{(m-1)i} \right]^\frac{1}{1-m} \]

where \( D \) is the constant of integration. With the help of Eq. (4.2.24), the line-element (4.2.1) reduces to

(4.2.25) \[ ds^2 = -\left[ \frac{\mathcal{K} \gamma^{1-m}}{(\mu^2+3\mu+3)} + D \gamma^{(m-1)i} \right]^{-2} dy^2 \]

\[ + H^2 \gamma^{2\mu+2} dx^2 + \gamma^{2\mu+2} e^{2x} dy^2 + \gamma^2 dz^2 \]

Under suitable transformation of coordinates, Eq. (4.2.25) reduces to

(4.2.26) \[ ds^2 = -\left[ \frac{\mathcal{K} \tau^{1-m}}{(\mu^2+3\mu+3)} + D \tau^{(m-1)i} \right]^{-2} d\tau^2 \]

\[ + H^2 \tau^{2\mu+2} dx^2 + \tau^{2\mu+2} e^{2x} dy^2 + \tau^2 dz^2 \]

For the model of Eq. (4.2.26), the other physical and geometrical parameters can easily be obtained. The expressions for the energy density \( \rho \), the string tension density, the particle density \( \rho_p \), the coefficient of bulk viscosity \( \xi \), the scalar of expression \( \theta \) and the shear scalar \( \sigma^2 \) are, respectively, given by
Investigations on some Bianchi type-III string cosmological models.

\[(4.2.27) \quad \rho = (\mu + 1)(\mu + 3). \left[ \frac{\kappa}{(\mu^2 + 3\mu + 3)} + D\mathcal{T}^{(m-1)(l+1)} \right]^{\frac{2}{1-m}} - \frac{\tau^{-(2\mu+2)}}{H^2} \]

\[(4.2.28) \quad \lambda = \frac{\mu}{1-m} \cdot \left[ \frac{\kappa\mathcal{T}^{1-m}}{(\mu^2 + 3\mu + 3)} + D\mathcal{T}^{(m-1)l} \right]^{\frac{1+m}{1-m}} - \frac{\tau^{-(2\mu+2)}}{H^2} \cdot \left[ (1-m)\kappa\mathcal{T}^{-(1+m)} + D(m-1)l\mathcal{T}^{(m-1)l-2} \right] + \mu(2\mu + 2). \left[ \frac{\kappa}{(\mu^2 + 3\mu + 3)} + D\mathcal{T}^{(m-1)(l+1)} \right]^{\frac{2}{1-m}}

\[(4.2.29) \quad \rho_p = \frac{-\mu}{1-m} \cdot \left[ \frac{\kappa\mathcal{T}^{1-m}}{(\mu^2 + 3\mu + 3)} + D\mathcal{T}^{(m-1)l} \right]^{\frac{1+m}{1-m}} + (\mu + 1)(3 - \mu). \left[ \frac{\kappa}{(\mu^2 + 3\mu + 3)} + D\mathcal{T}^{(m-1)(l+1)} \right]^{\frac{2}{1-m}} + (\mu + 1)(3 - \mu). \left[ \frac{\kappa}{(\mu^2 + 3\mu + 3)} + D\mathcal{T}^{(m-1)(l+1)} \right]^{\frac{2}{1-m}}\]
INVESTIGATIONS ON SOME BIANCHI TYPE-III STRING COSMOLOGICAL MODELS.

A STUDY ON SOME COSMOLOGICAL MODELS

\( (4.2.30) \quad \theta = (2\mu + 3) \left[ \frac{\mathcal{K}}{(\mu^2+3\mu+3)} + D\mathcal{T}^{(m-1)(l+1)} \right]^{\frac{1}{1-m}} \)

\( (4.2.31) \quad \sigma^2 = \frac{\mu^2}{3} \left[ \frac{\mathcal{K}}{(\mu^2+3\mu+3)} + D\mathcal{T}^{(m-1)(l+1)} \right]^{\frac{2}{1-m}} \)

From Eq. (4.2.27) it is observed that the standard condition \( \rho \geq 0 \) is fulfilled when

