Chapter 5

Asymmetric Laplace Stochastic Volatility Model

5.1 Introduction

It is known that return distributions of financial time series data, such as stock and foreign exchange returns, exhibit departure from the normality assumption as they are often skewed and have heavier tails than the normal distribution. Data also exhibit time varying volatility and volatility clustering over time. Accounting for these characteristics of data is crucial to make appropriate decisions for risk management. These aspects motivated the researchers to develop two main classes of models that capture the time-varying auto-correlated volatility process: the ARCH model, introduced by Engle (1982) and the SV model, introduced by Taylor (1986). In ARCH model, the time-varying variance is assumed to be a deterministic function
of the lagged values of the squared errors. For a comprehensive survey on this model and its various generalizations, such as Generalized ARCH (GARCH) by Bollerslev (1986), see Shephard (1996) and Tsay (2005). In SV model, the volatility at time $t$ is assumed to be a stochastic process in terms of some latent variables.

The asymmetric Laplace(AL) distribution demonstrates flexibility in fitting data with heavy tails and skewness, which make it a promising candidate for financial data modelling. Kozubowski and Podgorski (2000) and Kotz et al. (2012) studied many properties of asymmetric Laplace distributions. Jayakumar and Kuttykrishnan (2007) introduced a time series model using an asymmetric Laplace distribution for modelling data from financial contexts. Jose and Thomas (2011) developed a first order stationary autoregressive process with generalized Laplace marginal distribution. Although the theory and applications of asymmetric Laplace distributions is well developed and there is considerable literature in recent years, their applications in modelling stochastic volatility in financial time series is not developed. We consider SV model with log-volatility process have an asymmetric Laplace marginal distribution, rather than the Gaussian distribution.

The next section briefly discusses the asymmetric Laplace distribution and its properties. The construction AL-SV model and its second order properties are described in Section 5.3 and 5.4 of this chapter. We discussed the estimation procedure by the method of moments in Section 5.5. The asymptotic properties of estimators are established in Section 5.6. A simulation study is carried out in Section 5.7. In Section 5.8, we present the results on data analysis using our model.
5.2 Asymmetric Laplace distribution

A random variable $X$ is said to have an asymmetric Laplace distribution with parameters $\theta \in \mathbb{R}$, $\kappa > 0$ and $\sigma \geq 0$ \((AL(\theta, \kappa, \sigma))\) if its probability density function is (cf: Kotz et al. (2012)):

$$ f(x; \theta, \kappa, \sigma) = \frac{\sqrt{2}}{\sigma} \frac{\kappa}{1+\kappa^2} \begin{cases} \exp\left(-\frac{\sqrt{2}\kappa}{\sigma} |x-\theta|\right), & \text{if } x \geq \theta \\ \exp\left(-\frac{\sqrt{2}}{\kappa \sigma} |x-\theta|\right), & \text{if } x < \theta \end{cases} $$

(5.1)

or, the distribution function of the $X$ is the form

$$ F(x; \theta, \kappa, \sigma) = \begin{cases} 1 - \frac{1}{1+\kappa^2} \exp\left(-\frac{\sqrt{2}\kappa}{\sigma} |x-\theta|\right), & \text{if } x \geq \theta \\ \frac{\kappa^2}{1+\kappa^2} \exp\left(-\frac{\sqrt{2}}{\kappa \sigma} |x-\theta|\right), & \text{if } x < \theta \end{cases} $$

(5.2)

Hence the characteristic function of $AL(\theta, \kappa, \sigma)$ is obtained as

$$ \psi_X(t) = E\left(e^{itX}\right) = \frac{e^{i\theta t}}{1 + \frac{1}{2} \sigma^2 t^2 - i \frac{\sigma}{\sqrt{2}} \left(\frac{1}{\kappa} - \kappa\right) t}. $$

(5.3)

Using (5.3), the mean, variance and the coefficients of skewness and kurtosis can be respectively obtained as

$$ E(X) = \theta + \frac{\sigma}{\sqrt{2}} \left(\frac{1}{\kappa} - \kappa\right), \quad Var(X) = \frac{\sigma^2}{2} \left(\frac{1}{\kappa^2} + \kappa^2\right), $$

$$ \gamma = 2 \frac{1/\kappa^3 - \kappa^3}{(1/\kappa^2 + \kappa^2)^{3/2}}, \quad K = 6 - \frac{12}{(1/\kappa^2 + \kappa^2)^2}. $$

The absolute value of $\gamma$ is bounded by two, and as $\kappa$ increases within the interval $(0, \infty)$, then the corresponding value of $\gamma$ decreases monotonically from 2 to -2.
Similarly, the distribution is leptokurtic and $K$ varies from 3 (the least value for the symmetric Laplace distribution with $\kappa = 1$) to 6 (the greatest value attained for the limiting exponential distribution when $\kappa \to 0$ (see Kotz et al. (2012)).

The asymmetric Laplace random variable $X$ also admits the representation of the form

$$X \overset{d}{=} \rho X + (1 - \rho)\theta + \frac{\sigma}{\sqrt{2}} \left( \frac{1}{\kappa} I_1 E_1 - \kappa I_2 E_2 \right), \quad \rho \in [0, 1], \quad (5.4)$$

where $I_1, I_2$ are dependent Bernoulli random variables taking on values of either zero or one with probabilities,

$$P(I_1 = 0, I_2 = 0) = \rho^2, \quad P(I_1 = 1, I_2 = 1) = 0,$$

$$P(I_1 = 1, I_2 = 0) = (1 - \rho) \left( \rho + \frac{1 - \rho}{1 + \kappa^2} \right),$$

$$P(I_1 = 0, I_2 = 1) = (1 - \rho) \left( \rho + \frac{(1 - \rho)\kappa^2}{1 + \kappa^2} \right),$$

$E_1$ and $E_2$ are standard exponential variables, with all variables being mutually independent.

As shown in Kotz et al. (2012), all asymmetric Laplace laws are self decomposable for all values of the parameters. Gaver and Lewis (1980) proved that only self decomposable distributions can be marginal distributions of a first order autoregressive process. Hence the asymmetric Laplace distribution can be the marginal distribution of an AR(1) process. Various authors studied autoregressive models with non-Gaussian marginal distribution extensively in recent years due to wide applications of such models in socio-economic fields. Using the results in Jose and
Thomas (2011), a first order autoregressive process with asymmetric Laplace distribution is constructed in the next section.