\( (4.2.32) \quad (\mu + 1)(\mu + 3) \left[ \frac{\mathcal{K}}{(\mu^2+3\mu+3)} + D\mathcal{T}^{(m-1)(l+1)} \right]^{\frac{2}{1-m}} \geq \frac{T^{-2(\mu+2)}}{H^2} \)

It is seen that in the case \( m < 1 \), the scalar of expansion \( \theta \) tends to infinitely large and the energy density \( \rho \rightarrow \infty \) when \( T \rightarrow 0 \), but \( \theta \) tends to finite and \( \rho \) tends to finite when \( T \rightarrow \infty \) due to the presence of bulk viscosity (in the absence of bulk viscosity \( \mathcal{K} = 0 \), \( \theta \rightarrow 0 \) and \( \rho \rightarrow 0 \) when \( T \rightarrow \infty \)). Hence the model represents the shearing and non-rotating expanding universe with the big-bang start. However, in the case \( m > 1 \), it is observed that \( \theta \rightarrow 0 \) when \( T \rightarrow \infty \), but \( \theta \) tends to finite when \( T \rightarrow 0 \) due to the presence of bulk viscosity (in the absence of bulk viscosity \( \mathcal{K} = 0 \), \( \theta \rightarrow \infty \) when \( T \rightarrow 0 \)). Here \( \rho \rightarrow \infty \) when \( T \rightarrow 0 \) and \( \rho \rightarrow 0 \) when \( T \rightarrow \infty \). Therefore the model describes a shearing non rotating expanding universe without the big-bang start. We can see from the above discussion that the bulk viscosity plays a significant role in the evolution of universe [37,38]. Furthermore, since \( \lim_{T \rightarrow \infty} \sigma \frac{\theta}{\theta} \neq 0 \), the model does not approach isotropy for large values of \( T \). The shear scalar \( \sigma \) is zero when \( \mu = 1 \), hence \( \mu = 1 \)
is the isotropy condition.

\[ \textbf{SPECIAL CASE}:- \]

If we consider \( m = 0 \), then the model (4.2.26) reduces to the string model of constant coefficient of bulk viscosity. That is

\[\begin{align*}
(4.2.33) \quad d s^2 &= -\left[ \frac{\mathcal{K} \mathcal{T}}{(\mu^2+3\mu+3)} + D \mathcal{T}^{-l} \right]^2 d \mathcal{T}^2 \\
&\quad + H^2 \mathcal{T}^{2\mu+2} dx^2 + \mathcal{T}^{2\mu+2} e^{2x} dy^2 + \mathcal{T}^2 dz^2
\end{align*}\]

\[\begin{align*}
(4.2.34) \quad \rho &= (\mu + 1)(\mu + 3) \left[ \frac{\mathcal{K}}{(\mu^2+3\mu+3)} + D \mathcal{T}^{-(l+1)} \right]^2 \\
&\quad - \frac{\mathcal{T}^{-(2\mu+2)}}{H^2}
\end{align*}\]

\[\begin{align*}
(4.2.35) \quad \lambda &= \left[ \frac{(2\mu^2+3\mu)\mathcal{K}}{(\mu^2+3\mu+3)} + \frac{\mu(\mu+1)(\mu+3)}{\mu+2} D \mathcal{T}^{-(l+1)} \right] \\
&\quad \cdot \left[ \frac{\mathcal{K}}{(\mu^2+3\mu+3)} + D \mathcal{T}^{-(l+1)} \right] - \frac{\mathcal{T}^{-(2\mu+2)}}{H^2}.
\end{align*}\]
\[ \rho_p = \left[ \frac{(\mu-\mu^2+3)K}{(\mu^2+3\mu+3)} + \frac{2(\mu+1)(\mu+3)DT^{-(l+1)}}{\mu+2} \right] \cdot \left[ \frac{K}{(\mu^2+3\mu+3)} + DT^{-(l+1)} \right] \]

\[ \theta = (2\mu + 3) \left[ \frac{K}{(\mu^2+3\mu+3)} + DT^{-(l+1)} \right] \]

\[ \sigma^2 = \frac{\mu^2}{3} \left[ \frac{K}{(\mu^2+3\mu+3)} + DT^{-(l+1)} \right]^2 \]

In fact, this model is the result previously given by Bali and Dave [5] and has already been discussed by them, but the proper time was used there.

In the absence of bulk viscosity \( K = 0 \), the model (4.2.26) reduces to the string model without viscosity, that is

\[ ds^2 = -D^{-2}T^{2l} \, dt^2 + H^2 T^{2\mu+2} \, dx^2 \]
\[ + T^{2\mu+2}e^{2x} \, dy^2 + T^2 \, dz^2 \]

\[ \rho = (\mu + 1)(\mu + 3) \cdot D^2 T^{-2(l+1)} - \frac{T^{-(2\mu+2)}}{H^2} \]

\[ \lambda = \frac{\mu(\mu+1)(\mu+3)}{\mu+2} D^2 T^{-2(l+1)} - \frac{T^{-(2\mu+2)}}{H^2} \]

\[ \rho_p = \frac{2(\mu+1)(\mu+3)}{\mu+2} D^2 T^{-2(l+1)} \]
(4.2.43) \[ \theta = (2\mu + 3).D\mathcal{T}^{-(l+1)} \]

(4.2.44) \[ \sigma^2 = \frac{\mu^2}{3}.D^2\mathcal{T}^{-2(l+1)} \]

From Eq. (4.2.40) it is observed that the standard condition \( \rho \geq 0 \) is fulfilled when

(4.2.45) \[ (\mu + 1)(\mu + 3).D^2\mathcal{T}^{-2(l+1)} \geq \frac{\mathcal{T}^{-(2\mu+2)}}{H^2} \]

The scalar of expansion \( \theta \) tends to infinitely large when \( \mathcal{T} \to 0 \), and \( \theta \to 0 \) when \( \mathcal{T} \to \infty \), provided \( n > -1/2 \), and the scalar of expansion in the model is monotonically decreasing when \( 0 < \mathcal{T} < \infty \). Since \( \lim_{\mathcal{T} \to \infty} \frac{\sigma}{\theta} \neq 0 \), the model does not approach isotropy for large values of \( \mathcal{T} \). However the energy density \( \rho \to \infty \) when \( \mathcal{T} \to 0 \), and \( \rho \to 0 \) when \( \mathcal{T} \to \infty \), therefore the model describes continuously expanding shearing non-rotating universe with the big-bang start.

In the second special case, we assume that \( m = 1 \), then Eq. (4.2.21) reduces to

(4.2.46) \[ \frac{d\theta}{d\gamma} + f \frac{\theta}{\gamma} = 0 \]

where
\[(4.2.47) \quad f = \frac{(\mu + 1)^2 - \kappa}{\mu + 2}\]

Integration of Eq. (4.2.46) gives

\[(4.2.48) \quad \vartheta = D \gamma^{-f}\]

where \(D\) is the constant of integration. In the same way as performed above, we can easily obtain

\[(4.2.49) \quad ds^2 = -D^{-2}T^{-2f} dT^2 + H^2 T^{2\mu + 2} dx^2 + T^{2\mu + 2} e^{2x} dy^2 + T^2 dz^2\]

\[(4.2.50) \quad \rho = (\mu + 1) (\mu + 3). D^2 T^{-2(f + 1)} - \frac{T^{-(2\mu + 2)}}{H^2}\]

\[(4.2.51) \quad \lambda = \mu (2\mu - f + 2). D^2 T^{-2(f + 1)} - \frac{T^{-(2\mu + 2)}}{H^2}\]

\[(4.2.52) \quad \rho_p = [(\mu + 1)(3 - \mu) + \mu f]. D^2 T^{-2(f + 1)}\]

\[(4.2.53) \quad \theta = (2\mu + 3). DT^{-(f + 1)}\]

\[(4.2.54) \quad \sigma^2 = \frac{\mu^2}{3}. D^2 T^{-2(f + 1)}\]
From Eq. (4.2.50) it is observed that the standard condition $\rho \geq 0$ is fulfilled when

$$\text{(4.2.55)} \quad (\mu + 1)(\mu + 3)D^2T^{-2(f+1)} \geq \frac{T^{-(2\mu+2)}}{H^2}$$