5.3 First order Asymmetric Laplace Autoregressive Process

The first order asymmetric Laplace AR process is constituted by \( \{ h_t, t \geq 1 \} \), where \( h_t \) satisfies the equation,

\[
h_t = \rho h_{t-1} + \eta_t; \quad \rho \in [0, 1), \quad t > 0,
\]

where \( \{ h_t \} \) is a stationary Markov process with asymmetric Laplace marginal distribution with location parameter \( \theta \), shape parameter \( \kappa \) and scale parameter \( \sigma \) (\( AL(\theta, \kappa, \sigma) \)) and \( \{ \eta_t \} \) is a independent and identically distributed random variables independent of \( h_{t-\tau} \) for all \( \tau \geq 1 \). The basic problem is to find the distribution of \( \{ \eta_t \} \) such that \( \{ h_t \} \) has the asymmetric Laplace distribution \( AL(\theta, \kappa, \sigma) \) as the stationary marginal distribution. The following theorem proved by Jose and Thomas (2011) summarizes the result in this context.

**Theorem 5.1.** The stationary marginal distribution of \( \{ h_t \} \) in model (5.5) is asymmetric Laplace marginal distribution with parameters \( \theta \), \( \kappa \) and \( \sigma \) iff the distribution of \( \eta_t \) is specified as a convolution of the form \( \eta_t \overset{d}{=} U + (I_1 E_1 - I_2 E_2) \), as in (5.7) provided \( \eta_0 \overset{d}{=} h_0 \).
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Proof. In terms of characteristic function the model in (5.5) can be rewritten as

\[ \psi_h(t) = \psi_h(\rho t) \psi_\eta(t). \]

Under stationarity assumption, \( \psi_\eta(t) = \frac{\psi_\eta(t)}{\psi_h(\rho t)}. \)

Substituting the characteristic function given by (5.3) we get,

\[ \psi_\eta(t) = e^{i\theta t} e^{i\theta \rho t} \left( 1 + i \sigma \kappa \sqrt{2} \rho t \right) \left( 1 - i \frac{\sigma}{\sqrt{2} \kappa} \rho t \right) \]

This implies that \( \eta_t \) has a convolution structure of the following form

\[ \eta_t \overset{d}{=} U + (I_1 E_1 - I_2 E_2), \]

where \( U \) is degenerate at \( \theta (1 - \rho) \). \( E_1 \) and \( E_2 \) are independent exponential random variables with means \( \frac{\sigma}{\sqrt{2} \kappa} \) and \( \frac{\sigma \kappa}{\sqrt{2}} \) respectively and \( (I_1, I_2) \) is such that

\[ P(I_1 = 0, I_2 = 0) = \rho^2, \quad P(I_1 = 1, I_2 = 1) = (1 - \rho)^2, \]

\[ P(I_1 = 1, I_2 = 0) = P(I_1 = 0, I_2 = 1) = \rho (1 - \rho). \]

Further \( (I_1, I_2) \) independent of \( E_1 \) and \( E_2 \).
The converse part can be provided by the method of induction. We assume that \( h_{t-1} \) follows \( AL(\theta, \kappa, \sigma) \) with characteristic function (5.3).

\[
\psi_{h_t}(t) = \psi_{h_{t-1}}(\rho t) \psi_{\eta_t}(t)
\]

\[
= \left[ e^{i\theta \rho t} \left( \frac{e^{i\theta(1-\rho)t} \left( 1 + \frac{\sigma_\eta}{\sqrt{2}} \rho t \right)}{1 + i \frac{\sigma_\eta}{\sqrt{2}} \rho t} \left( 1 - \frac{\sigma_\eta}{\sqrt{2}} \rho t \right) \right) \right] \left[ e^{i\theta \rho t} \left( \frac{e^{i\theta(1-\rho)t} \left( 1 - \frac{\sigma_\eta}{\sqrt{2}} \rho t \right)}{1 + i \frac{\sigma_\eta}{\sqrt{2}} \rho t} \left( 1 - \frac{\sigma_\eta}{\sqrt{2}} \rho t \right) \right) \right]
\]

\[
= e^{i\theta t} \left( \frac{1}{1 + i \frac{\sigma_\eta}{\sqrt{2}} t} \right) \left( \frac{1}{1 - i \frac{\sigma_\eta}{\sqrt{2}} t} \right),
\]

which is same as the asymmetric Laplace characteristic function. This shows that \( \{h_t\} \) is strictly stationary with asymmetric Laplace marginals provided \( \eta_0 \overset{d}{=} h_0 \) which follows \( AL(\theta, \kappa, \sigma) \). Hence the theorem.

The mean and variance of \( \eta_t \) are given by

\[
E(\eta_t) = (1 - \rho) \left[ \theta + \frac{\sigma}{\sqrt{2}} \left( \frac{1}{\kappa} - \kappa \right) \right] \text{ and } Var(\eta_t) = (1 - \rho^2) \left[ \frac{\sigma^2}{2} \left( \frac{1}{\kappa} - \kappa \right)^2 + \sigma^2 \right].
\]

Hence the second order properties of the process \( \{h_t\} \) are summarized below.

\[
E(h_t) = \theta + \frac{\sigma}{\sqrt{2}} \left( \frac{1}{\kappa} - \kappa \right), \text{ Var}(h_t) = \frac{\sigma^2}{2} \left( \frac{1}{\kappa^2} + \kappa^2 \right) \text{ and the ACF, } \rho_k(h_t) = \rho^k, \ k = 1, 2, \ldots.
\]

The regression of \( h_t \) on \( h_{t-1} \) is given by

\[
E(h_t | h_{t-1}) = \rho h_{t-1} + E(\eta_t | h_{t-1})
\]

\[
= \rho h_{t-1} + (1 - \rho) \left[ \theta + \frac{\sigma}{\sqrt{2}} \left( \frac{1}{\kappa} - \kappa \right) \right]
\]

\[
= g(\Theta; h_{t-1}), \ \Theta = (\theta, \kappa, \sigma)'.
\]
Next we discuss the construction of SV model generated by first order AL autoregressive process discussed in this section.