### 4.3 DISCUSSION:

The scalar of expansion $\theta$ is infinitely large at $T=0$, and $\theta$ tends to zero when $T \to \infty$ provided $f + 1 > 0$, hence the model represents the shearing and non-rotating expanding universe with the big-bang start. The energy density $\rho \to \infty$ when $T \to 0$, $\rho$ tends to zero when $T \to \infty$. Furthermore, since $\lim_{T \to \infty} \frac{\sigma}{\theta} \neq 0$, the model does not approach isotropy for large values of $T$. In the absence of bulk viscosity $\mathcal{K}=0$, this model can reduce to the model (4.2.39). In addition, we observe from Eqs. (4.2.15), (4.2.52), and (4.2.53) that the relation between the coefficient of bulk viscosity and the energy density of particles $\xi = \rho_p^{1/2}$ in this model, and such a relation is believed to be reasonable and has been considered in many papers [5,15].

### 4.4 SUMMARY:

In this chapter, we study the Bianchi type-III string cosmological model with bulk viscosity. To obtain a determinate solution, we assume that the coefficient of the bulk viscosity is a power function of the scalar of expansion $\xi = k\theta^{a+bm}$ and the shear scalar is proportional to scalar of
expansion $\sigma \propto \theta$, which leads to the relation between metric potentials $\beta = a + b\gamma^{\mu+1}$. The physical and geometric features of the model are also discussed. It is found that the power index $m$ has significant influence on the string model. There is a big-bang start in the model when $m \leq 1$ but there is not any big-bang in the start when $m > 1$ [37, 38]. In particular when $m=0$ the model reduces to the string model of constant coefficient of bulk viscosity, which was previously given by Bali and Dave [5].
4.5. INTRODUCTION:

In recent years, there has been considerable interest in string cosmology, as strings are believed to have played an important role during early stages of the universe [30] and can generate density fluctuations which lead to galaxy formation [40]. The strings have stress-energy and they can couple to the gravitational field, hence it may be interesting to study the gravitational effects that arise from strings. In fact, the general relativistic treatment of strings was initiated by Letelier [14] and Satchel [26]. This model was used as a source for Bianchi type-I and Kantowski-Sachs cosmologies by Letelier. More recently, Krori et al. [12,13] and Wang [36,31,38] have discussed the solutions of Bianchi type I, VI, VIII and IX for a cloud string, and Tikekar and Patel [28] and Chackraborty and Chackraborty [9] have presented the exact solutions of Bianchi type-III and spherically symmetric cosmology respectively for a cloud string. It is well known that in an early stage of universe when neutrino decoupling occurs, the matter behaves like a viscous fluid. The cosmological models of a fluid with viscosity play a significant role in study the evolution of the universe. Recently, string cosmological models of Bianchi types I, II, III with bulk viscosity have been discussed by several authors [4, 32-37].

On the other hand, the magnetic field has an important role at the cosmological scale and is present in galactic and intergalactic spaces. The importance of the magnetic field for various astrophysical phenomena has been studied in many papers. Melvin [16] has pointed out that during
the evolution of the universe, the matter was in a highly ionized state and is smoothly coupled with the field; subsequently forming neutral matter as a result of universe expansion. Therefore the possibility of the presence of magnetic field in the cloud string universe is not unrealistic and has been investigated by many authors[10,21,35]. Bahera [3], Bali and Dave[5], Bali [7], Kibble[11], Takabayski [27], Yadav [39], Zimadahl [41] are the some workers in this line.

In this Chapter, we study Bianchi type-III string cosmological model in the presence of bulk viscosity and magnetic field. To obtain an explicit solution, an equation of state \( \rho = a + b\lambda \) and an assumption that the scalar of expansion is proportional to the shear scalar \( \sigma \propto \theta \), which leads to the relation between metric potentials \( \beta = a + b\gamma^{\mu+1} \). The physical and geometric aspects of the model in the presence and absence of magnetic field are also discussed.