### 5.4 Asymmetric Laplace SV Model

Let \( \{r_t\} \) be a sequence of returns on certain financial asset and the volatilities are generated by a Markov sequence \( \{\exp(h_t)\} \) of non-negative random variables. Define the SV model

\[
\begin{align*}
r_t &= \exp(h_t/2) \varepsilon_t, \\
h_t &= \rho h_{t-1} + \eta_t, \quad t = 1, 2, ..., \quad 0 \leq \rho < 1 \tag{5.8}
\end{align*}
\]

where \( \{\varepsilon_t\} \) is a sequence of independent and identically distributed standard Laplace random variables with mean zero and variance one. We assume that the sequence \( \{\varepsilon_t\} \) is independent of \( h_t \) and \( \eta_t \) for every \( t \). Here we assume that for every \( t \), the volatility, \( h_t \) is an asymmetric Laplace random variables. Since the sequence \( \{\varepsilon_t\} \) follows standard Laplace distribution, the odd moments of \( r_t \) are zero and its even moments are given by

\[
E(r_t^{2r}) = \frac{(2r)! \ e^{r\theta}}{2^r \left(1 - \frac{1}{2}r^2 \sigma^2 - \frac{r\sigma}{\sqrt{2}} \left(\frac{1}{\kappa} - \kappa\right)\right)}, \quad r = 1, 2, .... \tag{5.9}
\]

Then

\[
Var(r_t) = \frac{e^{\theta}}{\left(1 - \frac{a^2}{2} - \frac{a}{\sqrt{2}} \left(\frac{1}{\kappa} - \kappa\right)\right)}
\]

and the kurtosis of \( r_t \) becomes
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\[ K = 6 \frac{\left[ 1 - \frac{\sigma^2}{2} - \frac{\sigma}{\sqrt{2}} \left( \frac{1}{\kappa} - \kappa \right) \right]^2}{\left[ 1 - 2\sigma^2 - \sqrt{2}\sigma \left( \frac{1}{\kappa} - \kappa \right) \right]^2}. \]

(5.10)

By choosing different values for \( \sigma \) and \( \kappa \), one can get a distribution with larger kurtosis as shown in Figure 5.1.

Figure 5.1: The plot of kurtosis of \( r_t \)

The variance and covariance function of the squared return series are obtained as

\[ \text{Var}(r_t^2) = E(r_t^4) - (E(r_t^2))^2 \]

\[ = e^{2\theta} \left\{ 6 \left[ 1 - \frac{\sigma^2}{2} - \frac{\sigma}{\sqrt{2}} \left( \frac{1}{\kappa} - \kappa \right) \right]^2 - \left[ 1 - 2\sigma^2 - \sqrt{2}\sigma \left( \frac{1}{\kappa} - \kappa \right) \right] \right\} \]

\[ \frac{\left[ 1 - 2\sigma^2 - \sqrt{2}\sigma \left( \frac{1}{\kappa} - \kappa \right) \right]^2}{\left[ 1 - 2\sigma^2 - \sqrt{2}\sigma \left( \frac{1}{\kappa} - \kappa \right) \right]} \left[ 1 - \frac{\sigma^2}{2} - \frac{\sigma}{\sqrt{2}} \left( \frac{1}{\kappa} - \kappa \right) \right] \]

\[ \text{Cov} \left( r_t^2, r_{t-k}^2 \right) = E \left( r_t^2 r_{t-k}^2 \right) - E \left( r_t^2 \right) \left. E \left( r_{t-k}^2 \right) \right). \]
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For this purpose, we consider

\[ E(r_t^2r_{t-k}^2) = E(e^{h_t}e^{h_{t-k}}e^{h_{t-k}}) = E(e^{h_{t-k}}) \]

\[ = E(e^{\rho^k h_{t-k} + \sum_{i=1}^{k-1} \rho^{k-i} h_{t-(k-i)}} e^{h_{t-k}}) \]

\[ = E(e^{(1+\rho^k) h_{t-k}}) E(e^{\rho^{k-1} h_{t-(k-1)}}) E(e^{\rho^{k-2} h_{t-(k-2)}}) \ldots E(e^{\rho_{h_{t-1}}}) E(\eta_h) \]

\[ = E(e^{(1+\rho^k) h_{t-k}}) \frac{E(e^{\eta_h})}{E(e^{\rho^k h_{t-k}})} \]

\[ = e^{2\theta} \left[ 1 - \frac{1}{2} \sigma^2 \rho^{2k} - \frac{\sigma \rho^k}{\sqrt{2}} \left( \frac{1}{\kappa} - \kappa \right) \right] \]

\[ \left[ 1 - \frac{\sigma^2 (1+\rho^k)^2}{2} - \frac{\sigma}{\sqrt{2}} \left( \frac{1}{\kappa} - \kappa \right) (1 + \rho^k) \right] \left[ 1 - \frac{\sigma^2}{2} - \frac{\sigma}{\sqrt{2}} \left( \frac{1}{\kappa} - \kappa \right) \right] \]

and so

\[ \text{Cov} (r_t^2, r_{t-k}^2) = e^{2\theta} \sigma^2 \rho^k \left[ 1 + \frac{1}{2} \left( \frac{1}{\kappa} - \kappa \right)^2 + \frac{\sigma^2 \rho^k}{4} + \frac{\sigma}{2\sqrt{2}} \left( \frac{1}{\kappa} - \kappa \right) (1 + \rho^k) \right] \]

\[ \left[ 1 - \frac{\sigma^2 (1+\rho^k)^2}{2} - \frac{\sigma}{\sqrt{2}} \left( \frac{1}{\kappa} - \kappa \right) (1 + \rho^k) \right] \left[ 1 - \frac{\sigma^2}{2} - \frac{\sigma}{\sqrt{2}} \left( \frac{1}{\kappa} - \kappa \right) \right]^2. \]

Hence, the lag \( k \) autocorrelation of the squared sequence \( \{r_t^2\} \) is

\[ \rho_{r_t^2}(k) = C(\sigma, \kappa) \times \frac{\sigma^2 \rho^k \left[ 1 + \frac{1}{2} \left( \frac{1}{\kappa} - \kappa \right)^2 + \frac{\sigma^2 \rho^k}{4} + \frac{\sigma}{2\sqrt{2}} \left( \frac{1}{\kappa} - \kappa \right) (1 + \rho^k) \right]}{\left[ 1 - \frac{\sigma^2 (1+\rho^k)^2}{2} - \frac{\sigma}{\sqrt{2}} \left( \frac{1}{\kappa} - \kappa \right) (1 + \rho^k) \right]} , \quad (5.11) \]

where

\[ C(\sigma, \kappa) = \frac{\left[ 1 - 2\sigma^2 - \sqrt{2}\sigma \left( \frac{1}{\kappa} - \kappa \right) \right]}{6 \left[ 1 - \frac{\sigma^2}{2} - \frac{\sigma}{\sqrt{2}} \left( \frac{1}{\kappa} - \kappa \right) \right]^2 - \left[ 1 - 2\sigma^2 - \sqrt{2}\sigma \left( \frac{1}{\kappa} - \kappa \right) \right]} , \]
The ACF is an exponentially decreasing function of the lags for different values of the parameters, as can be seen in Figure 5.2.

Figure 5.2: The ACF of squared returns for different combinations of parameters

5.5 Parameter Estimation

One of the difficulties with the statistical inference for SV models is that the likelihood function involves the unobservable Markov dependent latent variables. These variables have to be integrated out using multiple integrals and this complicates the parameter estimation by the method of maximum likelihood. A number of methods
are proposed for estimating the parameters of a SV model and a comprehensive survey may be seen in Tsay (2005). For the SV model described above, we adopt the method of moments to estimate the parameters. Let \((r_1, r_2, ..., r_T)\) be a realization of length \(T\) from the AL-SV model (5.8) and \(\Theta = (\theta, \kappa, \sigma, \rho)'\) be the parameter vector to be estimated. We use the moments

\[
E(r_t^2) = \frac{e^\theta}{\left[1 - \frac{\sigma^2}{2} - \frac{\sigma}{\sqrt{2}} \left(\frac{1}{\kappa} - \kappa\right)\right]}, \quad E(r_t^4) = \frac{6e^{2\theta}}{\left[1 - 2\sigma^2 - \sqrt{2}\sigma \left(\frac{1}{\kappa} - \kappa\right)\right]},
\]

\[
E(r_t^6) = \frac{90e^{3\theta}}{\left[1 - \frac{9}{2}\sigma^2 - \frac{3}{\sqrt{2}}\sigma \left(\frac{1}{\kappa} - \kappa\right)\right]},
\]

\[
E(r_t^2 r_{t-1}^2) = \frac{e^{2\theta} \left[1 - \frac{\sigma^2\rho^2}{2} - \frac{\sigma\rho}{\sqrt{2}} \left(\frac{1}{\kappa} - \kappa\right)\right]}{\left[1 - \frac{\sigma^2}{2} - \frac{\sigma}{\sqrt{2}} \left(\frac{1}{\kappa} - \kappa\right)\right] \left[1 - \frac{\sigma^2(1+\rho)^2}{2} - \frac{\sigma(1+\rho)}{\sqrt{2}} \left(\frac{1}{\kappa} - \kappa\right) (1+\rho)\right]}
\]

to estimate the parameters.

We define

\[
f(r_t, r_{t-1}, \theta) = \begin{pmatrix}
r_t^2 - \frac{e^\theta}{\left[1 - \frac{\sigma^2}{2} - \frac{\sigma}{\sqrt{2}} \left(\frac{1}{\kappa} - \kappa\right)\right]}
r_t^4 - \frac{6e^{2\theta}}{\left[1 - 2\sigma^2 - \sqrt{2}\sigma \left(\frac{1}{\kappa} - \kappa\right)\right]}
r_t^6 - \frac{90e^{3\theta}}{\left[1 - \frac{9}{2}\sigma^2 - \frac{3}{\sqrt{2}}\sigma \left(\frac{1}{\kappa} - \kappa\right)\right]}
r_t^2 r_{t-1}^2 - \frac{e^{2\theta} \left[1 - \frac{\sigma^2\rho^2}{2} - \frac{\sigma\rho}{\sqrt{2}} \left(\frac{1}{\kappa} - \kappa\right)\right]}{\left[1 - \frac{\sigma^2}{2} - \frac{\sigma}{\sqrt{2}} \left(\frac{1}{\kappa} - \kappa\right)\right] \left[1 - \frac{\sigma^2(1+\rho)^2}{2} - \frac{\sigma(1+\rho)}{\sqrt{2}} \left(\frac{1}{\kappa} - \kappa\right) (1+\rho)\right]}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
r_t^2 - c_1 \\
r_t^4 - c_2 \\
r_t^6 - c_3 \\
r_t^2 r_{t-1}^2 - c_4
\end{pmatrix}, \quad (Say).
\]
Then the moment estimator $\hat{\Theta} = (\hat{\theta}, \hat{\kappa}, \hat{\sigma}, \hat{\rho})'$ of $\Theta$ may be obtained by solving
$$
\frac{1}{T} \sum_{t=1}^{T} f(r_t, r_{t-1}, \Theta) = 0.
$$
The resulting moment equations for $\Theta = (\theta, \kappa, \sigma, \rho)'$ are expressed as

$$
\hat{\sigma}^2 = \frac{2e^{\hat{\theta}}Y_4 - 6e^{2\hat{\theta}}Y_2 - Y_2Y_4'}{Y_2Y_4},
$$

$$
\hat{Y}_4 = \frac{6e^{2\theta}}{[1 - 2\sigma^2 - \sqrt{2}\sigma (\frac{1}{\kappa} - \kappa)]'},
$$

$$
\hat{Y}_6 = \frac{90e^{3\theta}}{[1 - \frac{9}{2}\sigma^2 - \frac{3}{\sqrt{2}}\sigma (\frac{1}{\kappa} - \kappa)]'},
$$

$$
\frac{\hat{Y}_{22}}{Y_2} = \frac{e^{\theta} \left[ 1 - \frac{\sigma^2}{2} - \frac{\sigma \rho}{\sqrt{2}} (\frac{1}{\kappa} - \kappa) \right]}{\left[ 1 - \frac{\sigma^2(1+\rho)^2}{2} - \frac{\sigma}{\sqrt{2}} (\frac{1}{\kappa} - \kappa) (1 + \rho) \right]},
$$

where $Y_2 = (1/T) \sum_{t=1}^{T} r_t^2$, $Y_4 = (1/T) \sum_{t=1}^{T} r_t^4$, $Y_6 = (1/T) \sum_{t=1}^{T} r_t^6$ and $Y_{22} = (1/T) \sum_{t=1}^{T} r_t^2 r_{t-1}^2$. These equations have to be solved by numerical methods and computational details are given in Appendix D.

### 5.6 Asymptotic Properties of Estimators

To prove that the moment estimators are consistent and asymptotically normal (CAN), we refer to Hansen (1982) under the following assumptions. Hansen (1982) proved results for GMM estimators. This theorem also valid for method of moment estimators.

(i) $\{r_t : -\infty < t < \infty\}$ is stationary and ergodic sequence.
(ii) The parameter space $\Theta$ is an open subset of $\mathbb{R}^q$ that contains the true parameter $\nu_0$.

(iii) $f(\cdot, \nu)$ and $\frac{\partial f}{\partial \nu}$ are Borel measurable for each $\nu \in \Theta$ and $\frac{\partial f}{\partial \nu}$ is continuous on $\Theta$ for each $r \in \mathbb{R}^q$.

(iv) $\frac{\partial f}{\partial \nu}$ is first moment continuous at $\nu_0$, $D = E\left[\frac{\partial}{\partial \nu} f(r, \nu_0)\right]$ exists, is finite, and has full rank.

(v) Let $\omega_t = f(r_t, \nu_0)$, $-\infty < t < \infty$ and

$$v_j = E(\omega_0|\omega_{-j}, \omega_{-j-1}, \ldots) - E(\omega_0|\omega_{-j-1}, \omega_{-j-2}, \ldots), \ j \geq 0.$$  

The assumptions are that $E(\omega_0, \omega_0')$ exists and is finite, $E(\omega_0|\omega_{-j}, \omega_{-j-1}, \ldots)$ converges in mean square to zero and $\sum_{j=0}^{\infty} E(v'_j v_j)^{1/2}$ is finite. Now we have the following result, proved by Hansen (1982).