### 4.6 THE FIELD EQUATIONS AND ITS SOLUTIONS:

The Bianchi type-III space-time metric we considered here is [37]

\[
(4.6.1) \quad ds^2 = -dt^2 + \alpha^2 dx^2 + \beta^2 e^{2x} dy^2 + \gamma^2 dz^2
\]

where \( \alpha, \beta, \) and \( \gamma \) are only the functions of time \( t \).

The energy-momentum tensor for a cloud of string with bulk viscosity and magnetic field [35].
(4.6.2) \[ T_{ij} = \rho u_i u_j - \lambda X_i X_j - \xi \theta (u_i u_j + g_{ij}) + E_{ij} \]

where \( \rho = \rho_p + \lambda \), is the rest energy density of the cloud of strings with particles attached to them, \( \rho_p \) is the rest energy density of particles, \( \lambda \) is the tension density of the cloud of strings, \( \theta = u_i^i \), is the scalar of expansion, and \( \xi \) is the coefficient of bulk viscosity. According to Letelier [14] the energy density for the coupled system \( \rho \) and \( \rho_p \) is restricted to be positive, while the tension density \( \lambda \) may be positive or negative. The vector \( u^i \) describes the cloud four-velocity and \( X^i \) represents a direction of anisotropy, i.e. the direction of string. They satisfy the standard relations [14].

(4.6.3) \[ u^i u_j = -X^i X_j = -1, \quad u^i X_i = 0 \]

\( E_{ij} \) is the energy-momentum tensor for the magnetic field.

(4.6.4) \[ E_{ij} = \frac{1}{4\pi} g^{hk} F_{ih} F_{jk} - \frac{1}{4} g_{ij} F_{hk} F^{hk} \]

where \( F_{ij} \) is the electromagnetic field tensor, which satisfies the Maxwell equations

(4.6.5) \[ F_{[ij;h]} = 0, \quad (F^{ij} \sqrt{-g})_{ij} = 0 \]
Einstein's equation we consider here is

\[ R_{ij} - \frac{1}{2} R g_{ij} = T_{ij} \]

where we have choose the units such that \( c = 1 \) and \( 8\pi G = 1 \). In the co-moving coordinates \( u^i = \delta^i_0 \) and \( u^i = -\delta^i_0 \), and the incident magnetic field taken along the \( z \)-axis, with the help of Maxwell equations (4.6.5), the only non-vanishing component of \( F_{ij} \) is [35].

\[ F_{ij} = \text{constant} = M \]

The Einstein equation (4.6.6) for the metric (4.6.1) can be written as following system of equations [35,37]:

\[ \frac{\dot{\beta}}{\beta} + \frac{\dot{y}}{y} + \frac{\dot{\beta} \dot{y}}{\beta y} = \xi \theta - \frac{M^2}{8\pi \alpha^2 \beta^2 e^{2x}} \]  

\[ \frac{\dot{\alpha}}{\alpha} + \frac{\dot{y}}{y} + \frac{\dot{\alpha} \dot{y}}{\alpha y} = \xi \theta - \frac{M^2}{8\pi \alpha^2 \beta^2 e^{2x}} \]  

\[ \frac{\dot{\alpha}}{\alpha} + \frac{\dot{\beta}}{\beta} + \frac{\dot{\alpha} \dot{\beta}}{\alpha \beta} - \frac{1}{\alpha^2} = \lambda + \xi \theta + \frac{M^2}{8\pi \alpha^2 \beta^2 e^{2x}} \]  

\[ \frac{\dot{\alpha} \dot{\beta}}{\alpha \beta} + \frac{\dot{\beta} \dot{y}}{\beta y} + \frac{\dot{\alpha} \dot{y}}{\alpha y} - \frac{1}{\alpha^2} = \rho + \frac{M^2}{8\pi \alpha^2 \beta^2 e^{2x}} \]
\[
(4.6.12) \quad \frac{\dot{\alpha}}{\alpha} - \frac{\dot{\beta}}{\beta} = 0 \]

where the dot denotes the differentiation with respect to time \( t \). From Eq. (4.6.12), we have

\[
(4.6.13) \quad \alpha = H \beta
\]

where \( H \) is the constant of integration. In order to obtain a more general solution, we assume Takabayasi’s equation of state [27]

\[
(4.2.14) \quad \rho = a + k \lambda
\]

where \( a \) and \( k \) are the positive constants.