**Theorem 5.2.** Suppose that the sequence $\{r_t : -\infty < t < \infty\}$ satisfies the assumptions (i) - (v). Then $\left\{\sqrt{T}(\hat{\Theta} - \Theta), T \geq 1\right\}$ converges in distribution to a normal random vector with mean 0 and dispersion matrix $\left[D S^{-1} D'\right]^{-1}$, where $D$ is as given in (iv) and $S = \sum_{k=-\infty}^{\infty} \Gamma_k, \Gamma_k = E(\omega_t \omega_{t-k}')$.

The sequence $\{r_t\}$ given in (5.8) is stationary, ergodic and has finite moments, due the fact that $\{h_t\}$ holds these properties. Therefore, the regularity conditions listed above hold good for our AL-SV model.
For computing the elements of the asymptotic dispersion matrix, the following observations become useful.

\[
m_{2;2}^{(k)} = E \left( r_{t}^{2} r_{t-k}^{2} \right) = \frac{E \left( e^{(p+1)h_{t-k}} \right) E \left( e^{h_{t}} \right)}{E \left( e^{ph_{t-k}} \right)};
\]

\[
m_{2;4}^{(k)} = E \left( r_{t}^{2} r_{t-k}^{4} \right) = 6 \frac{E \left( e^{(p+2)h_{t-k}} \right) E \left( e^{h_{t}} \right)}{E \left( e^{ph_{t-k}} \right)};
\]

\[
m_{2;6}^{(k)} = E \left( r_{t}^{2} r_{t-k}^{6} \right) = 90 \frac{E \left( e^{(p+3)h_{t-k}} \right) E \left( e^{h_{t}} \right)}{E \left( e^{ph_{t-k}} \right)};
\]

\[
m_{2;2,2}^{(k)} = E \left( r_{t}^{2} r_{t-k}^{2} r_{t-k-1}^{2} \right) = \frac{E \left( e^{(p+1+p+1)h_{t-k-1}} \right) E \left( e^{(p+1)h_{t-k}} \right) E \left( e^{h_{t}} \right)}{E \left( e^{(p+1+p+1)h_{t-k-1}} \right) E \left( e^{ph_{t-k}} \right)};
\]

\[
m_{4;2}^{(k)} = E \left( r_{t}^{4} r_{t-k}^{2} \right) = 6 \frac{E \left( e^{(2p+1)h_{t-k}} \right) E \left( e^{2h_{t}} \right)}{E \left( e^{2ph_{t-k}} \right)};
\]

\[
m_{4;4}^{(k)} = E \left( r_{t}^{4} r_{t-k}^{4} \right) = 36 \frac{E \left( e^{(2p+2)h_{t-k}} \right) E \left( e^{2h_{t}} \right)}{E \left( e^{2ph_{t-k}} \right)};
\]

\[
m_{4;6}^{(k)} = E \left( r_{t}^{4} r_{t-k}^{6} \right) = 540 \frac{E \left( e^{(2p+3)h_{t-k}} \right) E \left( e^{2h_{t}} \right)}{E \left( e^{2ph_{t-k}} \right)};
\]

\[
m_{4;2,2}^{(k)} = E \left( r_{t}^{4} r_{t-k}^{2} r_{t-k-1}^{2} \right) = 6 \frac{E \left( e^{(2p+1+p+1)h_{t-k-1}} \right) E \left( e^{(2p+1)h_{t-k}} \right) E \left( e^{2h_{t}} \right)}{E \left( e^{(2p+1+p+1)h_{t-k-1}} \right) E \left( e^{2ph_{t-k}} \right)};
\]

\[
m_{6;2}^{(k)} = E \left( r_{t}^{6} r_{t-k}^{2} \right) = 90 \frac{E \left( e^{(3p+1)h_{t-k}} \right) E \left( e^{3h_{t}} \right)}{E \left( e^{3ph_{t-k}} \right)};
\]

\[
m_{6;4}^{(k)} = E \left( r_{t}^{6} r_{t-k}^{4} \right) = 540 \frac{E \left( e^{(3p+2)h_{t-k}} \right) E \left( e^{3h_{t}} \right)}{E \left( e^{3ph_{t-k}} \right)};
\]

\[
m_{6;2,2}^{(k)} = E \left( r_{t}^{6} r_{t-k}^{2} r_{t-k-1}^{2} \right) = 90 \frac{E \left( e^{(3p+1+p+1)h_{t-k-1}} \right) E \left( e^{(3p+1)h_{t-k}} \right) E \left( e^{3h_{t}} \right)}{E \left( e^{(3p+1+p+1)h_{t-k-1}} \right) E \left( e^{3ph_{t-k}} \right)};
\]

\[
m_{2;2,2}^{(k)} = E \left( r_{t}^{2} r_{t-1}^{2} r_{t-k}^{2} \right) = \frac{E \left( e^{(p+p+1)h_{t-k}} \right) E \left( e^{(p+1)h_{t-1}} \right) E \left( e^{h_{t}} \right)}{E \left( e^{(p+p+1)h_{t-k}} \right) E \left( e^{ph_{t-1}} \right)};
\]
Chapter 5. Asymmetric laplace SV model

\[ m_{2,2;4}^{(k)} = E \left( t_{i}^{2} r_{i-1}^{2} t_{i-k}^{4} \right) = 6 \frac{E \left( e^{(\rho^{k}+\rho^{k-1}+2)h_{i-k}} \right) E \left( e^{(\rho^{k}+1)h_{i-1}} \right) E \left( e^{h_{i}} \right)}{E \left( e^{(\rho^{k}+\rho^{k-1})h_{i-1}} \right) E \left( e^{(\rho^{k}+\rho^{k-1})h_{i-k}} \right) E \left( e^{h_{i}} \right)}; \]

\[ m_{2,2;6}^{(k)} = E \left( t_{i}^{2} r_{i-1}^{2} t_{i-k}^{6} \right) = 90 \frac{E \left( e^{(\rho^{k}+\rho^{k-1}+3)h_{i-k}} \right) E \left( e^{(\rho^{k}+1)h_{i-1}} \right) E \left( e^{h_{i}} \right)}{E \left( e^{(\rho^{k}+\rho^{k-1})h_{i-1}} \right) E \left( e^{(\rho^{k}+\rho^{k-1})h_{i-k}} \right) E \left( e^{h_{i}} \right)}; \]

\[ m_{2,2;2,2}^{(k)} = E \left( t_{i}^{2} r_{i-1}^{2} t_{i-k}^{2} t_{i-k-1}^{2} \right) = \frac{E \left( e^{(\rho^{k}+\rho^{k-1}+\rho^{k-2})h_{i-k-1}} \right) E \left( e^{(\rho^{k}+\rho^{k-1}+1)h_{i-k}} \right) E \left( e^{(\rho^{k}+1)h_{i-1}} \right) E \left( e^{h_{i}} \right)}{E \left( e^{(\rho^{k}+\rho^{k-1})h_{i-1}} \right) E \left( e^{(\rho^{k}+\rho^{k-1})h_{i-k}} \right) E \left( e^{h_{i}} \right)}; \]

where

\[ E \left( e^{h_{i}} \right) = \frac{e^{\rho \theta}}{1 - \frac{\sigma^{2} \rho^{2}}{2} - \frac{\sigma \rho}{\sqrt{2}} \left( \frac{1}{\kappa} - \kappa \right)}. \]

Let \( \Gamma^{(k)} = \begin{pmatrix} \gamma_{11}^{(k)} & \gamma_{12}^{(k)} & \gamma_{13}^{(k)} & \gamma_{14}^{(k)} \\ \gamma_{21}^{(k)} & \gamma_{22}^{(k)} & \gamma_{23}^{(k)} & \gamma_{24}^{(k)} \\ \gamma_{31}^{(k)} & \gamma_{32}^{(k)} & \gamma_{33}^{(k)} & \gamma_{34}^{(k)} \\ \gamma_{41}^{(k)} & \gamma_{42}^{(k)} & \gamma_{43}^{(k)} & \gamma_{44}^{(k)} \end{pmatrix} \), \( k = 0, \pm 1, \pm 2, \ldots \). Then the \( 4 \times 4 \) matrix \( S \) is given by \( S = \Gamma^{(0)} + 2 \sum_{k=1}^{\infty} \Gamma^{(k)} \).

When \( k = 0 \), the elements of \( \Gamma^{(0)} = E \left( \omega_2 \omega_4' \right) \) are obtained as

\[ \gamma_{11}^{(0)} = m_{2,2}^{(0)} - c_1^2, \]
\[ \gamma_{12}^{(0)} = \gamma_{21}^{(0)} = m_{2,4}^{(0)} - c_1 c_2, \]
\[ \gamma_{13}^{(0)} = \gamma_{31}^{(0)} = m_{2,6}^{(0)} - c_1 c_3, \]
\[ \gamma_{14}^{(0)} = \gamma_{41}^{(0)} = m_{2,2;2}^{(0)} - c_1 c_4, \]
\[ \gamma_{22}^{(0)} = \gamma_{33}^{(0)} = \gamma_{44}^{(0)} = 0. \]
\[ \gamma_{22}^{(0)} = m_{4:4}^{(0)} - c_2^2, \]
\[ \gamma_{23}^{(0)} = \gamma_{32}^{(0)} = m_{4:6}^{(0)} - c_2 c_3, \]
\[ \gamma_{24}^{(0)} = \gamma_{42}^{(0)} = m_{4:2:2}^{(0)} - c_2 c_4, \]
\[ \gamma_{33}^{(0)} = \gamma_{33}^{(0)} = m_{6:6}^{(0)} - c_3^2, \]
\[ \gamma_{34}^{(0)} = \gamma_{43}^{(0)} = m_{6:2:2}^{(0)} - c_3 c_4, \]
\[ \gamma_{44}^{(0)} = m_{2:2:2:2}^{(0)} - c_4, \]

where \( c_1, c_2, c_3 \) and \( c_4 \) as in (5.12).

Similarly, the following are the elements of \( \Gamma_k \) for \( k = 1, 2, ... \)

\[ \gamma_{11}^{(k)} = m_{2:2}^{(k)} - c_1^2; \quad \gamma_{12}^{(k)} = m_{2:4}^{(k)} - c_1 c_2; \quad \gamma_{13}^{(k)} = m_{2:6}^{(k)} = c_1 c_3; \quad \gamma_{14}^{(k)} = m_{2:2:2}^{(k)} - c_1 c_4, \]
\[ \gamma_{21}^{(k)} = m_{4:2}^{(k)} - c_1 c_2; \quad \gamma_{22}^{(k)} = m_{4:4}^{(k)} - c_2^2; \quad \gamma_{23}^{(k)} = m_{4:6}^{(k)} - c_2 c_3; \quad \gamma_{24}^{(k)} = m_{4:2:2}^{(k)} - c_2 c_4, \]
\[ \gamma_{31}^{(k)} = m_{6:2}^{(k)} - c_1 c_3; \quad \gamma_{32}^{(k)} = m_{6:4}^{(k)} - c_2 c_3; \quad \gamma_{33}^{(k)} = m_{6:6}^{(k)} - c_3^2; \quad \gamma_{34}^{(k)} = m_{6:2:2}^{(k)} - c_3 c_4, \]
\[ \gamma_{41}^{(k)} = m_{2:2:2}^{(k)} - c_1 c_4; \quad \gamma_{42}^{(k)} = m_{2:2:4}^{(k)} - c_2 c_4; \quad \gamma_{43}^{(k)} = m_{2:2:6}^{(k)} - c_3 c_4; \quad \gamma_{44}^{(k)} = m_{2:2:2:2}^{(k)} - c_4^2, \]

The 4 \times 4 matrix \( D \) is evaluated using the form \( D = E \left( \frac{d}{dt} f (r_t, r_{t-1}, \nu) \right) \) and its elements are:

\[ D_{11} = - \frac{e^\theta}{1 - \frac{\sigma^2}{2} - \frac{\sigma}{\sqrt{2}} \left( \frac{1}{\kappa} - \kappa \right)} \]
\[ D_{12} = - \frac{12e^{2\theta}}{1 - 2\sigma^2 - \sqrt{2} \sigma \left( \frac{1}{\kappa} - \kappa \right)} \]
\[ D_{13} = - \frac{270e^{3\theta}}{1 - \frac{9}{2} \sigma^2 - \frac{3}{\sqrt{2}} \sigma \left( \frac{1}{\kappa} - \kappa \right)} \]
\[ D_{14} = -2e^{2\theta} \left[ 1 - \sigma^2 \left( \frac{1}{\kappa} - \kappa \right) \right] \left[ 1 - \frac{\sigma^2 (1 + \rho)^2}{2} - \frac{\sigma}{\sqrt{2}} \left( \frac{1}{\kappa} - \kappa \right) (1 + \rho) \right] \left[ 1 - \frac{\sigma^2}{2} - \frac{\sigma}{\sqrt{2}} \left( \frac{1}{\kappa} - \kappa \right) \right], \]