The expression for scalar of expansion and shear scalar are

\[
(4.6.15) \quad \theta = u_{i,i} = \frac{\dot{\alpha}}{\alpha} + \frac{\dot{\beta}}{\beta} + \frac{\dot{\gamma}}{\gamma}
\]

\[
(4.6.16) \quad \sigma^2 = \frac{1}{2} \sigma_{ij} \sigma^{ij}
\]

\[
= \frac{1}{3} \left( \frac{\dot{\alpha}^2}{\alpha^2} + \frac{\dot{\beta}^2}{\beta^2} + \frac{\dot{\gamma}^2}{\gamma^2} - \frac{\dot{\alpha} \dot{\beta}}{\alpha \beta} - \frac{\dot{\beta} \dot{\gamma}}{\beta \gamma} - \frac{\dot{\alpha} \dot{\gamma}}{\alpha \gamma} \right)
\]

We note that the five independent equations (4.6.9)-(4.6.12) and (4.6.14) connecting six unknown variables \((\alpha, \beta, \gamma, \lambda, \rho, \xi)\). Thus, one more
relation connecting these variables is needed to solve these equations. In order to obtain explicit solutions, one additional relation is needed and we adopt an assumption that the shear scalar of expansion is proportional to the shear scalar of expansion $\sigma \propto \theta$, which leads to

\[(4.6.17) \quad \beta = a + b\gamma^{\mu+1}\]

where $a$, $b$ and $\mu$ is a constant.

Now we consider $a = 0$ and $b = 1$, then Equation (4.6.14) and (4.6.17) reduces to-

\[(4.6.18) \quad \rho = k\lambda\]

\[(4.6.19) \quad \beta = \gamma^{\mu+1}\]

From equation (4.6.9), (4.6.10), (4.6.11), with the help of Eq. (4.6.18) eliminating $\rho$, $\lambda$ and $\xi\theta$, we obtain

\[(4.6.20) \quad k\frac{\dddot{\alpha}}{\alpha} - k\frac{\dddot{\gamma}}{\gamma} + (k - 1)\frac{\dddot{\alpha}\beta}{\alpha\beta} - (k + 1)\frac{\dddot{\beta}\gamma}{\beta\gamma} - \frac{\dddot{\alpha}\gamma}{\alpha\gamma} = (k - 1)\frac{1}{\alpha^2} + \frac{M^2}{8\pi} \frac{(2k-1)}{\alpha^2 \beta^2 e^{2x}}\]
Substituting Equation (4.6.13) and (4.6.19) into eq. (4.6.20), we have

\begin{equation}
\ddot{y} + \frac{(\mu + 1)[2k\mu - (\mu + 3)]}{k\mu} \frac{\dot{y}^2}{\gamma^2} = \frac{(k-1)}{k\mu H^2} \gamma^{-(2\mu+1)}
\end{equation}

\begin{equation}
+ \frac{M^2}{8\pi} \frac{(2k-1)}{k\mu H^2 e^{2x}} \gamma^{-(4\mu+3)}
\end{equation}

To solve Eq. (4.6.21), we denote \( \dot{y} = \vartheta \), then \( \ddot{y} = \vartheta \frac{d\vartheta}{dy} \), and the Eq. (4.6.21) can be reduced to the first-order differential equation in the following form:

\begin{equation}
\frac{\vartheta}{\gamma} \frac{d\vartheta}{dy} + l \frac{\vartheta^2}{\gamma} = \frac{(k-1)}{k\mu H^2} \gamma^{-(2\mu+1)}
\end{equation}

\begin{equation}
+ \frac{(2k-1)\kappa}{k\mu H^2 e^{2x}} \gamma^{-(4\mu+3)}
\end{equation}

where

\begin{equation}
l = \frac{(\mu+1)[2k\mu-(\mu+3)]}{k\mu}
\end{equation}

\begin{equation}
\kappa = \frac{M^2}{8\pi}
\end{equation}

Equation (4.6.22) can be written as
(4.6.25) \[
\frac{d}{d\gamma} (\vartheta^2 \gamma^{2l}) = \frac{2(k-1)}{k\mu H^2} \gamma^{2l-(2\mu+1)} + 2(2k-1)\kappa \frac{k\mu H^2 e^{2x}}{k\mu - (\mu^2 + 4\mu + 3)H^2 e^{2x}} \gamma^{2l-(4\mu+3)}
\]