\[ D_{21} = \frac{e^{\theta} \sigma \left( \frac{1}{\kappa^2} + 1 \right)}{\sqrt{2} \left[ 1 - \frac{\sigma^2}{2} - \frac{\sigma}{\sqrt{2}} \left( \frac{1}{\kappa} - \kappa \right) \right]^2}, \]

\[ D_{22} = \frac{6\sqrt{2}e^{2\theta} \sigma \left( \frac{1}{\kappa^2} + 1 \right)}{\left[ 1 - 2\sigma^2 - \sqrt{2}\sigma \left( \frac{1}{\kappa} - \kappa \right) \right]^2}, \]

\[ D_{23} = \frac{270e^{3\theta} \sigma \left( \frac{1}{\kappa^2} + 1 \right)}{\sqrt{2} \left[ 1 - \frac{9\sigma^2}{2} - \frac{3}{\sqrt{2}} \sigma \left( \frac{1}{\kappa} - \kappa \right) \right]^2}, \]

\[ D_{24} = -\frac{\sigma e^{2\theta} \left( \frac{1}{\kappa^2} + 1 \right)}{\left\{ d_1 (\sigma, \kappa, \rho) d_2 (\sigma, \kappa, \rho) \right\}}, \]

\[ + \frac{e^{2\theta} d_3 (\sigma, \kappa, \rho) \left\{ d_1 (\sigma, \kappa, \rho) \frac{\sigma}{\sqrt{2}} \left( \frac{1}{\kappa^2} + 1 \right) + d_2 (\sigma, \kappa, \rho) \frac{\sigma(1 + \rho)}{\sqrt{2}} \left( \frac{1}{\kappa^2} + 1 \right) \right\}}{\left\{ d_1 (\sigma, \kappa, \rho) d_2 (\sigma, \kappa, \rho) \right\}^2}, \]

\[ D_{31} = \frac{e^{\theta} \left[ \sigma + \frac{1}{\sqrt{2}} \left( \frac{1}{\kappa} - \kappa \right) \right]}{\left[ 1 - \frac{\sigma^2}{2} - \frac{\sigma}{\sqrt{2}} \left( \frac{1}{\kappa} - \kappa \right) \right]^2}, \]

\[ D_{32} = \frac{6e^{2\theta} \left[ 4\sigma + \sqrt{2} \left( \frac{1}{\kappa} - \kappa \right) \right]}{\left[ 1 - 2\sigma^2 - \sqrt{2}\sigma \left( \frac{1}{\kappa} - \kappa \right) \right]^2}, \]

\[ D_{33} = \frac{90e^{3\theta} \left[ 9\sigma + \frac{3}{\sqrt{2}} \left( \frac{1}{\kappa} - \kappa \right) \right]}{\left[ 1 - \frac{9\sigma^2}{2} - \frac{3}{\sqrt{2}} \sigma \left( \frac{1}{\kappa} - \kappa \right) \right]^2}, \]

\[ D_{34} = -\frac{e^{2\theta} \left[ \sigma \rho^2 + \frac{\rho}{\sqrt{2}} \left( \frac{1}{\kappa} - \kappa \right) \right]}{\left\{ d_1 (\sigma, \kappa, \rho) d_2 (\sigma, \kappa, \rho) \right\}} \]

\[ + \frac{e^{2\theta} \left[ d_3 (\sigma, \kappa, \rho) \left\{ d_1 (\sigma, \kappa, \rho) \left[ \sigma + \frac{1}{\sqrt{2}} \left( \frac{1}{\kappa} - \kappa \right) \right] + d_2 (\sigma, \kappa, \rho) \left[ \sigma(1 + \rho)^2 + \frac{1}{\sqrt{2}} \left( \frac{1}{\kappa} - \kappa \right) (1 + \rho) \right] \right\} \right\}}{\left\{ d_1 (\sigma, \kappa, \rho) d_2 (\sigma, \kappa, \rho) \right\}^2}, \]

\[ D_{41} = D_{42} = D_{43} = 0. \]
\[ D_{44} = - e^{2\theta} \left\{ -\sigma^2 \rho - \frac{\sigma}{\sqrt{2}} \left( \frac{1}{\kappa} - \kappa \right) \right\} \]
\[
\left\{ 1 - \frac{\sigma^2(1+\rho)^2}{2} - \frac{\sigma}{\sqrt{2}} \left( \frac{1}{\kappa} - \kappa \right) (1 + \rho) \right\} \left\{ 1 - \frac{\sigma^2}{2} - \frac{\sigma}{\sqrt{2}} \left( \frac{1}{\kappa} - \kappa \right) \right\} \}
\]
\[
+ e^{2\theta} \left\{ 1 - \frac{\sigma^2 \rho^2}{2} - \frac{\sigma \rho}{\sqrt{2}} \left( \frac{1}{\kappa} - \kappa \right) \right\} \left\{ 1 - \frac{\sigma^2}{2} - \frac{\sigma}{\sqrt{2}} \left( \frac{1}{\kappa} - \kappa \right) (1 + \rho) \right\} \left\{ 1 - \frac{\sigma^2}{2} - \frac{\sigma}{\sqrt{2}} \left( \frac{1}{\kappa} - \kappa \right) \right\}^2 ,
\]
where
\[
d_1 (\sigma, \kappa, \rho) = \left[ 1 - \frac{\sigma^2(1+\rho)^2}{2} - \frac{\sigma}{\sqrt{2}} \left( \frac{1}{\kappa} - \kappa \right) (1 + \rho) \right] ;
\]
\[
d_2 (\sigma, \kappa, \rho) = \left[ 1 - \frac{\sigma^2}{2} - \frac{\sigma \rho}{\sqrt{2}} \left( \frac{1}{\kappa} - \kappa \right) \right] ;
\]
\[
d_3 (\sigma, \kappa, \rho) = \left[ 1 - \frac{\sigma^2 \rho^2}{2} - \frac{\sigma \rho}{\sqrt{2}} \left( \frac{1}{\kappa} - \kappa \right) \right] .
\]

Hence the asymptotic dispersion matrix becomes \( \frac{1}{T} \Sigma \), where
\[
\Sigma = \left[ DS^{-1} D' \right]^{-1} .
\]

### 5.7 Simulation Study

We carry out a simulation study to evaluate the performance of the proposed estimators with sample sizes 1000 and 3000. First, we generate a sample of size \( T \) from AL Markov sequence specified in (5.5) using the innovation random variable described in (5.7). Then simulate the sequence \( \{r_t\} \) using (5.8) model. We use this simulated sample to obtain the estimates of the parameters by solving the moment
equations given in the Section 5.5. For each specified value of parameter, we repeat the experiment 1000 times for computing the estimates and then averaged them over the repetitions. The average estimates and the corresponding RMSEs (within parentheses) based on the simulated samples are reported in Table 5.1 and 5.2.

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<th>$\sigma$</th>
<th>$\hat{\rho}$</th>
<th>$\hat{\sigma}$</th>
<th>$\hat{\kappa}$</th>
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Table 5.1: The average estimates and the corresponding mean square error of moment estimates based on sample of size n=1000, when $\kappa=2$, $\theta=1$ and for different values of $\rho$ and $\sigma$.

From the above tables, we observe that the estimates are slightly biased. When the sample size is large, the estimators behave reasonably well and there is a significant reduction in bias of the estimates. Hence we claim that the method of moment estimation yields good estimates for the parameters involved.
### Table 5.2: The average estimates and the corresponding mean square error of moment estimates based on sample of size n=3000, when $\kappa=2$, $\theta=1$ and for different values of $\rho$ and $\sigma$

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<td>1.4751(0.1611)</td>
<td>1.9746(0.1845)</td>
<td>0.9793(0.1644)</td>
</tr>
</tbody>
</table>

#### 5.8 Data Analysis

To illustrate the application of the proposed model and the associated inferential results, we analyse two sets of financial data. The data sets used for this purpose are: (1) the daily exchange rate of Rupee/Pound Sterling, with the data consisting of 2399 observations from January 02, 2007 to December 15, 2016 obtained from Database on Indian Economy, Reserve Bank of India; (2) the daily average price of crude oil futures (USD/1 Barrel) traded in Multi Commodity Exchange of India Ltd (MCX), India with the data consisting of 1789 observations January 04, 2010 to December 16, 2016.
Chapter 5. Asymmetric laplace SV model

The time series plots of these data are given in Figure 5.3. The left panel show the plots of actual data series and the return series are on the right panels.

![Time series plots](image)

**Figure 5.3:** Time series plot of the original data and the returns

Table 5.3 summarizes the descriptive statistics of the return series, including the mean, median, standard deviation, skewness, kurtosis and Jarque-Bera statistic. The return series are slightly skewed and the kurtosis is well above three, indicating that the return distribution is asymmetric and leptokurtic for both the series. The Jarque-Bera statistic is calculated for the test of joint hypothesis of zero skewness and excess kurtosis and statistic value clearly indicates the return data is non-normal. $Q(20)$ and $Q^2(20)$ are the Ljung-Box statistic for return and squared return series with lag 20. The corresponding $\chi^2$ table value at 5% significance level is
Hence the test suggests that the return series is serially uncorrelated whereas the squared return series has significant serial correlation.

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Exchange rate</th>
<th>Crude oil futures</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sample Size</td>
<td>2398</td>
<td>1789</td>
</tr>
<tr>
<td>Mean</td>
<td>0.0001</td>
<td>-0.0002</td>
</tr>
<tr>
<td>Minimum</td>
<td>-0.0655</td>
<td>-0.0867</td>
</tr>
<tr>
<td>Maximum</td>
<td>0.0374</td>
<td>0.1232</td>
</tr>
<tr>
<td>Std. Dev.</td>
<td>0.0071</td>
<td>0.0212</td>
</tr>
<tr>
<td>Skewness</td>
<td>-0.6628</td>
<td>0.3535</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>9.9279</td>
<td>5.8937</td>
</tr>
<tr>
<td>$Q(20)$</td>
<td>3.2262</td>
<td>0.0723</td>
</tr>
<tr>
<td>$Q^2(20)$</td>
<td>11.7232</td>
<td>20.2069</td>
</tr>
<tr>
<td>Jarque-Bera</td>
<td>4971.2720</td>
<td>661.4428</td>
</tr>
</tbody>
</table>

Table 5.3: Summary statistics of return series

From the ACF of the returns plotted in Figure 5.4 (left panel), it is observed that serial correlations in the return series are insignificant whereas the ACF of the squared returns in the right panel are significant and declines with increasing lags very slowly. In Table 5.4, we present the parameter estimates for each of the return series. The values of the $\hat{\rho}$ in the Table suggest that there is a significant persistence of volatility in the above data series.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Exchange rate</th>
<th>Crude oil futures</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\theta}$</td>
<td>0.0003</td>
<td>-0.1030</td>
</tr>
<tr>
<td>$\hat{\kappa}$</td>
<td>0.5122</td>
<td>0.4971</td>
</tr>
<tr>
<td>$\hat{\sigma}$</td>
<td>0.2177</td>
<td>0.1076</td>
</tr>
<tr>
<td>$\hat{\rho}$</td>
<td>0.7810</td>
<td>0.8620</td>
</tr>
</tbody>
</table>

Table 5.4: Parameter estimates using method of moments
We now perform a diagnostic check of the model based on the residuals. For the AL-SV model, once the estimates of the parameters are obtained, the unobservable component \( \{h_t\} \) is estimated using an approximate Kalman filtering (for details see Jacquier et al. (1994)). We define the residuals of the model as \( \hat{\varepsilon}_t = r_t \exp(-\hat{h}_t/2) \), where \( \hat{h}_t \) is the estimator of \( h_t \) provided by the Kalman filter at the MM estimate. If the fitted model is adequate, then \( \{\hat{\varepsilon}_t\} \) should behave as an independent and identically distributed sequence of random variables with the assumed distribution. Since the model assumes that the residuals are independent, any dependence on either the residuals or their squares indicates misspecification of the model. In particular, if the fitted model is adequate, both series should have no autocorrelations.

Figure 5.5 gives the sample autocorrelation of the residuals for the fitted AL-SV model.
model. From the figures, the residuals appear to be random and their ACFs fail to indicate any significant serial dependence. Further, we also checked the significance of ACF in the residues by computing the Ljung-Box statistic for the series \( \{ \hat{\varepsilon}_t \} \) and \( \{ \hat{\varepsilon}^2_t \} \), which are summarized in the Table 5.5. All these values are less than the 5\% chi-square critical value 10.117 at degrees of freedom 19. Hence we conclude that there is no significant serial dependence among the residuals and the squared residuals.

<table>
<thead>
<tr>
<th>Data</th>
<th>Ljung-Box statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Residuals</td>
</tr>
<tr>
<td>Exchange rate</td>
<td>2.4065</td>
</tr>
<tr>
<td>Crude oil futures</td>
<td>0.8789</td>
</tr>
</tbody>
</table>

Table 5.5: Ljung-Box Statistic for the residuals and squared residuals
Thus the data analysis illustrates that the proposed model is capable of capturing the stylized features of financial return series.

The results of this chapter are reported in Balakrishna and Rahul (2017b).