Thus we obtained:

(4.6.26) \[
dt = \left[ \frac{(k-1)}{\{k(\mu^2+2\mu)-(\mu^2+4\mu+3)\}H^2} \gamma^{-2\mu} + \frac{(2k-1)\kappa}{\{k\mu-(\mu^2+4\mu+3)\}H^2 e^{2x}} \gamma^{-(4\mu+2)} + N\gamma^{-2l} \right]^{-\frac{1}{2}} d\gamma
\]

where \( N \) is the constant of integration. For this solution, the geometry of the universe is described by the metric

(4.6.27) \[
ds^2 = -\left[ \frac{(k-1)}{\{k(\mu^2+2\mu)-(\mu^2+4\mu+3)\}H^2} \gamma^{-2\mu} + \frac{(2k-1)\kappa}{\{k\mu-(\mu^2+4\mu+3)\}H^2 e^{2x}} \gamma^{-(4\mu+2)} + N\gamma^{-2l} \right]^{-1} d\gamma^2 + H^2 \gamma^{2\mu+2} dx^2 + \gamma^{2\mu+2} e^{2x} dy^2 + \gamma^2 dz^2
\]

Under suitable transformation of coordinates, Eq. (4.6.27) reduces to
(4.6.28) \[ ds^2 = -\left[\frac{(k-1)}{\{k(\mu^2+2\mu)-(\mu^2+4\mu+3)\}H^2} \right]^{-2\mu} T^{-2l} \]

\[ \quad + \frac{(2k-1)\mathcal{K}}{\{k\mu-(\mu^2+4\mu+3)\}H^2e^{2x}} T^{-(4\mu+2)} + N^{-2l} \] \[ dT^2 \]

\[ + H^2 T^{2\mu+2} dx^2 + T^{2\mu+2} e^{2x} dy^2 + T^2 dz^2 \]

For the model of Eq. (4.6.28), the order physical and geometrical parameters can easily be obtained. The expressions for the energy density \( \rho \), the string tension density \( \lambda \), the particle density \( \rho_p \), the coefficient of bulk viscosity \( \xi \), the scalar of expression \( \theta \) and the shear scalar \( \sigma^2 \) are, respectively, given by

(4.6.29) \[ \rho = \frac{k(2\mu+3)}{\{k(\mu^2+2\mu)-(\mu^2+4\mu+3)\}H^2} T^{-2(\mu+1)} \]

\[ + \frac{[2k(\mu+1)^2 + 3k\mu + 3k+3]\mathcal{K}}{\{k\mu-(\mu^2+4\mu+3)\}H^2 e^{2x}} T^{-(4\mu+4)} \]

\[ + (\mu + 1)(\mu + 3)N^{-2(l+1)} \]

(4.6.30) \[ \lambda = \frac{\rho}{k} \]

(4.6.31) \[ \rho_p = \rho - \lambda = \left(1 - \frac{1}{k}\right) \rho \]
**In Investigations on Some Bianchi Type-III String Cosmological Models**

\( \xi \theta = \frac{(k-1)}{\{k(\mu^2+2\mu)-(\mu^2+4\mu+3)\}H^2T^{-2(\mu+1)}} + \frac{(\mu^2+3\mu+1)(1-2k)}{\{k\mu-(\mu^2+4\mu+3)\}H^2e^{2x}} \mathcal{K}T^{-2(\mu+4)} \)

\( + \frac{(\mu+1)(\mu+2)-k\mu(\mu+1})(\mu+3)}{k\mu} N T^{-2l+2} \)

\( \theta = (2\mu + 3) \left[ \frac{(k-1)}{\{k(\mu^2+2\mu)-(\mu^2+4\mu+3)\}H^2} T^{-2(\mu+1)} \right] \)

\( + \frac{(2k-1)\mathcal{K}}{\{k\mu-(\mu^2+4\mu+3)\}H^2e^{2x}} T^{-2(\mu+4)} + N T^{-2l+2} \right]^{1/2} \)

\( \sigma^2 = \frac{\mu^2}{3} \left[ \frac{(k-1)}{\{k(\mu^2+2\mu)-(\mu^2+4\mu+3)\}H^2} T^{-2(\mu+1)} \right] \)

\( + \frac{(2k-1)\mathcal{K}}{\{k\mu-(\mu^2+4\mu+3)\}H^2e^{2x}} T^{-2(\mu+4)} + N T^{-2l+2} \]
4.7 DISCUSSION:

From Equations (4.6.29) and (4.6.31) it is observed that the energy condition \( \rho \geq 0 \) and \( \rho_p \) are fulfilled, provided

\[
N \geq 0, \mu > 0 \text{ and } k > (\mu + 1)(\mu + 3)/\mu
\]

Or

\[
N \geq 0, \mu > 0 \text{ and } k < -1/(\mu + 1)(2\mu + 5)
\]

when \( N \geq 0, \mu > 0 \text{ and } k > (\mu + 1)(\mu + 3)/\mu \), the string tension density \( \lambda > 0 \); however, \( \lambda < 0 \) when \( N \geq 0, \mu > 0 \text{ and } k < -1/(\mu + 1)(2\mu + 5) \).

The above expressions (4.6.29 - 4.6.32) indicate that the magnetic field is related with \( \rho, \lambda, \rho_p \) and \( \xi \). Here a term of \( K \) is involved in the expression for \( \rho, \lambda, \rho_p, \xi, \theta \) and \( \sigma^2 \) respectively, and it represents the effect of magnetic field on the model.

It is seen that in the case \( \mu > 0 \), whether \( k > (\mu + 1)(\mu + 3)/\mu \) or \( k < -1/(\mu + 1)(2\mu + 5) \), we have \( l + 1 > 0 \). Hence equation (4.6.33) shows that the scalar of expansion \( \theta \) tends to infinitely large when \( T \to 0 \), but \( \theta \to 0 \) or tends to finite when \( T \to \infty \). The energy density \( \rho \) tends to finite when \( T \to \infty \) and \( \rho \to \infty \) when \( T \to 0 \), therefore the model describes a shearing non rotating expanding universe with the big-bang start. We can see from the above discussion that the bulk viscosity plays a significant role in the evolution of universe [1, 15]. Furthermore, since \( \lim_{T \to \infty} \frac{\sigma}{\theta} \neq 0 \), the model does not approach isotropy for large values of \( T \).
In the special case $k = 1$, the model represents a geometric string model [38]. In the absence of magnetic field, $K = 0$, the metric (4.6.28) reduces to the string model with bulk viscosity, i.e.,

\[(4.7.1) \quad ds^2 = -\left[\frac{(k-1)}{(k^2+2\mu)-(\mu^2+4\mu+3)}H^2T^{-2\mu} + NT^{-2l}\right]^{-1}dT^2 + H^2T^{2\mu+2}dx^2 + T^{2\mu+2}e^{2x}dy^2 + T^2dz^2\]

4.8 **SUMMARY:**

In this Chapter, we study Bianchi type-III string cosmological model in the presence of bulk viscosity and magnetic field. To obtain an explicit solution, an equation of state $\rho = a + b\dot{\lambda}$ and an assumption that the scalar of expansion is proportional to the shear scalar $\sigma \propto \theta$, which leads to the relation between metric potentials $\beta = a + b\gamma^{\mu+1}$. Then the cosmological model for a cosmic string with bulk viscosity and magnetic field is obtained. The physical and geometric aspects of the model in the presence and absence of magnetic field are also discussed. Our model describes a shearing non-rotating continuously expanding universe with a big-bang start. In the absence of magnetic field it reduces to the string model with bulk viscosity.
4.9 REFERENCES:


